INVESTIGATIONS INTO SOME OF THE PROPERTIES
OF THE CONIC SECTIONS IN THREE DIMENSIONS

by

Richard Bieberich

Senior Honors Thesis

ID 499

Advisor: Luane E. Deal

June 5, 1966
The author's motivation should be presented for the mathematics which is to follow. In a classroom demonstration given during the author's student teaching, a stringed model of a cone was being used to illustrate the origin of the conic sections. A student asked whether the focus of a parabola lay on the axis of the cone from which it is produced. He was answered positively, but the author, being in a somewhat subordinate position, disagreed quietly. While attempting to prove that this was not generally the case—that the focus of a parabola need not lie on the axis of the cone from which it is produced—the author's interest in and appreciation for the many aspects of that family of curves known as the conic sections increased.

While attempting to solve the problem dealing with the focus of a parabola, the spatial aspects—the physical intersection of planes and cones—drew the author's attention. The rotation of figures in space led to an attempt to describe some three-dimensional interpretations of the asymptotes of a hyperbola. This led to a successful attempt to describe, given a hyperbola, a cone and a plane which can produce this hyperbola. The next obvious
question was whether there existed another situation, involving a completely different combination of plane and cone, which would produce this same hyperbola. After this existence was proved, several unsuccessful attempts were made to display an infinite number of combinations of cone and plane which would all duplicate the same hyperbola.

A mental time-out was taken, and a proof that a specific hyperbola may be produced from a given cone by a unique section was forged.

The problem referred to above—that of displaying an infinite number of combinations of cone and plane which would produce the same hyperbola—has not been solved. There are several approaches, each of which requires much algebra and time.

The basic reference for this paper has been the third edition of John Cell's Analytic Geometry. This book was published in New York by John Wiley and Sons, Inc., in 1950. Only basic mechanical devices were obtained there.
CONTENTS

1. The locus of the foci of all parabolas produced on a specific cone .......................... 1
2. A general source for a hyperbola ............... 13
3. Three-dimensional interpretations of the asymptotes of a hyperbola .................. 16
4. The non-uniqueness of the source of a hyperbola .................................................. 22
5. The uniqueness of hyperbola production in a fixed cone ............................. 37
6. Projections of further work to be done ........ 43
1. The locus of the foci of all parabolas produced on a specific cone.

Consider the equation of a right circular cone

\[(1.01) \quad b^2z^2 = a^2x^2 + a^2y^2 \]

whose vertex is at the origin, and whose axis is the \(z\)-axis. Consider the family of planes

\[(1.02) \quad z = - \frac{ax}{b} + ka. \]

In order to produce a parabola, a plane must intersect a cone and be parallel to one of the elements of the cone. Each member of the family (1.02) is obviously parallel to

\[(1.03) \quad z = - \frac{ax}{b}, \quad y = 0 \]

which is an element of the cone (1.01). Therefore, the intersection of each member of this family of planes with (1.01) will produce a parabola.

In Figure 1, plane BCDE represents

\[(1.04) \quad z = - \frac{ax}{b} + 2a \]

which is a member of the family (1.02), passes through
(b,0,a), and is parallel to (1.03). The intersection of the plane (1.04) with the cone (1.01) produces the parabola which will be considered throughout the remainder of this section. This parabola's vertex is at (b,0,a) and its axis is

\[(1.05) \quad z = -\frac{ax}{b} + 2a, y = 0.\]

From Figure 1, the distance from the vertex (b,0,a) to the z-axis along (1.05), the axis of the parabola, is \(\sqrt{a^2 + b^2}\), or c, from the Pythagorean relationship. If it can be shown that the distance from the vertex of this parabola to its focus is also c, then the locus of the foci of all parabolas obtained by sectioning a fixed cone will be the axis of that cone. If this locus is not the axis of the cone, the proper locus will be displayed.
In order to determine the distance from this parabola's vertex to its focus, the parabola will undergo a sequence of transformations which will place it in a coordinate plane, where the desired distance may be obtained by inspection of the coefficients of the parabola's transformed equation. The vertex will be placed at the origin by the following translation:

\[ \begin{align*}
x' &= x - b \\
z' &= z - a \\
y' &= y.
\end{align*} \tag{1.06} \]

Under the equations of transformation (1.06), the cone (1.01) becomes

\[ b^2z'^2 + 2ab^2z' = a^2x'^2 + 2a^2bx' + a^2y'^2. \tag{1.07} \]

Under the same equations (1.06), the plane (1.04) becomes

\[ z' = - \frac{ax'}{b}. \tag{1.08} \]

The parabola is now in the situation described in Figure 2.
The parabola will now be placed in the x"y"-plane by a rotation of the axes around the y'-axis. The equations of transformation are:

\[
\begin{align*}
x' &= x''\cos(180^\circ + \theta) - z''\sin(180^\circ + \theta) \\
z' &= x''\sin(180^\circ + \theta) + z''\cos(180^\circ + \theta) \\
y' &= y''.
\end{align*}
\]

The trigonometric functions of \( \theta \) can be determined from Figure 2. Therefore, the above equations become

\[
\begin{align*}
x' &= -\frac{b x''}{c} - \frac{a z''}{c} \\
z' &= \frac{a x''}{c} - \frac{b z''}{c} \\
y' &= y''.
\end{align*}
\]

After applying equations (1.09) and some algebraic simplification, the cone (1.07) becomes

\[
\begin{align*}
(\phi^4 - a^4)z''^2 + (2a^3bc - 2ab^3c)z'' + \\
4a^2b^2c x'' - a^2c^2y''^2 - (2ab^3 + 2a^3b)x''z'' &= 0.
\end{align*}
\]

The plane (1.05) quite easily becomes

\[
z'' = 0.
\]

The parabola being discussed has been moved to a situation where it may be obtained by intersecting (1.10)
with (1.11). If this is performed algebraically, the equation of the parabola will be obtained, and will be in the two variables, $x''$ and $y''$. This equation is

$$a^2c^2y''^2 = 4a^2b^2cx''$$

$$y''^2 = \frac{4b^2x''}{c}$$

(1.12)

The parabola is now in the position illustrated in Figure 3.

![Figure 3](image)

Look at Figure 1, Figure 2, and Figure 3 in that order. Notice that the parabola has not moved, but that the axes have moved. As a result, Figure 3 is very unclear. Figure 3 is redrawn from a better point of view in Figure 4.
From Cell's *Analytic Geometry*, page 98, the distance from a parabola's vertex to its focus is equal to one-fourth of the coefficient of the single-powered variable, if the parabola is of the following form:

\[(\text{one variable})^2 = \text{constant}(\text{second variable}).\]

But one-fourth of the coefficient on the \(x''\) term in (1.12) is

\[(1.13) \quad \frac{b^2}{c} = \frac{b^2}{\sqrt{a^2 + b^2}}.\]

(1.13) expresses the distance from the vertex to the focus* of the parabola produced by the intersection of (1.01) with (1.04). Set (1.13) equal to the distance from the vertex of this parabola to the axis of (1.01)* and inspect the equality.

*Both of these distances were measured along the same line, the axis of the parabola, so they could be properly compared.
\[
\frac{b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}
\]

\[
b^2 = a^2 + b^2
\]

\[
a = 0
\]

(1.14) is not an identity, and can only be an equality when \( a \) is zero. In general, then, (1.14) is not an equality. In other words, the locus of the foci of the parabolas obtained from a fixed cone is not the axis of that cone. The true locus will now be displayed.

Consider the family of planes (1.02). The intersection of each member of this family with (1.01) defines a parabola whose focus is in the \( xz \)-coordinate plane. The locus of the foci in the \( xz \)-plane will be one element of the locus of the foci of all parabolas obtained from (1.01). If this first locus—in the \( xz \)-coordinate plane—is found and rotated around the \( z \)-axis, the locus of the foci of all parabolas obtained from this cone will be described.

Consider Figure 5.

\[\text{Figure 5}\]
The larger parabola, whose vertex is point P, is produced by the intersection of (1.01) with

\[ z = -\frac{ax + 2\lambda a}{b} \]

while the smaller parabola, whose vertex is point E, is produced by the intersection of (1.01) with (1.04), and is the parabola which has been considered throughout this section.

In Figure 5, B and A are the foci of these two parabolas respectively. Inspect points O, A, and B to see if they lie on a straight line.

Figure 5 is partially redrawn in Figure 6 for clarity.

Figure 6

Triangles OEB and OFD are similar. If it can be shown that

\[ \frac{EA}{EC} = \frac{FB}{FD} \]

then points O, A, and B will have been shown to lie on a straight line.
From the previous development,

\[(1.17) \quad \overline{EA} = \frac{b^2}{c} \]

and

\[(1.18) \quad \overline{EC} = c = \sqrt{a^2 + b^2} \]

When the results in (1.17) and (1.18) are substituted into (1.16), (1.16) becomes

\[\frac{b^2}{a^2 + b^2} = \frac{\overline{EF}}{\overline{FD}}.\]

\(\overline{EF}\) and \(\overline{FD}\) may be found in exactly the same manner as were \(\overline{EA}\) and \(\overline{EC}\). By the Pythagorean relationship,

\[(1.19) \quad \overline{FD} = \sqrt{L^2b^2 + L^2a^2} = Lc.\]

If the intersection of (1.15) and (1.01) is considered after these equations have been transformed by (1.06) and (1.09), inspection of the equation of this parabola will yield

\[(1.20) \quad \overline{FB} = \frac{b^2L}{c}.\]

When the results from (1.17), (1.18), (1.19), and (1.20) are substituted into (1.15), (1.16) becomes the following:
\[
\frac{EC}{FD} = \frac{EA}{FB}
\]

(1.21)

\[
\frac{b^2}{c^2} = \frac{\frac{b^2}{c}}{} \cdot \frac{1}{Lc}
\]

\[
\frac{b^2}{c^2} = \frac{b^2}{c^2}
\]

Therefore, \( O, A, \) and \( B \) do lie on a straight line.

The locus of the focus of the parabolas obtained from
the intersection of (1.01) with the family of planes (1.02)
is a straight line which passes through points \( O, A, \) and \( B.\)
If the equation of this line is determined and rotated
around the \( z \)-axis, the locus of the focus of all parabolas
obtained from (1.01) will be displayed.

Find point \( A. \) The equation of line \( EA \) is (1.05),
and point \( E \) has coordinates \((b,0,0)\). The distance from
\( E \) to \( A \) is (1.15). By the distance formula,

\[
\sqrt{(b - x)^2 + (z - a)^2} = EA = \frac{b^2}{\sqrt{a^2 + b^2}}
\]

\[
\frac{1}{b} \sqrt{(b^2 + a^2)(b - x)^2} = \frac{b^2}{\sqrt{a^2 + b^2}}
\]

\[
\frac{b - x}{b} \sqrt{a^2 + b^2} = \frac{b^2}{\sqrt{a^2 + b^2}}
\]

\[
x = \frac{a^2b}{a^2 + b^2}
\]

(1.22)
(1.22) is the x-coordinate of point A. Substitute this value into the equation of the line, (1.05).

\[ z = - \frac{a \left( \frac{a^2 b}{a^2 + b^2} \right)}{b} + 2a \]

\[ z = - \left( \frac{a^3 b}{a^2 + b^2} \cdot \frac{1}{b} \right) + 2a \]

\[ z = - \frac{a^3 + 2a^3 + 2ab^2}{a^2 + b^2} \]

\[ z = \frac{a(a^2 + 2b^2)}{a^2 + b^2} \]  

(1.23) is the z-coordinate of point A. Therefore, one element of the desired locus passes through the origin and point A, whose coordinates are:

\[ \left( \frac{a^2 b}{a^2 + b^2}, 0, \frac{a(a^2 + 2b^2)}{a^2 + b^2} \right) \]  

(1.24)

The equation of this element is

\[ z = \left( \frac{a(a^2 + 2b^2)}{a^2 + b^2}, \frac{a^2 + b^2}{a^2 b} \right) x \]

\[ = \left( \frac{a^2 + 2b^2}{ab} \right) x. \]  

(1.25)
When this element, (1.25), is rotated around the z-axis according to Figure 7, a cone will be obtained.*

\[
Z = \left( \frac{a^2 + 2b^2}{ab} \right) X
\]

\[
Z = z = \left( \frac{a^2 + 2b^2}{ab} \right) X
\]

\[
z^2 = \left( \frac{a^2 + 2b^2}{ab} \right)^2 \cdot x^2
\]

\[
x^2 = (\overline{PM})^2 = (\overline{NM})^2 = x^2 + y^2
\]

\[
z^2 = \left( \frac{a^2 + 2b^2}{ab} \right)^2 (x^2 + y^2)
\]

\[(1.26) \quad \frac{z^2}{(a^2 + 2b^2)^2} = \frac{x^2}{a^2b^2} + \frac{y^2}{a^2b^2}
\]

The locus of the focus of all parabolas obtained from (1.01) is (1.26), a right circular cone whose axis is the z-axis.

*This is a standard rotation mechanism. It comes from Cell's Analytic Geometry, page 271, and will be used several more times in the remainder of this paper. The reference is mentioned here because, in the instances where this mechanism is used in the following pages, an explanatory figure similar to Figure 7 and the development similar to (1.26) will not be included.
2. A general source for a hyperbola.

Consider a general hyperbola of the following form:

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.
\]

The transverse axis of this hyperbola is the z-axis, and its center is the origin. \((b,0)\) and \((-b,0)\) are end-points of the conjugate axis of this hyperbola, and \(a\) and \(-a\) are its z-intercepts. The graph of (2.1) has its foci at \((0,c)\) and \((0,-c)\), where \(c^2 = a^2 + b^2\).

Figure 8

A hyperbola may be produced by intersecting a right circular cone with a plane which is parallel to the axis of the cone.
(2.2) Theorem: A hyperbola which is of the same form as (2.1) may be produced by intersecting

\[(2.3) \quad b^2 z^2 = a^2 x^2 + a^2 y^2\]

with

\[(2.4) \quad x = b.\]

Proof. Perform the intersection algebraically.

\[
b^2 z^2 = a^2 (b)^2 + a^2 y^2
\]

\[
1 = \frac{z^2}{a^2} - \frac{y^2}{b^2}.
\]

(2.5) Theorem: The distance from the vertex of a cone to a vertex of a hyperbola obtained from the cone is equal to c, where c is the distance from the intersection of the asymptotes of the hyperbola to a focus of the hyperbola.

Proof. Consider the hyperbola (2.1) produced by the intersection of (2.3) with (2.4). From Figure 8, \(c = \sqrt{a^2 + b^2}\), where c is the distance from the origin to the focus. The hyperbola in Figure 8 is now redrawn in its proper three-dimensional position in Figure 9.
From Figure 9,

\[ \overline{AF}^2 + \overline{FO}^2 = \overline{AO}^2 \]
\[ a^2 + b^2 = \overline{AO}^2 = c^2 \]
\[ \sqrt{a^2 + b^2} = \overline{AO} = c. \]
3. Three-dimensional interpretations of the asymptotes of a hyperbola.

Consider the hyperbola

\[ \frac{z^2}{a^2} - \frac{x^2}{b^2} = 1, \quad x = b, \]

whose graph is drawn in Figure 3.

The asymptotes of (3.01) are \( y = \frac{bz}{a} \) and \( y = -\frac{bz}{a} \). The proof of this consists of solving (3.01) for \( y \) and showing that, as \( z \) increases without limit, \( y \) approaches \( \pm \frac{bz}{a} \). The proof is completed by showing that the perpendicular distance from a general point on the hyperbola to \( y = \pm \frac{bz}{a} \) approaches zero as \( z \) gets infinitely large.*

Two three-dimensional interpretations for the asymptotes of a hyperbola will now be displayed.

The hyperbola (3.01) is produced by intersecting

\[ b^2z^2 = a^2x^2 + a^2y^2 \]

with

\[ x = b. \]

*Cell, Analytic Geometry, page 115.
The asymptotes of (3.01) are

\[(3.04) \quad y = \pm \frac{bz}{a}, \quad x = b.\]

Observe that (3.01) and (3.04) are written as pairs of equations. The reason for this is that (3.01) and (3.04) each describe a line (or lines) in space. A line in a three-dimensional coordinate system must be described as the intersection of two surfaces. When the equations of two surfaces are written down to describe a line in space, they are called parametric equations for that line.

Remove the second parametric equation from both (3.01) and (3.04). The equations are now

\[(3.05) \quad \frac{z^2}{a^2} - \frac{y^2}{b^2} = 1\]

and

\[(3.06) \quad y = \pm \frac{bz}{a}.\]

(3.05) and (3.06) are cylinders, and (3.05), a hyperbolic cylinder, approaches the cylinders described by (3.06) in the same manner as the hyperbola (3.01) approaches its asymptotes (3.04). (3.05) and (3.06) are drawn in Figure 10.
The situation described in Figure 10 is the first three-dimensional interpretation of the asymptotes of a hyperbola. The second interpretation will now be displayed.

Consider the traces of (3.05) and (3.06) in the yz-plane. The yz-trace of (3.05) has the following parametric equations:

\[(3.07) \quad \frac{z^2}{a^2} - \frac{y^2}{b^2} = 1, \ x = 0.\]

The yz-trace of (3.06) is described by

\[(3.08) \quad y = \pm \frac{b}{a} z, \ x = 0.\]

The graphs of (3.07) and (3.08) are drawn in Figure 11.
rotate (3.07) and (3.08) around the z-axis. (3.08), when rotated, becomes (3.02). When (3.07) is rotated, a circular hyperboloid of two sheets,

$$\frac{z^2}{a^2} - \frac{x^2}{b^2} - \frac{y^2}{b^2} = 1,$$

is obtained.

The hyperboloid of two sheets (3.09) is now "resting" inside the cone (3.02), and has much the same relationship to the cone as a hyperbola has to its asymptotes.

The fact that the cone (3.02) is an asymptote to (3.09) may be proven in the same way in which $y = \pm \frac{b}{a}$
were proven to be asymptotes for (3.01). Pass a plane

(3.10) \[ z = k \]

through the cone and the hyperboloid of two sheets. The results of this will be the equations of two circles. Obtain the radius of each of these circles and show that, as \( k \) approaches infinity, the difference between the radii of the two circles approaches zero.

Intersect (3.09) with (3.10).

\[
\frac{x^2}{a^2} - \frac{x^2}{b^2} - \frac{y^2}{b^2} = 1
\]

\[
b^2k^2 - a^2x^2 - a^2y^2 = a^2b^2
\]

\[
x^2 + y^2 = \frac{b^2(k^2 - a^2)}{a^2}
\]

(3.11) \[ r_{hp} = \sqrt{x^2 + y^2} = \frac{b}{a}\sqrt{k^2 - a^2} \]

Intersect (3.02) with (3.10).

\[
b^2k^2 = a^2x^2 + a^2y^2
\]

\[
\frac{b^2k^2}{a^2} = x^2 + y^2
\]

(3.12) \[ r_c = \frac{bk}{a} = \sqrt{x^2 + y^2} \]

\( r_c \) is the radius of the circle of intersection of (3.10) with the cone, while \( r_{hp} \) is the radius of the circle of intersection of (3.10) with the hyperboloid.
Express the difference between these two radii.

\[(3.13) \quad r_c - r_{np} = b k - \frac{b}{a} \sqrt{k^2 - a^2}\]

Clearly, as \(k\) gets very large, \(\sqrt{k^2 - a^2}\) approaches \(k\).
Therefore, \(r_c - r_{np}\) approaches zero as \(k\) gets very large.

The cone \((3.02)\) does act as a three-dimensional asymptote for the hyperboloid of two sheets \((3.09)\).
4. The non-uniqueness of the source of a hyperbola.

It will be displayed in this section that, for any hyperbola, there exist at least two distinct combinations of plane and cone which can produce this hyperbola at their intersections.*

The procedure will be to consider a cone and a plane (the plane not necessarily parallel to an axis) and the intersection of these two surfaces. This is graphically described in Figure 13a. The axes will be rotated around

*It is possible to display a trivial infinite number of cases which will produce the same hyperbola. For example, rotate (2.4) around the z-axis. The intersection of (2.4) with (2.3) will produce (2.1), no matter where (2.4) is stopped in its rotation. This trivial case will not be considered anywhere in this paper. The distinctness of the two combinations of plane and cone to be displayed will be guaranteed when the cones have different vertex angles.
the \( y \)-axis until the \( zy \)-plane is parallel to the plane of the hyperbola (Figure 13b). A simple translation will bring the hyperbola into such a position (Figure 13a) that a second combination of cone and plane may be displayed, using the information developed in Section 2 of this paper.

Consider the cone

\[(4.01)\]

\[b^2z^2 = a^2x^2 + a^2y^2\]

and the plane

\[(4.02)\]

\[z = x(tan \Theta) + H.\]

Inspect the following diagram.\(^\dagger\)

\[\text{Figure 14}\]

\[\text{Notice that, if } \Theta \text{ does not satisfy the inequality } -a/b > \tan \Theta > a/b, \text{ a hyperbola is not produced. If } \tan \Theta = \pm a/b, \text{ a parabola is produced. An ellipse or a circle occurs if } -a/b < \tan \Theta < a/b. \text{ These limits on } \Theta \text{ are not essential to what follows, but in an application they would have to be realized.}\]
Solve for $H$. Point 3 has coordinates $(Kb, 0, 0)$.

\[ z = x(\tan \theta) + H \]
\[ 0 = Kb(\tan \theta) + H \]

(4.03) \hspace{1cm} H = -Kb(\tan \theta)

(4.02) may be rewritten now as follows:

(4.04) \hspace{1cm} z = x(\tan \theta) - Kb(\tan \theta).

The equations of the plane and cone being considered are now (4.04) and (4.01).

The angle of rotation may be found by inspecting Figure 15.

Let angle XEA be $\theta$. Then angle OED is also $\theta$. Construct OD perpendicular to AE. Then angle ODE is $(90^\circ - \theta)$.

In a rotation, however, the axes move—not the figure.
The x-axis will move through an angle of \((\theta - 90^\circ)\) in order for the x-axis to be coincidental with \(\overline{OD}\). When the x-axis is coincidental with \(\overline{OD}\), the zy-plane will be parallel to the plane of the hyperbola. Therefore, the angle of rotation is

\[(4.05) \quad (\theta - 90^\circ).\]

![Diagram](image)

**Figure 16**

\(\theta\) is always considered to be the negative or positive acute angle measured from the x-axis. In Figure 16a, \(\theta\) is a positive acute angle. In Figure 16b, \(\theta\) is a negative acute angle. The motivation for this method of consideration of \(\theta\) is twofold. First, consideration of \(\theta\) and \(\tan \theta\)

*The angle of rotation would be different if \(\theta\) were a negative acute angle. However, the situation where \(\theta\) is a positive acute angle is the only one which needs to be considered. This will be displayed shortly.*
in this manner will always yield the slope of \( \overrightarrow{AD} \). Secondly, it will be shown that the intersecting of (4.01) with (4.04) will yield the same result as the intersecting of (4.01) with

\[
(4.06) \quad z = x[tan (-\theta)] - Kb[tan (-\theta)].
\]

This will limit the span of consideration considerably.

(4.07) Lemma: The same hyperbola is obtained from intersecting (4.01) with either (4.04) or (4.06).

The second situation is merely the first situation after the axes have been rotated through an angle of 180° around the x-axis. The equations of rotation will be the following:

\[
\begin{align*}
(4.08) \quad &y = y'(\cos 180°) - z'(\sin 180°) = -y' \\
&z = y'(\sin 180°) + z'(\cos 180°) = -z'.
\end{align*}
\]

If the cone (4.01) is rotated through an angle of 180° around the x-axis, the same cone will again be obtained. If the plane (4.04) is rotated around the x-axis using equations of rotation (4.08), the result is (4.06).

\[
\begin{align*}
(4.09) \quad &z = x(tan \theta) - Kb(tan \theta) \\
&-z' = x'(tan \theta) - Kb(tan \theta) \\
&z' = -x'(tan \theta) + Kb(tan \theta) \\
&z' = x'[tan (-\theta)] - [Kb tan (-\theta)].
\end{align*}
\]
The span of consideration has now been limited to planes of the following form:

\[ z = x(tan \theta) - kb(tan \theta), \quad 90^0 < \theta < \arctan \frac{2}{5}. \]  

From the discussion preceding (4.05), the angle of rotation of the axes in order to place them in the proper relationship to the hyperbola (that is, the relationship pictured in Figure 13b) is \((\theta - 90^0)\). The equations of rotation are

\[
\begin{align*}
    x &= x'[\cos (\theta - 90^0)] - z'[\sin (\theta - 90^0)] \\
    z &= x'[\sin (\theta - 90^0)] + z'[\cos (\theta - 90^0)].
\end{align*}
\]

When several well-known trigonometric identities are applied, (4.11) becomes

\[
\begin{align*}
    x &= x'(\sin \theta) + z'(\cos \theta) \\
    z &= -x'(\cos \theta) + z'(\sin \theta).
\end{align*}
\]

Algebraically intersect (4.01) with (4.04).

\[
\begin{align*}
    b^2z^2 &= a^2\left(\frac{z}{\tan \theta} + kb\right)^2 + a^2y^2 \\
    b^2z^2 &= a^2z^2 + \frac{2a^2bkz}{\tan \theta} + a^2b^2k^2 + a^2y^2 \\
    a^2b^2k^2 + a^2y^2 &= \left(\frac{b^2 - a^2}{\tan^2 \theta}\right)z^2 - \left(\frac{2a^2bk}{\tan \theta}\right)z
\end{align*}
\]
(4.13) is a hyperbolic cylinder containing the hyperbola at a \(\theta\)-angled section. This is drawn in Figure 17.

\[
\begin{align*}
\text{Figure 17} \\
\text{Rotate} \ (4.13) \ \text{through an angle of} \ (90^\circ - \theta) \ \text{around the} \ y-\text{axis using the equations of rotation} \ (4.12).
\end{align*}
\]

\[
(4.14) \left( b^2 - \frac{a^2}{\tan^2 \theta} \right) \left[ - x'(\cos \theta) + z'(\sin \theta) \right]^2 - \\
\left( \frac{2a^2bk}{\sin \theta} \right) \left[ - x'(\cos \theta) + z'(\sin \theta) \right] = a^2b^2x^2 + a^2y^2
\]

(4.14) is a hyperbolic cylinder containing the hyperbola in a section parallel to the \(zy\)-plane. Since (4.14) contains the hyperbola in a section parallel to the
zy-plane, the easiest way to obtain the equation of the hyperbola is to pass the plane, \( x' = 0 \), through (4.14).

\[
\begin{align*}
\left( b^2 - \frac{a^2}{\tan^2 \theta} \right) [z'(\sin \theta)]^2 - 2a^2 b \xi (\cos \theta) z' &= a^2 b^2 k^2 + a^2 y^2 \\
\left[ b^2 (\tan^2 \theta) - a^2 \right] (\cos^2 \theta) z'^2 - 2a^2 b \xi (\cos \theta) z' &= a^2 b^2 k^2 + a^2 y^2 \\
\left[ b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta) \right] z'^2 - 2a^2 b \xi (\cos \theta) z' &= a^2 b^2 k^2 + a^2 y^2
\end{align*}
\]

\[(4.15)\]

(4.15) is the hyperbola in the situation described in Figure 18.

Figure 18 displays the hyperbola in the zy-plane. With a translation, the center of the hyperbola will be moved to the origin. The equations of transformation are the following:

\[
\begin{align*}
z'' &= z' - \frac{a^2 b \xi (\cos \theta)}{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)} \\
y'' &= y'
\end{align*}
\]

\[(4.16)\]
Translate (4.15) with equations of translation (4.16).

\[
\left( z' - \frac{a^2 b K(\cos \theta)}{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)} \right)^2 = \frac{a^2 b^2 k^2}{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)} + \frac{a^2 y' \, y^2}{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)} + \frac{a^4 b^2 k^2 (\cos^2 \theta)}{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)^2}
\]

(4.17)

\[
\frac{z''^2}{\frac{ab^2 K(\sin \theta)}{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)}}^2 - \frac{y''^2}{\frac{b^2 K(\sin \theta)}{\sqrt{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)}}^2} = 1
\]

(4.17) is a hyperbola of form

\[
\frac{z^2}{a^2} - \frac{y^2}{b^2} = 1.
\]

By Theorem (2.2), (4.17) may be produced by intersecting the cone

(4.18)

\[
\left( \frac{b^2 K(\sin \theta)}{\sqrt{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)}} \right)^2 z'^2 = \left( \frac{ab^2 K(\sin \theta)}{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)} \right)^2 (x^2 + y^2)
\]

with the plane

(4.19)

\[
x = \pm \frac{b^2 K(\sin \theta)}{\sqrt{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)}}.
\]
Two distinct combinations of plane and cone can now be displayed which produce the same hyperbola.

(1) Cone: \( b^2z^2 = a^2y^2 + a^2x^2 \)
Plane: \( z = x(\tan \theta) - K_b(\tan \theta) \)

(4.20)

(2) Cone: \( 4.13 \)
Plane: \( 4.19 \)

These two situations are obviously distinct. The vertex angle of the cone from (4.20)(1) is \( 2 \arctan \frac{b}{a} \), while the vertex angle of the cone from (4.20)(2) is

\[
2 \arctan \frac{\sqrt{b^2(\sin^2 \theta) - a^2(\cos^2 \theta)}}{a}
\]

Consider Figure 19.

Figure 19
\[
\frac{D_1E_1^2}{D_1C_1^2} = \frac{b^2 \sin^2 (\theta)}{b^2 \sin^2 (\theta) - a^2 \cos^2 (\theta)} \cdot \frac{a^2 (\cos^2 (\theta) - b^2)}{a^2}
\]

(4.23)

\[
\frac{D_2E_2^2}{D_2C_2^2} = \frac{b^2 \sin^2 (\theta) - a^2 \cos^2 (\theta)}{a^2}
\]

The ratio \(\frac{D_2E_2^2}{D_2C_2^2}\) must now be found.

Consider again Figure 19b. Pass a line through \(D_2\) and \(E_2\). The equations of this line are

\[
x = r \tan \theta + Xb, \quad z = r.
\]

(4.24)

\(D_2E_2\) will be approximated by the distance from \(D_2\) to the point where (4.24) intersects the hyperbola. As \(r\) gets larger, the line will slide upward through the plane of the hyperbola. The approximation to \(D_2E_2\) will be better as \(r\) gets larger. If it can be shown that

\[
\lim_{r \to \infty} \frac{D_2E_2^2}{D_2C_2^2} = \frac{D_1E_1^2}{D_1C_1^2} = \frac{a^2 (\cos^2 (\theta) - b^2)}{a^2}
\]

(4.25)

then the two hyperbolas will be identical.

In order to find expressions for \(D_2E_2\) and \(D_2C_2\), the coordinates of \(D_2, C_2, F_2, E_2,\) and \(G_2\) must be found. The methods used to find these coordinates will now be explained, and the coordinates will be displayed.
Face the line (4.24) through the cone (4.01).

\[ a^2y^2 = b^2z^2 - e^2 \left( \frac{r^2 + K^2b^2 + 2\pi Kb}{\tan^2 \theta} \right) \]

(4.26) \[ y = \frac{\sqrt{a^2r^2 \left( \tan^2 \theta \right) - a^2r^2 - a^2K^2b^2 \left( \tan^2 \theta \right) - 2ra\pi Kb \left( \tan \theta \right)}}{a^2 \left( \tan^2 \theta \right)} \]

(4.26) is the approximate \( y \)-coordinate of point \( B_2 \), and the approximation becomes better as \( r \) becomes large.

The coordinates of \( F_2 \) and \( G_2 \) may be found by intersecting the \( xz \)-trace of \( z = x(\tan \theta) - Kb(\tan \theta) \) with \( x = \pm b^2/a \). The coordinates of \( G_2 \) may be found with the midpoint formulas, since \( G_2 \) is the midpoint of \( F_2F_2 \).

(4.27) \[ D_2 = \left( \frac{r}{\tan \theta} + Kb, 0, r \right) \]

(4.28) \[ E_2 = \left( \frac{r}{\tan \theta} + Kb, (4.26), r \right) \]

(4.29) \[ F_2 = \left[ - \frac{Kb^2(\tan \theta)}{a - b(\tan \theta)} , 0, - \frac{Kb(\tan \theta)}{1 - b(\tan \theta)} \right] \]

(4.30) \[ G_2 = \left[ \frac{Kb^2(\tan \theta)}{a + b(\tan \theta)} , 0, - \frac{Kb(\tan \theta)}{1 + b(\tan \theta)} \right] \]

(4.31) \[ C_2 = \left[ - \frac{Kb^2(\tan^2 \theta)}{a^2 - b^2(\tan^2 \theta)} , 0, - \frac{Kb^2a(\tan \theta)}{a^2 - b^2(\tan^2 \theta)} \right] \]

*These procedures involve very long and extensive applications of algebraic manipulations. It would serve no purpose if these were reproduced here.
\( \overline{D_2E_2} \) is approximately the y-coordinate of \( E_2 \), and 
\( \overline{D_2C_2} \) may be found from the distance formula.

\[
\lim_{r \to \infty} \frac{\overline{D_2E_2}^2}{\overline{D_2C_2}^2} = \lim_{r \to \infty} \left( \frac{\left( \frac{b^2 - 1}{a^2 \tan^2 \theta} \right) r^2 + \text{terms in } r \text{ and constants}}{r^2 \left[ \frac{1}{\tan^2 \theta} + 1 \right] + \text{terms in } r \text{ and constants}} \right)^* \\
= \lim_{r \to \infty} \left( \frac{2 \left( \frac{b^2}{a^2} - \frac{1}{\tan^2 \theta} \right) r + \text{constants}}{2 \left[ \frac{1}{\tan^2 \theta} + 1 \right] r + \text{constants}} \right) \\
= \lim_{r \to \infty} \left( \frac{\frac{b^2}{a^2} - \frac{1}{\tan^2 \theta}}{\frac{1}{\tan^2 \theta} + 1} \right) \\
= \lim_{r \to \infty} \left( \frac{\frac{a^2 b^2 - 1}{a^2 \tan^2 \theta}}{1 + \tan^2 \theta} \right) \\
= \frac{b^2 (\tan^2 \theta) - a^2}{a^2 (\tan^2 \theta)} \cdot \frac{\tan^2 \theta}{1 + \tan^2 \theta}
\]

(4.32) \[ = \frac{b^2 (\tan^2 \theta) - a^2}{a^2 + a^2 (\tan^2 \theta)} \]

From (4.25), the proof that hyperbolas (4.20)(1) and (4.20)(2) are the same will be complete if the following can be proven:

(4.33) \[ \frac{b^2 (\tan^2 \theta) - a^2}{a^2 + a^2 (\tan^2 \theta)} = \frac{b^2 (\sin^2 \theta) - a^2 (\cos^2 \theta)}{a^2} \]

\[ a^2 \]
Consider the following.

\[ \Theta = 0 \]

\[ b^2(\sin^2\Theta) - a^2(\sin^2\Theta) = b^2(\sin^2\Theta) - a^2(\sin^2\Theta) \]

\[ b^2(\sin^2\Theta) - a^2(\sin^2\Theta) = b^2(\tan^2\Theta)(1 - \sin^2\Theta) - a^2(1 - \cos^2\Theta) \]

\[ b^2(\tan^2\Theta) - a^2 = b^2(\sin^2\Theta) + b^2(\sin^2\Theta)(\tan^2\Theta) - \]

\[ a^2(\cos^2\Theta) - a^2(\sin^2\Theta) \]

\[ a^2 b^2(\tan^2\Theta) - a^4 = a^2 \left[ b^2(\sin^2\Theta) + b^2(\sin^2\Theta)(\tan^2\Theta) - \right. \]

\[ a^2(\cos^2\Theta) - a^2(\sin^2\Theta) \]

\[ \frac{b^2(\tan^2\Theta) - a^2}{a^2 + a^2(\tan^2\Theta)} \]

\[ a^2 \]

This completes the proof. Therefore, (4.20)(1) and (4.20)(2) are two distinct combinations of plane and cone which produce, at their intersections, the same hyperbola.
Figure 19a is a representation of (4.20)(2), and Figure 19b is a representation of (4.20)(1). If it can be shown that

\[ (4.21) \quad \overline{F_1G_1} = \overline{F_2G_2} \]

and

\[ (4.22) \quad \frac{D_1E_1^2}{D_1C_1^2} = \frac{D_2E_2^2}{D_2C_2^2}, \]

then the eccentricity of (1) will have been shown to be the same as the eccentricity of (2), and the two hyperbolas will definitely be the same.*

(4.21), the distances between the vertices of these hyperbolas, is easily proven. The first hyperbola considered was (1). It was rotated and translated—both distance-preserving transformations.

(4.22) will be proven in the following manner. Procure the ratio \( \frac{D_1E_1^2}{D_1C_1^2} \). This is easily seen to be the following:

*This is allowable. The mere preservation of (4.21) would have guaranteed that hyperbolas (1) and (2) were the same shape, or similar hyperbolas. Since the distance between vertices has also been preserved, some algebraic manipulations would show that the eccentricity ratio has been preserved. This form is used here because it eases the algebra somewhat.
5. The uniqueness of hyperbola production in a fixed cone.

It will be displayed in this section that the hyperbola obtained by sectioning the cone

\[(5.01) \quad b^2 z^2 = a^2 y^2 + a^2 x^2\]

with the plane

\[(5.02) \quad z = x(\tan \theta) - K_1 b(\tan \theta)\]

can not be obtained by any other section of (5.01), excluding the trivial case where (5.02) is merely rotated.

The procedure will be as follows. Another sectioning plane will be chosen.

\[(5.03) \quad z = x(\tan \phi) - K_2 b(\tan \phi)\]

The distance between the vertices of the hyperbola produced by (5.01) and (5.02) will be forced to be the same as the distance between the vertices of the hyperbola produced by (5.01) and (5.03). Ratios of the form $\overline{DE}/\overline{DC}$, as used in the preceding section, will be found for each hyperbola. It will then be demonstrated that
the only way in which these ratios may be equal (and hence, the only way in which the hyperbolas may be the same) is for $\theta$ to be equal to $\phi$. Therefore, uniqueness of hyperbola production in a fixed cone will be proven.

Consider Figure 20.

Figure 20

If $z = \pm ax/b$ are algebraically intersected with (5.02), the coordinates of $A_1$ and $B_1$ are found to be

\[(5.04) \quad A_1 = \left( -\frac{K_1 b^2 (\tan \theta)}{a - b (\tan \theta)}, 0, -\frac{aK_1 b (\tan \theta)}{a - b (\tan \theta)} \right) \]

\[(5.05) \quad B_1 = \left( \frac{K_1 b^2 (\tan \theta)}{a + b (\tan \theta)}, 0, \frac{aK_1 b (\tan \theta)}{a + b (\tan \theta)} \right) \]

The Pythagorean relationship then yields

\[(5.06) \quad \overline{A_1B_1} = \left( \frac{2aK_1 b (\tan \theta)}{a^2 - b^2 (\tan^2 \theta)} \right) \sqrt{a^2 + b^2} \]
Consider Figure 21.

The coordinates of \( A_2 \) and \( B_2 \) may be found with the same procedure that was used to find the coordinates of \( A_1 \) and \( B_1 \).

\[
(5.07) \quad A_2 = \left( -\frac{K_0 b^2 (\tan \phi)}{a - b (\tan \phi)}, 0, -\frac{a K_0 b (\tan \phi)}{a - b (\tan \phi)} \right)
\]

\[
(5.08) \quad B_2 = \left( \frac{K_0 b^2 (\tan \phi)}{a + b (\tan \phi)}, 0, \frac{a K_0 b (\tan \phi)}{a + b (\tan \phi)} \right)
\]

Another application of the Pythagorean relationship yields

\[
(5.09) \quad \overrightarrow{A_2 B_2} = \left( \frac{2a K_0 b (\tan \phi)}{a^2 - b^2 (\tan^2 \phi)} \right) \sqrt{a^2 + b^2}.
\]

Notice from Figure 20 and Figure 21 the functions of \( K_1 \) and \( K_2 \). They are the parameters which, if made larger or smaller, have the ability to slide the planes.
in a direction perpendicularly away from or toward the zy-plane.

Set \( \overline{A_1B_1} \) equal to \( \overline{A_2B_2} \), and find out what \( K_2 \) must be in order to insure that the distances between the vertices of these two hyperbolas will be equal.

\[
\frac{\overline{A_1B_1}}{\overline{A_2B_2}} = \frac{2aK_1b(\tan \Theta) \sqrt{a^2 + b^2}}{a^2 - b^2(\tan^2 \Theta)} = \frac{2aK_2b(\tan \Phi) \sqrt{a^2 + b^2}}{a^2 - b^2(\tan^2 \Phi)}
\]

\[
\frac{K_1(\tan \Theta)}{a^2 - b^2(\tan^2 \Theta)} = \frac{K_2(\tan \Phi)\ast}{a^2 - b^2(\tan^2 \Phi)}
\]

(5.10)

\[
K_2 = \frac{[a^2 - b^2(\tan^2 \Phi)]K_1(\tan \Theta)}{(\tan \Phi)[a^2 - b^2(\tan^2 \Theta)]}
\]

(5.10) is essentially a piece of "mathematical machinery", which will, upon our choice of an angle \( \Phi \) for a sectioning plane, slide this plane to the proper position—the position where \( \overline{A_1B_1} = \overline{A_2B_2} \).

If the hyperbola obtained at the intersection of (5.01) and (5.02) is considered to be fixed, and if the intersection of (5.01) with (5.03) is considered to be a family of hyperbolas since \( \Phi \) is arbitrary, then the following situation exists.

\ast \text{Observe that, if } \Theta = \Phi, \text{ then } K_1 = K_2 \ast
(5.11) Cone: \( b^2z^2 = a^2x^2 + a^2y^2 \)

Plane: \( z = x(\tan \theta) - K_1b(\tan \theta) \)

(5.12) Cone: \( b^2z^2 = a^2x^2 + a^2y^2 \)

Plane: \( z = x(\tan \phi_n) - K_n b(\tan \phi_n) \),

where \( K_n = \left[ \frac{a^2 - b^2(\tan^2 \phi_n)}{(\tan \phi_n)[a^2 - b^2(\tan^2 \theta)]} \right] \)

(5.11) defines a single hyperbola. (5.12) defines a family of hyperbolas. All hyperbolas from (5.11) and (5.12) have the distances between their vertices equal.

Find the ratios \( \frac{D_2E_2^2}{D_2C_2^2} \) and \( \frac{D_1E_1^2}{D_1C_1^2} \) as used in the preceding section. Let the subscript "1" correspond to (5.12), and let the subscript "2" correspond to (5.11).

From (4.32),

(5.13) \[ \frac{D_2E_2^2}{D_2C_2^2} = \frac{b^2(\tan^2 \theta) - a^2}{a^2 + a^2(\tan^2 \theta)} \]

Again from (4.32),

(5.14) \[ \frac{D_1E_1^2}{D_1C_1^2} = \frac{b^2(\tan^2 \phi_n) - a^2}{a^2 + a^2(\tan^2 \phi_n)} \]

Inspect the following equality.

(5.15) \[ \frac{D_2E_2^2}{D_2C_2^2} = \frac{D_1E_1^2}{D_1C_1^2} \]
It has been guaranteed that the distances between vertices of all of the hyperbolas being considered will be equal. It will now be shown, by a consideration of (5.15), that, given a hyperbola produced by a $\theta$-angled section and a hyperbola produced by a $\phi$-angled section, both sectioning the same cone and having the distances between their respective vertices equal, $\theta$ must equal $\phi$.

\begin{equation}
(5.16) \quad \frac{b^2(\tan^2\theta) - a^2}{a^2 + a^2(\tan^2\theta)} = \frac{b^2(\tan^2\phi_n) - a^2}{a^2 + a^2(\tan^2\phi_n)}
\end{equation}

\begin{equation}
\left[ \tan^2\theta + (\tan^2\theta)(\tan^2\phi_n) \right] b^2 - (\tan^2\phi_n)a^2 = \left[ \tan^2\phi_n + (\tan^2\phi_n)(\tan^2\theta) \right] b^2 - (\tan^2\theta)a^2
\end{equation}

Equating coefficients yields

\begin{equation}
\tan^2\phi_n = \tan^2\theta
\end{equation}

(5.17)

\begin{equation}
\phi_n = \theta.
\end{equation}

(5.17) completes the proof and tells the reader that a specific hyperbola may only be obtained from a cone with a unique section.

* A corollary of the Fundamental Theorem of Algebra allows this to be done.

** Recall that these angles have been restricted to being positive acute angles.
6. Projections of further work to be done.

A general source for a hyperbola has been found in Section 2. This source was proved not to be unique in Section 4, and the existence of at least two sources for every hyperbola was displayed. In Section 5, it was proven that a specific hyperbola may be obtained from a specific cone with a unique section.

The next question is whether or not an infinite number of combinations of plane and cone which produce a specific hyperbola may be displayed. The author is certain that this situation does exist. Several approaches to the solution of the problem have been attempted, but it has not yet been solved.

The work in this paper could be extended to include all of the conic sections. This was the author's original intent. The most difficult case—the hyperbola—was attacked first. It should now be very easy to deal with the rest of the conic sections.