The Steiner-Lehmus Theorem

An Honors Thesis (ID 499)

by

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Proposition: Any triangle having two equal internal angle bisectors (each measured from a vertex to the opposite side) is isosceles. (Steiner-Lehmus Theorem)

"That's easy for you to say!" These being the possible words of a suspicious mathematician after listening to some assuming person state the Steiner-Lehmus Theorem. To be sure, the Steiner-Lehmus Theorem is "simply stated, but notoriously difficult to prove."[1] Its converse, the bisectors of the base angles of an isosceles triangle are equal, is dated back to the time of Euclid and is easy to prove. The Steiner-Lehmus Theorem appears as if a proof would be simple, but it is definitely not.[2] The proposition was sent by C. L. Lehmus to the Swiss-German geometry genius Jacob Steiner in 1840 with a request for a pure geometrical proof. The proof that Steiner gave was fairly complex. Consequently, many inspired people began searching for easier methods. Papers on the Steiner-Lehmus Theorem were printed in various journals in 1842, 1844, 1848, almost every year from 1854 until 1864, and as a frequent occurrence during the next hundred years.[3]

In terms of fame, Lehmus did not receive nearly as much as Steiner. In fact, the only time the name Lehmus is mentioned in the literature is when the title of the theorem is given. His name would have been completely forgotten if he had not sent the theorem to Steiner. Steiner, on the other hand, is a different case. He is considered to be "the greatest geometer of modern times."[4] His
contemporaries called him "the greatest pure geometer since Apollonius" of Perga (c. 262-190 BC). Some even went as far as to substitute Euclid of Alexandria (c. 300 BC) for Apollonius in order to satisfy their deep admiration for Steiner's genius.

The purpose of this paper is to present samples of various proofs of the Steiner-Lehmus Theorem. There is some standard notation which shall be used throughout the paper. The figure which shall be referred to frequently is triangle ABC having two angle bisectors BM and CN, with M and N on AC and AB respectively (See Figure 1). The context will make clear when to refer to this figure.

Without further delay, let us examine the first proof. It makes use of the following two lemmas.

**Lemma 1:** If two chords of a circle subtend different acute angles at points on the circle, the smaller angle corresponds to the shorter chord.

**Proof of Lemma 1:** Two equal chords subtend equal angles at the center and equal angles (half as big) at suitable points on the circumference. Of two unequal chords, the shorter, being farther from the center, subtends a smaller angle there and consequently a smaller acute angle at the circumcenter.

**Lemma 2:** If a triangle has two different angles, the smaller angle has the longer internal bisector.
Proof of Lemma 2: Let triangle ABC be labeled as stated in the aforementioned manner. Also, let angle B < angle C. Here then, the object of the proof is to show that BM > CN.

Take point P on segment BM such that angle PCN = (1/2)angle B. Since this is equal to angle PBN, and since angle PBN and angle PCN are subtended by the same segment NP which can be referred to as a chord of a circle containing points N and P, the four points N, B, C, and P are concyclic (See Figure 2).

Angle B < (1/2)(angle B + angle C) < (1/2)(angle A + angle B + angle C). So, angle CBN < angle PCB < 90 degrees. By Lemma 1, CN < BP. Hence, BM > BP > CN gives the desired conclusion.\[8\]

Proof of the Steiner-Lehmus Theorem:
Express the theorem in contrapositive form. Thus, the work of the proof is to show that if a triangle is not isosceles, it does not have two equal angle bisectors. That is, prove that if in triangle ABC labeled in the manner previously suggested, angle B does not equal angle C, then BM does not equal CN.

The proof is quite easy. It is simply an immediate consequence of Lemma 2.\[9\]

Another proof of the Steiner-Lehmus Theorem:
Another proof utilizing much of the same development as the preceding one employs the indirect method of proof. That is, an assumption is made at the beginning that leads to a contradiction. The contradiction indicates that the assumption is false and the desired conclusion is true. In this case, the proof begins with the assumption that the triangle is not isosceles and aims for a contradiction to occur in the progress of the proof.
Let triangle ABC be labeled in the usual manner of this paper with equal angle bisectors BM and CN. Now, if the triangle is not isosceles, the angles B and C are not equal. So, one must be smaller than the other, say angle B < angle C.

Take point P on segment BM such that angle PCN is equal to (1/2)angle B (Note the similarity to the proof of Lemma 2). It is important to note here that BP < BM = CN. Since angle PCN is equal to angle PBN, the four points N, B, C, and P lie on a circle (See Figure 2).

Since angle B < (1/2)(angle B + angle C) < (1/2)(angle A + angle B + angle C), angle CBN < angle PCB < 90 degrees. Also, since BP < CN, and since smaller chords of a circle subtend smaller acute angles, angle PCB < angle CBN. However, angle PCB is not smaller than angle CBN. Thus, the contradiction proves that the Steiner-Lehmus Theorem holds.

Yet Another Proof of the Steiner-Lehmus Theorem:

It is necessary to point out that this proof does not have a reference in the bibliography of this paper as a proof of the Steiner-Lehmus Theorem. However, the proof does derive a large part of its development from an alternate proof of Lemma 2 found in the literature.

Let triangle ABC be labeled in the usual way of this paper with equal angle bisectors BM and CN. Now, use the indirect method of proof. That is, assume that the triangle is not isosceles, say angle C < angle B.

Take point D on segment AM such that angle DBM = angle ACN = angle NCB. Let E and F be the points of intersection of the bisector CN with segment BD and bisector BM respectively (See Figure 3).

The two triangles DBM and DCE are similar because of Angle-Angle Similarity (angle BDM = angle CDE, angle DCE = angle DBM). Hence, BD : CD = BM : CE (*). Now, in triangle BDC, the angle at vertex B is larger than the one at C since (1/2)(angle B + angle C) > C. Therefore, CD > BD. Furthermore, from (*), CE > BM. Consequently, since CN > CE, one can conclude by the transitive property of inequality that CN > BM. However, this contradicts the given information that BM = CN. Thus, the Steiner-Lehmus Theorem holds as stated.
The next proof is presented in the literature as a proof of the Internal Angle Bisector Theorem. Of course, the Internal Angle Bisector Theorem and the Steiner-Lehmus Theorem both denote the same theorem. This synthetic proof begins with an assumption of the following theorem.

Theorem A: If two legs each of two triangles are respectively equal, then the one triangle has the greater included angle if and only if it has the greater opposite side. [12]

Lemma 3: For all points A, B, C, and O, if OA = OB, then angle AOC > angle BOC if and only if the measure of the distance from A to OC is greater than the measure of the distance from B to OC.

Proof of Lemma 3: Let P and Q be the feet of the perpendiculars from A and B respectively to OC. Draw BP and AQ (See Figure 4).

By Theorem A, using triangles AOP and BOP, it suffices to show that if AP > BP, then AP > BQ (case one), and conversely, using triangles AOP and BQ, that if AP > BQ, then AO > BQ (case two).

These cases are easy to show. In the first case, AP > BP > BQ by definition of Q. In the second case, AO > AP > BQ by definition of P. [13]

Figure 4

Here, two corollaries arise.

Corollary 1: If two equal segments have one endpoint each on a third line l, then one making the greater angle with l has its other endpoint at the greater distance from l.

Proof of Corollary 1: Corollary 1 is simply a consequence of Lemma 3. [14]
Corollary 2: If two isosceles triangles have equal bases, then the one having the greater remaining sides has the greater base angle.

Proof of Corollary 2: Let the triangles be ABC and DEF with AB = DE, and AC > DF. Take point X on segment AC such that AX = DF. Let P, Q, and R be the feet of the perpendiculars from X, C, and F respectively to the base of the appropriate triangle (See Figure 5).

It suffices to show that XP > FR, for then Lemma 3 applies. Use the indirect method; suppose false: XP < FR. Triangle ACQ is similar to triangle AXP because of Angle-Angle Similarity (angle CAQ = XAP, angle AQC = angle APX). Upon further inspection, AP < AQ = DR. Now, AP² + XP² < DR² + FR², contradicting the Pythagorean Theorem by the definition of X. Therefore, XP > FR, and by Lemma 3, Corollary 2 holds as stated.

![Figure 5](image)

Proof of the Steiner-Lehmus Theorem:

Let triangle ABC be labeled in the standard way of this paper with equal angle bisectors BM and CN. Let b = angle ABM = angle MBC and c = angle ACN = angle NCB. Draw NX and YM parallel to BC (See Figure 6).

Use the indirect method of proof; suppose the triangle is not isosceles, say angle C > angle B. Thus, c > b. Applying Corollary 1 to CN, BM, and BC, it is the case that N is further from BC than is M. Hence, NX is closer to vertex A than is YM. Moreover, triangle ANX is similar to triangle AYM because of Angle-Angle Similarity (angle NAX = angle YAM, angle ANX = angle AYM). Upon inspection, NX < YM. Triangles NXC and BYM are both isosceles (angle XNC = angle XCN and angle YMB = angle YMB since alternate interior angles of a transversal are equal). Thus by Corollary 2, c < b, which is a contradiction. Indeed, the Steiner-Lehmus Theorem holds once again.
The next proof of the Steiner-Lehmus Theorem has been referred to as the best one of them all.\cite{17}

Proof of the Steiner-Lehmus Theorem:

Let triangle $ABC$ be labeled in the continuing manner presented in this paper with equal angle bisectors $BM$ and $CN$. Now, use the indirect method of proof; that is, if angle $B$ does not equal angle $C$, then one must be less, say angle $B < angle C$.

Take point $P$ on segment $AN$ such that angle $PCN = (1/2)(angle ABC)$. Take point $Q$ on segment $PN$ such that $BO = CP$ (See Figure 7).

Triangle $BMO$ is congruent to triangle $CNP$ by Side-Angle-Side Congruency ($BM = CN$, angle $PCN = angle QBM$, $BO = CP$). At this point, compare the corresponding angles $CPN$ and $BQM$ of triangles $CNP$ and $BMO$ respectively. The claim is that angle $CPN = angle BQM$. Given this claim, line $CP$ and line $MQ$ form equal corresponding angles with transversal $AB$. However, if this is true, then line $CP$ is parallel to line $MQ$. Obviously, this is not the case since $M$ and $Q$ are on opposite sides of $CP$. Thus, the contradiction proves that the Steiner-Lehmus Theorem holds as stated.\cite{18}
The following proof of the Steiner-Lehmus Theorem is an interesting example of an analytic proof. The proof relies heavily upon the fact that the square of the bisector of an angle of a triangle equals the product of the sides forming the angle, diminished by the product of the segments of the third side formed by the bisector.

In the literature, given triangle $ABC$ with the lengths of the sides opposite vertices $A$, $B$, and $C$ being $a$, $b$, and $c$ respectively, the square of the length of the angle bisector $BM$ (measured from the vertex to the opposite side) is given as $ac(1-(b/(a+c))^2)$ (See Figure 8).[19]

In this analytic proof, given the same triangle $ABC$ labeled not in the previous way of Figure 8, but rather in the continuing manner of this paper with equal angle bisectors $BM$ and $CN$ and letting segment $AN$ be designated by $p$, segment $BN$ by $q$, segment $AM$ by $r$, segment $CM$ by $s$, and segment $BC$ by $x$, the square of the length of the angle bisector $BM$ is given as $(p + q)x - rs$ (See Figure 9).[20]
It is possible to compare both representations. First, notice that $(p + q)$ in Figure 9 corresponds to $c$ in Figure 8. Next, observe that $x$ in Figure 9 corresponds to $a$ in Figure 8. Now, use the fact that an angle bisector in a triangle divides the opposite side into segments that have the same ratio as the other two sides. Also, remember that $AM = r$ and $CM = s$. So, $CM/AM = s/r = a/c$, and thus $ra = sc$. Now use algebra to show that $rs$ does indeed equal $ac(b/(a + c))^2$. That is,

$$rs = \frac{rs(a + c)^2}{(a + c)^2}$$

$$= \frac{rs(a^2 + 2ac + c^2)}{(a + c)^2}$$

$$= \frac{(raas/(a + c)^2) + (2rsac/(a + c)^2) + (sccr/(a + c)^2)}{(by \ substitution)}$$

$$= \frac{(sca + 2rsac + racr)/(a + c)^2}{(note \ that \ b = AM + CM = r + s)}$$

Hence, both representations, although written differently, involve the same expression.

Presented next is an analytic proof of the Steiner-Lehmus Theorem.[21]

Proof of the Steiner-Lehmus Theorem:

Given: Triangle ABC; BM bisects angle ABC; CN bisects angle ACB; BM = CN (See Figure 9)

Prove: Triangle ABC is isosceles ($BA = AC$)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Let $AN$ be designated by $p$, $BN$ by $q$, $AM$ by $r$, $CM$ by $s$, $BC$ by $x$, $BM$ by $t$, and $CN$ by $u$. Then $t = u$.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $t^2 = (p + q)x - rs$</td>
<td>2. The square of the bisector of an angle of a triangle equals the product of the sides forming the angle, diminished by the product of the segments of the third side formed by</td>
</tr>
</tbody>
</table>

page 9
3. In like manner, 
\[ u^2 = (r + s)x - pq \]
4. But \( t = u \)
5. Therefore, \( t^2 = u^2 \)

6. Therefore, \( (p + q)x - rs = (r + s)x - pq \)
7. Therefore, \( px + qx - rs = rx + sx - pq \)

8. Also, \( q/p = x/(r + s) \), 
   \[ s/r = x/(q + p) \]

9. Therefore, \( q(r + s) = px \), 
   \[ s(q + p) = rx \]

10. Therefore, \( qr + qs = px \), 
    \[ sq + sp = rx \]

11. \( qr + qs + qx - rs = sq + sp + sx - pq \)

12. \( qr + qx + pq = rs + sp + sx \)
13. \( q(r + x + p) = s(r + p + x) \)
14. Therefore, \( q = s \)

15. In triangles BNC and BMC, 
    CN = BM

16. BN = q = MC = s
17. BC = BC
18. Therefore, triangle BNC is 
    congruent to triangle BMC

19. Therefore, angle ABC = angle ACB
20. Therefore, BA = AC

Well, the next proof is the final proof of the paper and serves as a definite complement to the introductory tone of the paper. Earlier, it was said that the Steiner-Lehmus Theorem is easily stated, but difficult to prove. It just so happens that the following proof was printed in a mathematics textbook. Now, there would have been nothing wrong with that except for the fact that the proof was a fallacy, albeit a subtle one.
"Proof" of the Steiner-Lehmus Theorem:

Let \(\triangle BAC\) be the triangle and \(AN, CM\) the two equal bisectors, with \(N\) and \(M\) on \(BC\) and \(AB\) respectively. Suppose the perpendicular bisectors of \(AN\) and \(CM\) meet at \(O\) (See Figure 10). The circle with center \(O\) passes through \(A, M, N, C\). Angles \(MAN\) and \(MCN\), subtended by \(MN\), are equal. Hence, the angles \(\angle BAC\) and \(\angle BCA\) are equal, and the result follows. [22]

![Figure 10](image)

In order to show that this proof is a fallacy, it is necessary to examine the claim that the circle with center \(O\) passes through \(A, M, N, C\). Is this true? Let us find out.

First, illustrate a necessary property of the circumcenter. Take triangle \(\triangle ABC\) with the midpoint \(E\) of \(AC\) and the midpoint \(F\) of \(AB\). From \(E\) and \(F\) draw perpendiculars to \(AC\) and \(AB\) respectively. These perpendicular bisectors meet at point \(O\) (See Figure 11). Is \(O\) the circumcenter of triangle \(\triangle ABC\)? If \(QA = QB = QC\), then it is the circumcenter. So, show \(QA = QB = QC\).

Triangle \(\triangle AQF\) is congruent to triangle \(\triangle BQF\) by Side-Angle-Side Congruency (\(QF = QF\), angle \(QFA = \angle QFB\), \(AF = BF\)). Also, triangle \(\triangle CQE\) is congruent to triangle \(\triangle AQE\) by Side-Angle-Side Congruency (\(QE = QE\), \(\angle QEC = \angle QEA\), \(AE = CE\)). \(QA\) is shared by triangle \(\triangle AQE\) and
AQF. Thus, because of the congruency between triangle AQE and triangle CQE, the corresponding sides QC and QA are equal. Likewise, because of the congruency between triangle AQF and BQF, the corresponding sides QA and QB are equal. Hence, QA = QB = QC proving that Q is the circumcenter of triangle ABC.

![Figure 11](image)

Now, closely examine the questionable claim given in the proof. Again, the claim is that the circle with center O passes through A, M, N, and C. Take segments RS and TU with midpoints G and H respectively. From G and H draw perpendiculars which intersect at P. Connect P to R, S, T, and U with line segments (See Figure 12). At this point, compare the disputable claim to this setup.

![Figure 12](image)
As if in a theatrical production put to geometry, RS plays the part of angle bisector CM, TU stands in for angle bisector AN, and P stars as 0, the debated center of the circle containing points A, M, N, and C. It is important to note that the placement of the segments in the plane, otherwise referred to as the "Euclidean Stage" in this geometrical drama, are of no consequence to the validity of the results at the closing curtain. In the continuing tradition of Greek dramas, point P, the uncertain center, is the central character with a tragic flaw. Is P really the center of the circle containing R, S, T, and U or just very well disguised? Even before the play begins, Teiresias, the prophet and servant to the god Apollo, has his doubts. Remember, if P is not the center of the circle, then neither is 0. Watch and see if PR = PS = PT = PU. Without further adieu, let the show begin!

Triangle TPH is congruent to triangle UPH by Side-Angle-Side Congruency (PH = PH, angle PHT = PHU, TH = UH). Also, triangle RPG is congruent to triangle SPG by Side-Angle-Side Congruency (PG = PG, angle PGR = angle PGS, RG = SG). So far the storyline has been the same as that of the earlier case concerning the circumcenter Q. However, this is where the comparison of the plots stop, and the contrast begins. Before, QA was shared by triangle AQF and triangle AQE. Here though, PT and PS are distinct segments not necessarily equal. Hence, there is no way to show that PR = PS = PT = PU all the time. The tragic flaw has been discovered. Consequently, it would be fitting to describe this climactic scene as the catharsis of the play. Point P, and thus point 0 of the questionable proof just proved false, can only sing a simple soliloquy of being the intersection of two perpendicular bisectors. The chorus can no longer hail point P as the
center of the circle containing R, S, T, and U. Even the mighty god Zeus of Mount Olympus must concede to this fact. For only if it were possible to always change the length of the sides of the triangles, would P be the center. In the entertainment business, they simply say, "Break a leg!" Fortunately, geometry is much less tolerant than that.

In conclusion, it must be stated that the Steiner-Lehmus Theorem has a very rich history. For over a hundred years it has fascinated people with its deceptiveness to proof. As we saw with the last proof, anybody can make mistakes. After all, the Steiner-Lehmus Theorem is one thing to state, and a far different thing to prove. Still, when proven, it has a beautiful quality to it; an exquisite work of art to admire and share with others. And this is what the author's ultimate purpose was in preparing this paper. With the pen as his paintbrush and the paper as his canvas, he hopes to have given the reader a cherished glimpse of the truth and beauty of geometry, and in particular, the Steiner-Lehmus Theorem.
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