A SYSTEM OF EPISTEMIC LOGIC

by

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This paper examines two systems of logic that have been proposed for the notions of belief and/or knowledge. The first section looks at and criticizes a system proposed by Robert Ackermann. The second section is concerned with explicating the major points of a system put forth by Jaakko Hintikka. In the third section, an original attack is made upon the "KK-Thesis," which is a theorem provable by both of the systems mentioned above. The fourth and final section of this paper is an attempt by the author to develop a system of epistemic logic which will avoid the criticisms leveled at the other two.

Special thanks are extended to Dr. David Annis, whose inspiration and guidance have made the following study possible.
In his book *Belief* and *Knowledge*, Robert Ackerman proposes in the second chapter a method of testing a set of one's conscious beliefs for consistency. Such a set is called a "distinguished belief set," and consists of "the set of formulas...each member of which represents one of a's conscious beliefs in the circumstances." These formulae must be prefixed by one of the following operators:

- \( B_a \) (a believes that)
- \( \sim B_a \) (a doesn't believe that)
- \( B_a B_a \) (a believes that a believes that)
- \( B_a \sim B_a \) (a believes that a doesn't believe that)
- \( \sim B_a B_a \) (a doesn't believe that a believes that)
- \( \sim B_a \sim B_a \) (a doesn't believe that a doesn't believe that)

The test for consistency consists of two steps, or procedures, which are designed to produce "model belief sets." These model sets may be thought of as representing "possible worlds" or "alternative worlds" in which a's conscious beliefs might indeed turn out to be all true. If there is no such alternative possible, then a's set of distinguished beliefs will be inconsistent.

The first of Ackermann's procedures is called "Belief Augmentation," and is based on the criterion for consistency that one's beliefs are consistent only if they could all be true. If we label a's distinguished belief set B, then the set resulting from Belief Augmentation will be labeled B*. Set B* will consist of: (1) all of the members of B; and (2) those formulae which are obtained from the members of B "by deleting the prefix Ba, or by changing the prefix
Ba~Ba to ~Ba, or by both changing the prefix BaBa to Ba and deleting it altogether (thus adding two formulae if neither of the resulting formulae are already in the set). Thus, if we have a set $B = (\text{Ba} \text{Ba}, \text{Ba} \text{p}, \text{Ba} \text{~p})$, then by Belief Augmentation we will obtain the set $B^* = (\text{Ba} \text{Ba}, \text{Ba} \text{p}, \text{Ba} \text{~p}, \text{Ba}, q, p, \text{~p})$. Since $B^*$ is inconsistent according to the rules of standard Sentential Logic, $B$ is inconsistent.

This procedure, however, will only handle three of the six operators: $\text{Ba}$; $\text{Ba}\text{Ba}$; and $\text{Ba}~\text{Ba}$. In order to test those formulae prefixed by the other three operators ($\text{~Ba}$, $\text{~Ba}\text{Ba}$, $\text{~Ba}~\text{Ba}$), Ackermann introduces another step, called the "Disbelief Procedure." This procedure is applied to the members of $B^*$ in the following manner: (1) if there are \(n\) formulae prefixed by the operator $\text{~Ba}$ (including those prefixed by $\text{~Ba}\text{Ba}$ and $\text{~Ba}~\text{Ba}$) in $B^*$, then the Disbelief Procedure is applied \(n\) times to $B^*$ to obtain \(n\) new sets, $B^*_{1}$, $B^*_{2}$,...,$B^*_{n}$; (2) the denial of the sentence coming after same prefix $\text{~Ba}$ (including $\text{~Ba}\text{Ba}$ $\text{~Ba}~\text{Ba}$) is an element of same set $B^*_{1}$; (3) all of the members of $B^*$ are members of all sets $B^*_{1}$, $B^*_{2}$,...,$B^*_{n}$. Thus, if we have a set $B = (\text{Ba} \text{p}, \text{~Ba} \text{q}, \text{~Ba} \text{r}, \text{~Ba} \text{Ba} \text{p})$, then by Belief Augmentation we will obtain the set $B^* = (\text{Ba} \text{p}, \text{~Ba} \text{q}, \text{~Ba} \text{r}, \text{~Ba} \text{Ba} \text{p}, \text{p})$, and by three separate applications of the Disbelief Procedure we will obtain the sets $B^*_{1} = (\text{Ba} \text{p}, \text{~Ba} \text{q}, \text{~Ba} \text{r}, \text{~Ba} \text{Ba} \text{p}, \text{p}, \text{~q})$, $B^*_{2} = (\text{Ba} \text{p}, \text{~Ba} \text{q}, \text{~Ba} \text{r}, \text{~Ba} \text{Ba} \text{p}, \text{p}, \text{~r})$, and $B^*_{3} = (\text{Ba} \text{p}, \text{~Ba} \text{q}, \text{~Ba} \text{r}, \text{~Ba} \text{Ba} \text{p}, \text{p}, \text{~Ba} \text{p})$.

The reason for the stipulation that a different set be created for every formula prefixed by $\text{~Ba}$ (or $\text{~Ba}\text{Ba}$ or $\text{~Ba}~\text{Ba}$) is to preserve the consistency of agnosticism. For example, suppose that $a$ is agnostic with respect to some statement $p$. Then $(\text{~Ba}p, \text{~Ba} \text{~p})$ would seem to be a correct rendering of his position regarding $p$; he
does not believe \( p \), nor does he believe \( \neg p \). If both of the \( \neg \text{Ba} \) operators were eliminated in the same application of the Disbelief Procedure, then the resulting set \( (\neg \text{Bap}, \text{Ba} \neg p, p, \neg p) \) would be obviously inconsistent. We would not want a system to prove such a position inconsistent, for it clearly does not seem to be contrary to our notion of "rationality" to suppose that one might be agnostic towards a great many statements.

This, then is the basic structure of Mr. Ackermann's system of doxastic logic (the term "doxastic" will be used hereafter in relation to the notion of belief in the same manner that the term "epistemic" is used in relation to the notion of knowledge\(^4\)). I believe that there is at least one flaw in Mr. Ackermann's system. Suppose that some person \( a \) believes that \( p \), but does not believe that he believes that \( p \) (he may not have the concept of belief, for instance). Let us take the set \( B = (\text{Bap}, \neg \text{BaBap}) \) as a representation of the state of \( a \)'s beliefs in the circumstances.\(^5\) An application of Ackermann's Belief Augmentation would yield the following set:

\[
B^* = (\text{Bap}, \neg \text{BaBap}, p),
\]

and an application of the Disbelief Procedure would produce:

\[
B^*_1 = (\text{Bap}, \neg \text{BaBap}, p, \neg \text{Bap}).
\]

This set is clearly inconsistent according to the rules of Sentential Logic, so \( B \) must be inconsistent. This would mean that, in Ackermann's system, \( \text{Bap} \models \text{BaBap} \), since, assuming \( \text{Bap}, \neg \text{BaBap} \) cannot be the case without producing a contradiction. However, because of the example cited above, I do not think that this is something which we would want to fall out of a correct system of doxastic logic.\(^6\)

Also, while it is true that \( \text{Kap} \models p \), it is not true that
In fact, Jaakko Hintikka, in his much discussed book *Knowledge and Belief* points out that this one difference between the logics of the notions of belief and knowledge "serves to explain...many subtle divergencies in the logical behavior of the two notions." It is obvious that one can (and most of us probably do) believe quite a few things that are not true. This is why $B_a \not\Rightarrow p$. Now suppose that we substitute $B_a$ for $p$ in this implication. We would then have $B_a B_a \not\Rightarrow B_a$.

Yet in Ackermann's system, believing that one believes does imply that one believes. (Performing Belief Augmentation on the set $(B_a B_a, \sim B_a)$ yields $(B_a B_a, \sim B_a, B_a, p)$). Here again, it appears that Ackermann's system may not be correct. However, before any tempering with his system is attempted, it might be instructive to take a closer look at the system(s) proposed by Hintikka in the book mentioned above, as that work constitutes a landmark in the fields of epistemic and doxastic logic.

II.

The many praises and criticisms of Professor Hintikka's *Knowledge and Belief* are as diverse as such a seminal and somewhat difficult work merits. An attempt to mention and evaluate all of them would be the task of an essay much longer and of a different purpose than the present one. This would also be true of an effort to present and evaluate everything that Hintikka attempts in the book. Therefore we shall only discuss in this section those things in *Knowledge and Belief* which relate most directly to our present purpose, which is to develop a relatively simple and, hopefully, correct system of logic for the notions of knowledge and belief. (By "simple" we mean a system excluding quantification, identity, and so forth).
There are several notions whose understanding is central to an understanding of Hintikka's system (hereafter we will follow the lead of Charles Pailtharp and refer to Hintikka's proposed system as "SE4"), and these are model sets and systems, virtual implication, indefensibility, doxastic and epistemic alternatives, and self-sustaining sentences.

Suppose that we begin, similar to the way Ackermann begins, by considering a set of sentences. These sentences might represent a's belief claims, knowledge claims, things which a does not believe or know, or any combination of these. Now suppose that, again analogous to Ackermann, we have certain rules which will enable us to construct other sets on the basis of the members of the first set. The new sets which we have constructed would be called "alternatives" to the first; the new sets and the original set are called "model sets," and such an original set together with its alternatives is called a "model system." Whether an alternative is "doxastic" or "epistemic" depends upon which rule or rules are used in deriving its membership from the original set, and these are discussed below.

Chisholm says of the notions of "virtual implication" and "self-sustaining" that "Hintikka tells us that one sentence 'virtually implies' another provided that the conditional having the former sentence as antecedent and the latter sentence as consequent is a sentence that is 'self-sustaining.' And he tells us that a sentence is 'self-sustaining' provided that its negation is 'indefensible.'" Now what of the notion of "indefensibility?" Hintikka characterizes 'defensibility' as follows:

What my notion of consistency amounts to in typical cases is immunity to certain kinds of criticism. In
order to see this, suppose that a man says to you
"I know that \( p \) but I don't know whether \( q \)" and
suppose that \( p \) can be shown to entail logically
\( q \) by means of some argument which he would be willing
to accept. Then you can point out to him that what
he says he does not know is already implicit in what
he claims he knows. If your argument is valid, it is
irrational for our man to persist in saying that he
does not know whether \( q \) is the case. If he is reason­
able, you can thus persuade him to retract one of his
statements without imparting to him any fresh information
beyond certain logical relationships... 11

Hintikka goes on to say that "the general characteristic of indefensible
statements is, therefore, that they depend for their truth on somebody's
failure (past, present, or future) to follow the implications of what
he knows far enough." 12 (In these definitions the word "know" has
been used exclusively only because they are presented in Hintikka's
book at a time when he is discussing the logic of knowledge. These same
definitions hold if "knows" is replaced by "knows or believes." ) Thus,
in general terms, Hintikka replaces "consistent" with "defensible,
"valid" with "self-sustaining," "implies" with "virtually implies" and
"equivalent" with "virtually equivalent." In his article on "The Logic
of Knowing" (The Journal of Philosophy, Vol. 60, No. 25), Chisholm
points out, for example, that what Hintikka calls "virtual implication"
would be the same thing as implication in a world of "logically
omniscient beings," that is, a world where people actually do follow
the implications of what they claim to know or believe as far as
possible. The same conditions hold for the substitution of "defensible"
for consistent, "self-sustaining" for valid, and so on.

The most important part of Hintikka's book is, however, the
conditions and rules which he develops as a logical system. Three of
the most important conditions presented are, it seems to me, the
conditions \((C.P^*)\), \((C.\sim K)\) and \((C.\sim P)\). 13 They are formulated in the
following manner:

(C. P*) If "P_" ∈ μ and if μ belongs to a model system S, then there is in S at least one epistemic alternative μ* to μ (with respect to a) such that P ∈ μ*.

(C. ~K) If "~ Ka_" ∈ μ, then "Fa~P" E μ.

(C. ~P) If "~ Pa_" ∈ μ, then "Ka~P" E μ.

(It might be helpful to note, for those unfamiliar with Hintikka's book, that "Ka" stands for "a knows that", "Ba" for "a believes that", "Pa" for "it is possible, for all that a knows, that", and "Ca" for "it is compatible with everything a believes that". This notation will be used throughout this essay.)

These conditions can be intuitively justified. If it is possible for all I know that P, then there must be at least one "epistemic alternative" in which P is, in fact true. (An "epistemic alternative" is a state of affairs in which a knows at least as much as he does in the present state of affairs, which is described by μ.) That is to say, if I say truly of a that Pa_, then it must be possible for P to turn out to be true without overthrowing any of a's knowledge claims. Similarly, if I claim not to know that P, then it must be possible for all I know that ~P. If it were not possible, then something that I claim to know must virtually imply P, which means that my claim that I do not know that P is indefensible. For example, if I claim that I do not know that there is someone in the next room then it must be possible, for all I know that there is no one in that room. To say that it is not possible, for all I know, that no one is in the next room is to say that something(s) that I claim to know, for example, that the light is on in the next room, that I can hear voices coming from that room, and that whenever the light is on and I can hear voices there is someone in that room, imply that someone is in the next room. Thus,
my claim that I do not know that there is someone in the next room
is indefensible, as it is open to just the sort of criticism described
above (pp. 5-6). The same sort of argument will show (C.¬P).

To these three conditions, Hintikka adds two others in order to
form the "basis" of SE4; that is to say, any other condition which
might be needed can be derived from these five. These two new conditions
are (C.KK*) and (C.K), and are formulated as follows.

(C.KK*) If "KaE." ∈ µ and if µ* is an epistemic alternative
to µ (with respect to a) in some Model System, then
"KaE." ∈ µ*.

(C.K) If "KaE." ∈ µ, then p ∈ µ.

The justification for the latter condition is fairly simple. By
definition, KaE. ⊨ p. So, in any world where KaE. is the case, p must be
true. The justification for the first condition is not so simple.
Condition (C.KK*) has the effect of saying: if a knows that p in this
world (i.e., µ) then a knows that p in every epistemic alternative to
this world. This is part of Hintikka's "strong" definition of the verb
"to know." He views the statement "I know that p" as, in his words, a
"conversation stopper," in that it means that no further evidence can
possible persuade me to change my mind. Hence the fact that in every
epistemic alternative I must know at least as much as I do now; I may
gain some knowledge, but no evidence could cause me to give up any of
my knowledge.14

When dealing with the logic of the notion of belief, Hintikka
maintains that all of the five conditions above except (C.K) can be
made to work by replacing "K" with "B", "P" with "C", and "epistemic"
with "doxastic". These conditions thus re-worded result in doxastic
alternatives to µ, rather than epistemic alternatives. The reason why
(C.B), which would be the doxastic counterpart of (C.K), won't work for belief was mentioned above in our criticism of Ackermann's doxastic system, namely, that Bap \( \not \vdash p \). In place of (C.B), however, a "weaker" condition (C.b*) will work. It is formulated as follows.

\[
(C.b*) \quad \text{If } \text{"Bap" } \in \mu \text{ and if } \mu \text{ belongs to a model system } Q, \text{ then there is in } Q \text{ at least one doxastic alternative } \mu^* \text{ to } \mu \text{ (with respect to } a) \text{ such that } p \in \mu^*.
\]

We would at the very least want it to be possible that \( p \) turn out to be true in some possible alternative, if I believe that \( p \). This at least insures us that \( p \) itself is logically consistent.

Hintikka notes that the two epistemic conditions (C.KK*) and (C.K) imply a third condition, which he labels (C.K*) and formulates in the following manner.

\[
(C.K*) \quad \text{If } \text{"Bap" } \in \mu \text{ and if } \mu^* \text{ is an epistemic alternative to } \mu \text{ (with respect to } a) \text{ in some model system, then } p \in \mu^*.
\]

Thus, what I know is not only true in this world (\( \mu \)), but in every alternative world as well. Hintikka goes on to tell us that, even though (C.B) does not hold for belief as (C.K) does for knowledge, the doxastic counterpart of (C.K*), which would be (C.B*) does hold. It must be possible for what that I believe to be true in all alternative worlds, even though it may not be true in this one. This, again, shows the one big difference in the logic of the two notions.

This description of Hintikka's system, although very brief and somewhat superficial will, I think, suit our purposes. Now that we have at least the basic conditions for \( S4 \), let us take a look at an interesting theorem that follows. First of all, it is important to note one thing that does not follow as a theorem of \( S4 \), and that is that Bap \( \not \rightarrow Bap \).

(Technical note: I will use the arrow \( \rightarrow \) to indicate Hintikka's "virtual
implication," and reserve the use of the horseshoe \( \rightarrow \) for cases of standard material implication.) When Hintikka shows, for example, that \( A \rightarrow B \), he first begins with the set \( \mu \) having \( A \) and \( \sim B \) as its members. If a contradiction can be derived, then \( A \rightarrow B \). At one point in Knowledge and Belief a proof that \( BaBaE. \rightarrow BaE. \) is attempted, and it in deed fails. \( KaKap \rightarrow Kap \) will, however, follow as a theorem of SE\( \frac{4}{4} \) as it should, since \( Kap \rightarrow p \).

There is one proof in Knowledge and Belief that has been the object of much discussion in the literature. It is the proof of the KK-Thesis, or that \( Kap \rightarrow KaKap \). In Hintikka's notation, the proof is surprisingly simple and looks like this. 16

\[
\begin{align*}
(1) & \quad "Kap" \in \mu & \text{assumption} \\
(2) & \quad "\sim Kap" \in \mu & \text{assumption} \\
(3) & \quad "Pa \sim Kap" \in \mu & (C., \sim K), (2) \\
(4) & \quad "\sim Kap" \in \mu^* & (C.P^*), (3) \\
(5) & \quad "Kap" \in \mu^* & (C.KK^*), (1)
\end{align*}
\]

Since the conjunction of (4) and (5) results in a contradiction, \( Kap \rightarrow KaKap \). Hintikka's critics usually attack this proof by an attack upon the condition \( (C.KK^*) \) or \( (C.P^*) \). 17 In what follows I shall offer an independent attack upon the KK-Thesis itself, and then endeavor to show what can be done in terms of formulating a system of epistemic logic which will not result in the KK-Thesis.

III.

First of all, let us begin with a look at a modern day, practicing epistemological skeptic (we have in mind here the skepticism that would be advocated by someone like Keith Lehrer 18). He would deny all knowledge on the basis of a "Skeptical Hypothesis" (any set of conditions under which most, if not all, of our beliefs are false although very nearly correct). Those things which most of us would claim to know, our skeptic
would claim to believe, perhaps very strongly.

Now, although the skeptic would never say of some statement \( p \) "I know that \( p \)" (unless such a remark were to slip out "by accident"), there may be times when "we", meaning non-skeptics, would want to say of him that "He knows that \( p \)". By this we would mean that he has satisfied the following conditions for knowing that \( p \): (a) he believes that \( p \); (b) he is justified in believing that \( p \) (i.e., he has evidence to support his belief that \( p \)); (c) his justification for believing that \( p \) is undefeated\(^{19}\); and (d) furthermore, \( p \) is in fact true. For one who believes skepticism to be false—as defeating every knowledge claim—it would be correct to say of our skeptic, "He knows that \( p \)".

If we add to the notation already presented in this essay the symbols "\( J \)" for "\( \_ \) is justified in believing \( \_ \)\)", and "\( \sim D \)" for "\( \_ \)'s justification for believing that \( \_ \)\) is undefeated", we could symbolize our necessary (and here we assume them to be also sufficient) conditions above in the following manner.

\[
\text{(a) } Ka_p \equiv Da_p \cdot Ja_p \cdot \sim Da_p \cdot p
\]

Substituting '\( Ka_p \)' for the statement letter \( p \) in this formula would yield the following equivalence.

\[
\text{(b) } Ka_Ka_p \equiv Da_Ka_p \cdot Ja_Ka_p \cdot \sim Da_Ka_p \cdot Ka_p
\]

When we said of our skeptic "He knows that \( p \)”, we meant that he had satisfied those conditions in formula (a) for knowledge. Likewise, if we were to say truly of him "He knows that he knows that \( p \)", we would have to say that he has satisfied those conditions in formula (b) for knowing that one knows. But clearly this not be the case, for our skeptic would not believe that he knows that \( p \), because he does not believe that he knows anything at all, and would therefore not satisfy the necessary condition \( Da_Ka_p \).
Since, as was shown, it is true that $\text{K}a\text{K}_p \Rightarrow \text{K}_p$, one who holds that the KK-Thesis is true maintains that $\text{K}_p \equiv \text{K}_p a\text{K}_p$. This is to say that the truth conditions for knowing and knowing that one knows are the same; whenever one is true so is the other. But our counter example with the skeptic would seem to show this to be false, since he has satisfied all of the necessary conditions for $\text{K}_p$ without satisfying those for $\text{K}_p a\text{K}_p$. (Notice that his skeptical position would not keep him from consistently maintaining any of those necessary conditions in formula (a) for knowledge. As Lehrer says "...all his position debars him from is believing such things as would entail that we have knowledge," and one's believing the conditions on the right-hand side of (a) would not entail his believing that he knows anything.)

Hintikka points out that the KK-Thesis is nothing new under the philosophical sun, so one would expect that this is probably not the first attack ever made upon it. I will try to show presently that this counter example can succeed where others have failed. First we shall look at a couple of other proposed counter examples wherein one allegedly knows that $\text{p}$ but doesn't believe that he knows that $\text{p}$.

The first of these is an argument advanced by Colin Radford against the "Entailment Thesis", that is, that $\text{K}_p \Rightarrow \text{K}_p a\text{p}$. His example involves an oral examination wherein a person is asked "When did Queen Elizabeth die?" The answer given is "I don't know, 1603?" Radford tells us that the examinee thinks he is guessing. However, since he did learn some English history (he forgot that he learned it), and since most of his answers to other questions on the subject are correct, Radford concludes that the person did know that Elizabeth died in 1603, even though he did not believe that he knew it (he believes that he is guessing).
The second counter example, advanced by E. J. Lemmon, involves a man who is asked what the value of pi to ten decimal places is and replies "I don't know," waits a few minutes and (suddenly remembering that he had learned it at school) say "3.1415926536." Lemmon says that at the time of the first answer, the man knew the answer (as his second utterance demonstrates), but did not know that he knew it, whereas at the time he says "3.1415926536" he both knows and knows that he knows the answer.

In an analysis of these two examples, Keith Lehrer concludes that they both "...take conscious conviction and a readiness to report as a condition of the application of the epistemic term they wish to prove not to apply while rejecting these as conditions of the epistemic term they wish to assume does apply." In other words, according to Lehrer, both Radford and Lemmon assume that a conscious conviction that \( p \) and a readiness to report that \( p \) are not conditions for saying that \( \text{Kap} \), but that such conditions are required for saying \( \text{BaKap} \) or \( \text{KaKap} \).

I submit that because our counter example does not involve any guessing or forgetting, it is free from just this sort of criticism. We can, for example, consistently maintain that a conscious conviction that \( p \) and a readiness to report that \( p \) are conditions for knowledge and knowledge that one knows. If, during an oral examination, our skeptic was asked "When did Queen Elizabeth die?", he might immediately reply "1603". There is no hesitation before his answer, there is no tone of uncertainty in his voice. He would maintain that he strongly believes that Queen Elizabeth died in 1603 because, for example, all eight of the reputable histories of England which he has read reported 1603 as
the date of Elizabeth's death. We would say that he knows that Queen Elizabeth died in 1603 because: (a) he believes it; (b) he is justified in believing it; (c) his justification for believing it is undefeated; and (d) it is in fact true. He, however, would not say that he knows it because he does not believe that he knows anything (and for good philosophical reasons—not as a sort of joke, game or pathological obsession), and for this reason we would not say that he knows that he knows it. Therefore, a "conscious conviction and a willingness to report" can be maintained as conditions for both knowing and knowing that one knows without damage to our counter example.

The second objection that we might like to anticipate also comes from Keith Lehrer. In his paper "Belief and Knowledge", Mr. Lehrer argues in favor of the Entailment Thesis, mentioned above. Embedded in his proof is another proof which runs as follows:

\[
\begin{align*}
(1) & \quad \sim B_K a \rightarrow s, \\
(2) & \quad s \supset \sim K_p, \\
(3) & \quad \sim B_K a \rightarrow \sim K_p,
\end{align*}
\]

Wherein the statement letter 's' stands for "even though a correctly says that p and knows that he has said that p, a does not know that he correctly says that p." Since (3) is logically equivalent to \( K_p \supset B_K a \), and since we maintain in our counter example both \( K_p \) and \( \sim B_K a \), it is important that we find fault with this proof.

Let us add the following to the notation listed above:

\[
\begin{align*}
C_S a & \quad : \quad a \text{ correctly says that } p, \\
S_a & \quad : \quad a \text{ says that } p.
\end{align*}
\]

It should be obvious, I think, that \( C_S a \equiv S_a \cdot p \). It would seem to follow, then that \( K_a C_S a \equiv K_a S_a \cdot K_a p \). Let us now take the sentences that comprise \( s \) and call them members of the set \( \Delta \). Thus:
Since both $\neg KaSaE.$ and $\neg KaCSaE.$ are members of $\Delta$, and since the conjunction of $Ka_e$ and $KaSaE.$ would yield $KaCSaE.$, it should be fairly obvious that $\Delta \supset \neg Ka_e$, which is exactly what premise (2) of Mr. Lehrer's argument maintains. We will, then, grant Lehrer this premise. It is with premise (1) that we must find fault if we are to cast doubt upon the proof.

Premise (1) states that $\neg BaKap \supset s$. Now if $\neg s$ were conjoined with $\neg BaKap$ and a contradiction were to result, we would have to maintain that $\neg BaKap \supset s$. We first need to see what the difference between $s$ and $\neg s$ would be. I think that, because of the words "even though" in the English version of $s$ given above, the only difference between $s$ and $\neg s$ would be that the last part of $\neg s$ would read "...he knows that he has correctly said that $e$". The formulation of premise (2) would seem to bear this out, because if $Ka_e \supset \neg s$ (which is logically equivalent to $s \supset \neg Ka_e$) then $KaCSaE.$ would have to be a member of $\neg s$ instead of $\neg KaCSaE.$ So, if we were to construct a set called $\Delta^*$ for $\neg s$ in the same manner in which we made one called $\Delta$ for $s$ we would get the following:

$$\Delta^* = (CSaP, KaSaP, \neg KaCSaP).$$

Now, if the addition of $\neg BaKap$ to the membership of $\Delta^*$ results in a contradiction, then $\Delta^* \supset BaKap$, which is to say that $\neg s \supset BaKap$. This is logically equivalent to $\neg BaKap \supset s$, and this, in turn, is premise (1). It is clear that the addition of $\neg BaKap$ to $(CSaP, KaSaP)$ cannot result in a contradiction, because this is a sub-set of both $\Delta$ and $\Delta^*$, and we want $\neg BaKap$ to be inconsistent with $\Delta^*$ but not $\Delta$. Therefore, the contradiction must be between $\neg BaKap$ and $KaCSaP$. We said above that $KaCSaP \equiv KaSaP \cdot Ka_e$, so the contradiction must be because $\neg BaKap$ is
inconsistent with (\(\neg B \neg A \neg K \neg P\)). But we, just noted that \(\neg B \neg A \neg K \neg P\) could not be inconsistent with \(B \neg A \neg K \neg P\), so that leaves \(B \neg A \neg K \neg P\). If \(\neg B \neg A \neg K \neg P\) is inconsistent with \(B \neg A \neg K \neg P\), then \(\Delta^* \Rightarrow B \neg A \neg K \neg P\), but the fact that \(\Delta^* \Rightarrow B \neg A \neg K \neg P\) is in turn being used as a premise in support of the fact that \(\neg B \neg A \neg K \neg P\) is inconsistent with \(B \neg A \neg K \neg P\) (i.e., that \(\neg B \neg A \neg K \neg P \Rightarrow \neg B \neg A \neg K \neg P\)). In other words, premise (1) of Mr. Lehrer's argument cannot be supported unless (3), the conclusion, is already assumed.

IV.

In the previous three sections we took a look at two systems of epistemic logic, and found reasons to criticize both. In this section, we shall attempt to formulated a third system of logic which will hopefully include the advantages of the two systems discussed above while eliminating the parts of those systems whose results we criticized.

Out of sheer personal preference, I shall set my system up in a format similar to Ackermann's. Assuming the existence of some set \(\Delta\), a series of operations shall be outlined which can be applied to \(\Delta\). Whether or not the sets resulting from the application of these procedures are consistent will determine whether or not \(\Delta\) itself is consistent. We shall, however, adopt Hintikka's operators. In this context, let us lay the groundwork for our system with the following "rule."

(1) \(\Delta\) is the set of sentences, each member of which represents some claim about a's conscious beliefs or knowledge, or about a's lack of conscious beliefs or knowledge, or about the possibility of a having certain conscious beliefs or knowledge. Each member of \(\Delta\) is prefixed by one of the following operators:
The first procedure which we shall introduce shall be similar to Ackermann's "Belief Argumentation," but will avoid producing $\neg B \vdash B \neg p$ as a theorem of our system. Notice (see section I above) that the reason that $B \vdash B \neg p$ falls out of Ackermann's system is that his Belief Augmentation stipulates that every member of $B$ shall also be a member of $B^*$, and that, if $B \vdash B \neg p$ is an element of $B$ both $B \neg p$ and $p$ are added to $B^*$. In an effort to correct this, we shall begin our second rule in the following manner.

(2) Doxastic Augmentation (D.A.) is an operation applied to the members of $\Delta$ to form a new set, $\Delta_{dox}$, which is a doxastic alternative to $\Delta$ with respect to $a$.

(a) For every element of the form $'B \neg p'$ in $\Delta$, include an element of the form $'p'$ in $\Delta_{dox}$.

Thus, if $\Delta$ was $(B \vdash B \neg p)$, $\Delta_{dox}$ would be $(B \neg p)$, showing no inconsistency. In Hintikka's doxastic logic, there is a condition $(C \vdash p)$, which was discussed in section II above. In order to incorporate this condition into our system, we need to add another part to Rule (2) as follows:

(b) For every element of the form $'\neg C \neg p'$ in $\Delta$, include both $'B \vdash p'$ and $'\neg p'$ in $\Delta_{dox}$. 
By including both Ba ~ p and ~ p in Δdox, we absorb into our system two of Hintikka's conditions: (C. ~ F), which would include Ba ~ p in Δ, and (C.3*), which is the basis for (a) in our Rule (2) in the first place. Here we include Ba ~ p in Δdox because, assuming that p v ~ p is true and p is incompatible with everything that we believe, we will believe ~ p if we carry out the implications of our beliefs far enough.

Our next procedure is our counterpart to Ackermann's "Disbelief Procedure," the only change being an addition to include the operators Cap and Ca ~ p.

(3) Doxastic Compatibility Procedure (D.C.P.) is an operation applied to the members of Δ and Δdox to form new sets, Δdox1, Δdox2, ..., Δdoxn, which are doxastic alternatives to Δ and Δdox with respect to a.

(a) If there are n elements of the form 'ba ~ p' or 'ca ~ p' (or 'ba ~ p' or 'cap') in Δ and/or Δdox, then n different sets, Δdox1, Δdox2, ..., Δdoxn, shall be formed.

(b) Every element of Δdox is an element of every set Δdox1, Δdox2, ..., Δdoxn.

(c) If either 'ba ~ p' or 'ca ~ p' (or both) is an element of Δ or Δdox, then 'p' is an element of some set Δdoxi.

(d) If either 'ba ~ p' or 'cap' (or both) is an element of Δ or Δdox, then 'p' is an element of some set Δdoxi.

In the manner in which we have developed these two procedures for doxastic logic, we will develop two similar procedures for epistemic logic. The only difference between our first epistemic procedure, which we shall call "Epistemic Augmentation," and Doxastic Augmentation is that the former should prove the theorem KaKap ⊨ Kap, while the latter will not prove the
Theorem \( BaBaP \supset BaP \). The parenthetical clause in part (a) of the following will serve that purpose.

(4) Epistemic Augmentation (E.A.) is an operation applied to the members of \( \Delta \) to form a new set, \( \Delta_{ep} \), which is an epistemic alternative to \( \Delta \) with respect to \( a \).

(a) For every element of the form '\( KaP \)' in \( \Delta \), include an element of the form '\( p \)' in \( \Delta_{ep} \) (if \( KaP \) is an element of \( \Delta \), add both '\( KaP \) and '\( p \)' to \( \Delta_{ep} \)).

(b) For every element of the form '\( \sim PaP \)' in \( \Delta \), add both '\( Ka \sim p \)' and '\( \sim p \)' to \( \Delta_{ep} \).

The second epistemic procedure, the "Epistemic Possibility Procedure," corresponds exactly with the Doxastic Compatibility Procedure.

(5) Epistemic Possibility Procedure (E.P.P.) is an operation applied to the members of \( \Delta \) and \( \Delta_{ep} \) to form new sets, \( \Delta_{ep1} \), \( \Delta_{ep2} \), \( \ldots \), \( \Delta_{epn} \), which are epistemic alternatives to \( \Delta \) and \( \Delta_{ep} \) with respect to \( a \).

(a) If there are \( n \) elements of the form '\( \sim KaP \)' or '\( Pa \sim p \)' (or '\( \sim Ka \sim p \) or '\( Pa \sim p \)') in \( \Delta \) and/or \( \Delta_{ep} \), then \( n \) different sets, \( \Delta_{ep1} \), \( \Delta_{ep2} \), \( \ldots \), \( \Delta_{epn} \), shall be formed.

(b) Every element of \( \Delta_{ep} \) is an element of every set \( \Delta_{ep1} \), \( \Delta_{ep2} \), \( \ldots \), \( \Delta_{epn} \).

(c) If either '\( \sim KaP \)' or '\( Pa \sim p \)' (or both) is an element of \( \Delta \) or \( \Delta_{ep} \), then '\( \sim p \)' is an element of some set \( \Delta_{ep1} \).

(d) If either '\( \sim Ka \sim p \) or '\( Pa \sim p \)' (or both) is an element of \( \Delta \) or \( \Delta_{dox} \), then '\( p \)' is an element of some set \( \Delta_{dox1} \).

One last "rule" is needed in our system, which will constitute the actual test of consistency.

(6) Consistency Requirement (C.R.) If \( \Delta \) is consistent, then any doxastic or epistemic alternative resulting from the application(s) of (2) thru (5) to \( \Delta \) (or \( \Delta_{dox} \) or \( \Delta_{ep} \)) will be consistent according to the rules of standard sentential logic.
As it stands, our system provides a means of checking whether or not certain doxastic or epistemic sets are consistent, but will tell us nothing about the consistency of a set containing both doxastic and epistemic operators. For example, assuming the standard definition of knowledge, we would not want to be incapable of proving \((\text{K}p, \sim \text{B}p)\) inconsistent. In Hintikka's system when certain rules and conditions are used "every doxastic alternative is also an epistemic alternative."

Suppose that we have three sets, A, B and C. Now let us suppose that B is an epistemic alternative to A, and that C is a doxastic alternative to A. According to Hintikka, then, C is also an epistemic alternative to A. This does not say, however, that B is also a doxastic alternative to A. The reason why this is the case is that we want our beliefs to be restricted by our knowledge, but not vice versa. This is to say that, if I know that \(p\), it would be inconsistent for me to claim that I believe that \(\sim p\). Given the two claims \(\text{K}p\) and \(\text{B}a \sim p\), we would want the knowledge claim to overrule the belief claim, but not vice versa. That is, given these two claims, we would not force the knowledge claim to conform to the belief claim, but rather the reverse. It can be seen that this aspect of the logic of belief and knowledge is again due to the fact that \(\text{K}p \Rightarrow p\), while \(\text{B}p \Rightarrow p\).

We can reflect this logical behavior in our system by including in the procedure Doxastic Augmentation all of the steps which from the procedure Epistemic Augmentation, and we will show that the reverse cannot be done. Let us add the steps to Doxastic Augmentation which are labeled (a) and (b) under Epistemic Augmentation, and call them (c) and (d). Thus, if we have \((\text{K}p, \text{B}a \sim p)\), by Doxastic Augmentation we will obtain \((p, \sim p)\), which is obviously inconsistent according to our Consistency
Requirements. What about the set \((Kp \sim Kp)\)? By an application of Doxastic Augmentation we obtain \((p)\), and by an application of the Doxastic Compatibility Procedure we have \((p, \sim p)\). These results square with our intuitive notions of the relationship between belief and knowledge.

Yet, we cannot similarly include as (c) and (d) under Epistemic Augmentation those steps which are (a) and (b) under Doxastic Augmentation. Suppose for a moment that we have done so. We could then construct the following proof:

\[
\begin{align*}
(1) & \quad (p, \sim p) \quad \Delta \text{ assumption} \\
(2) & \quad (p) \quad \Delta_{ep} \text{ (E.A.)} \\
(3) & \quad (p, \sim p) \quad \Delta_{ep1} \text{ (E.P.P.)}
\end{align*}
\]

Certainly we do not want \(p \implies \sim p\) to be a theorem provable by our system. Thus, Doxastic Augmentation can include (a) and (b) of Epistemic Augmentation, but not vice versa. Again, this is because our beliefs must conform to the "world" described by our knowledge, but not vice versa.

Thus, the epistemic conditions can be imposed upon the doxastic alternatives, making them also epistemic alternatives, but the doxastic conditions cannot be imposed upon the epistemic alternatives producing counter-intuitive results.

We have constructed a system which will: (a) check doxastic sets for consistency; (b) check epistemic sets for consistency; and (c) check sets containing both doxastic and epistemic operators for consistency.

Given our system, we can prove the following:

\[
\begin{align*}
(1) & \quad (\sim p, \sim p) \text{ and } (p, \sim p) \text{ consistent to preserve doxastic agnosticism;} \\
(2) & \quad (\sim p, \sim p) \text{ and } (p, \sim p) \text{ consistent to preserve epistemic agnosticism;} \\
(3) & \quad (p, \sim p) \text{ consistent because } p \not\implies p; \\
(4) & \quad (p, \sim p) \text{ consistent because } p \not\implies p.
\end{align*}
\]
(5) \((Kap, \sim KaKap)\) consistent because \(Kap \not\rightarrow KaKap\);

(6) \((\sim Kap, \sim Ka \sim P, BaP)\) (and all other cases of believing, not believing or disbelieving anything about which one know nothing) consistent;

(7) \((Kap, \sim Bap)\) and \((Kap, Ba \sim P)\) inconsistent because \(Kap \not\rightarrow P\) and \(Kap \not\rightarrow Bap\); and

(8) \((Kakap, \sim Kap)\) inconsistent because \(Kakap \not\rightarrow Kap\).
Appendix: A Proposed System of Epistemic Logic

(1) \( \Delta \) is the set of sentences, each member of which represents some claim about a's conscious beliefs or knowledge, or about a's lack of conscious beliefs or knowledge, or about the possibility of a having certain conscious beliefs or knowledge. Each member of \( \Delta \) is prefixed by one of the following operators:

- \( \text{Ba} \) - "a believes that:"
- \( \sim \text{Ba} \) - "a does not believes that:"
- \( \text{Ka} \) - "a knows that:"
- \( \sim \text{Ka} \) - "a does not know that:"
- \( \text{Ca} \) - "it is compatible with everything that a believes that:"
- \( \sim \text{Ca} \) - "it is not compatible with everything that a believes that:"
- \( \text{Pa} \) - "it is possible for all a knows that:"
- \( \sim \text{Pa} \) - "it is not possible for all a knows that:"

(2) Doxastic Augmentation (D.A.) is an operation applied to the members of \( \Delta \) to form a new set, \( \Delta_{dox} \), which is a doxastic alternative to \( \Delta \) with respect to a.

(a) For every element of the form 'Ba\(p\)' in \( \Delta \), include an element of the form 'p' in \( \Delta_{dox} \).

(b) For every element of the form 'Ba \( \sim p\)' in \( \Delta \), include both 'Ba \( \sim p\)' and ' \( \sim p\)' in \( \Delta_{dox} \).

(c) For every element of the form 'Kap' in \( \Delta \), include an element of the form 'p' in \( \Delta_{dox} \) (if Kap is an element of \( \Delta \), add both 'Kap' and 'p' to \( \Delta_{dox} \).)
(d) For every element of the form ' \sim \mathcal{P} \mathcal{A} \mathcal{P}' in $\Delta$, add both $'\mathcal{K} \sim \mathcal{P}'$ and '$\sim \mathcal{P}'$ to $\Delta_{\text{dox}}$.

(3) Doxastic Compatibility Procedure (D.C.P.) is an operation applied to the members of $\Delta$ and $\Delta_{\text{dox}}$ to form new sets, $\Delta_{\text{dox}_1}$, $\Delta_{\text{dox}_2}$, $\ldots$ $\Delta_{\text{dox}_n}$, which are doxastic alternatives to $\Delta$ and $\Delta_{\text{dox}}$ with respect to $a$.

(a) If there are $n$ elements of the form ' $\sim \mathcal{B} \mathcal{A} \mathcal{P}'$ or ' $\mathcal{C} \mathcal{A} \mathcal{N} \mathcal{P}'$ (or ' $\sim \mathcal{B} \sim \mathcal{P}'$ or ' $\mathcal{C} \mathcal{A} \mathcal{P}'$) in $\Delta$ and/or $\Delta_{\text{dox}}$, then $n$ different sets, $\Delta_{\text{dox}_1}$, $\Delta_{\text{dox}_2}$, $\ldots$ $\Delta_{\text{dox}_n}$, shall be formed.

(b) Every element of $\Delta_{\text{dox}}$ is an element of every set $\Delta_{\text{dox}_1}$, $\Delta_{\text{dox}_2}$, $\ldots$ $\Delta_{\text{dox}_n}$.

(c) If either ' $\sim \mathcal{B} \mathcal{A} \mathcal{P}'$ or ' $\mathcal{C} \mathcal{A} \mathcal{N} \mathcal{P}'$ (or both) is an element of $\Delta$ or $\Delta_{\text{dox}}$, then '$ \sim \mathcal{P} $' is an element of some set $\Delta_{\text{dox}_1}$.

(d) If either ' $\sim \mathcal{B} \sim \mathcal{P}'$ or ' $\mathcal{C} \mathcal{A} \mathcal{P}'$ (or both) is an element of $\Delta$ or $\Delta_{\text{dox}}$, then '$ \mathcal{P} $' is an element of some set $\Delta_{\text{dox}_1}$.

(4) Epistemic Augmentation (E.A.) is an operation applied to the members of $\Delta$ to form a new set, $\Delta_{\text{ep}}$, which is an epistemic alternative to $\Delta$ with respect to $a$.

(a) For every element of the form ' $\mathcal{K} \mathcal{A} \mathcal{P}$' in $\Delta$, include an element of the form '$ \mathcal{P} $' in $\Delta_{\text{ep}}$ (if $\mathcal{K} \mathcal{A} \mathcal{P}$ is an element of $\Delta$, add both ' $\mathcal{K} \mathcal{A} \mathcal{P}$' and '$ \mathcal{P} $' to $\Delta_{\text{ep}}$).

(b) For every element of the form ' $\sim \mathcal{P} \mathcal{A} \mathcal{P}$' in $\Delta$, add both ' $\mathcal{K} \sim \mathcal{P} $' and '$ \sim \mathcal{P} $' to $\Delta_{\text{ep}}$.

(5) Epistemic Possibility Procedure (E.P.P.) is an operation applied to the members of $\Delta$ and $\Delta_{\text{ep}}$ to form new sets, $\Delta_{\text{ep}_1}$, $\Delta_{\text{ep}_2}$, $\ldots$ $\Delta_{\text{ep}_n}$, which are epistemic alternatives to $\Delta$ and $\Delta_{\text{ep}}$ with respect to $a$.

(a) If there are $n$ elements of the form ' $\sim \mathcal{K} \mathcal{A} \mathcal{P}$' or ' $\mathcal{P} \mathcal{A} \sim \mathcal{P}$' (or ' $\sim \mathcal{K} \mathcal{A} \sim \mathcal{P}$' or ' $\mathcal{P} \mathcal{A} \mathcal{P}$') in $\Delta$ and/or $\Delta_{\text{ep}}$, the $n$ different sets, $\Delta_{\text{ep}_1}$, $\Delta_{\text{ep}_2}$, $\ldots$ $\Delta_{\text{ep}_n}$, shall be formed.
(b) Every element of $\Delta_{ep}$ is an element of every set $\Delta_{ep_1}$, $\Delta_{ep_2}$, ... $\Delta_{ep_i}$.

(c) If either $\neg \text{Ka}$ or $\neg \text{Pa}$ (or both) is an element of $\Delta$ or $\Delta_{ep}$, then $\neg p$ is an element of some set $\Delta_{ep_i}$.

(d) If either $\neg \text{Ka}$ or $\neg \text{Pa}$ (or both) is an element of $\Delta$ or $\Delta_{ep}$, then $\neg p$ is an element of some set $\Delta_{ep_i}$.

(6) Consistency Requirement (C.R.). If $\Delta$ is consistent, then any doxastic or epistemic alternative to $\Delta$ will be consistent according to the rules of standard Sentential Logic.
REFERENCE NOTES


(2) Ibid., p. 24. The wording could be improved somewhat here. For example, 'Bap' is not a conscious belief.

(3) Ibid., p. 25.


(5) Technical Note: Bap and \( \sim Bap \) don't have to be the only members of a's distinguished belief set. Obviously, if A is a subset of B, and A is inconsistent, so is B.

(6) See David Annis' article "A Note on Lehrer's Proof That Knowledge Entails Belief" (Analysis Vol. 29, no. 6, pp. 207-208) for a more complete development of a similar sort of counterexample; the only difference is that in Annis' example one does not have the concept of knowledge, and we mention a case in which one does not have the concept of belief.

(7) J. Hintikka, op. cit., p. 49.

(8) On pp. 123-125 of Knowledge and Belief, Hintikka says that the set \( \{Bap, \sim Bap\} \) is possible (i.e., consistent), and on pp. 62-63, he says that the virtual implication \( \sim Bap \rightarrow Bap \) is not self-sustaining. (The notions "virtual implication" and "self-sustaining" are explained in section II of this essay.

(9) Paillthorp, Charles, "Hintikka and Knowing That One Knows," The Journal of Philosophy, Vol. LXIV, no. 16, 1967. He chooses the label "SEP", because Hintikka's system is an epistemic counterpart to Lewis's modal system \( L^4 \).


(11) J. Hintikka, op. cit., p. 31.

(12) Ibid., p. 32.

(13) Hintikka's system also includes some "logical" conditions, such as \( (C.A) \) which will be taken for granted here. These conditions merely insure formally that logical notions like conjunction, addition, the law of non-contradiction, and so forth are included.
14) Hintikka has eliminated "forgetting" by limiting us to speaking about one person at one time, etc. Hintikka's essay "Epistemic Logic and the Methods of Philosophical Analysis" in his Models for Modolites, D. Reidel Publishing Co., Dordrech-Holland, 1969, pp. 3-19, explains clearly, I think, Hintikka's conception of knowledge and epistemic logic in general.

15) Hintikka, Knowledge and Belief, p. 124.

16) This proof is found on p. 105 of Hintikka's Knowledge and Belief. My numeration differs from the original, as Hintikka introduces this proof in the middle of a discussion on knowing that one knows.

17) Chisholm, for example, attacks (C.P*) in his "The Logic of Knowing," and Failthorpe in his "Hintikka and Knowing That One Knows" points out some stipulations that must be made in order to accept (C.KK*).


19) For a discussion of defeasibility, see David Annis' "Knowledge and Defeasibility," Philosophical Studies 24 (1973), pp. 199-203. Actually, all that is important to our discussion is that Kaa \supset Baa.

20) K. Lehrer, "Why Not Skepticism?" (See bibliographical information above, note 18.)


26) For our purposes, 'Baa\supset Baa' is a substitution-instance of 'Baa' for 'P'. Therefore, if Baa\supset Baa is an element of \Delta, Baa becomes an element of \Delta dox by Doxastic Augmentation.