Counting Techniques: A Sixth Grade Unit

An Honors Thesis (HONRS 499)

by

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Ball State University
Muncie, IN
May 1995

Expected Date of Graduation
December 1995
Purpose of Thesis

This unit was designed as a guide for teaching various counting techniques in a sixth grade classroom. The unit includes five lesson plans. Generally, there is no order to the use of these plans. However, it would be helpful to teach the lesson **Fundamental Counting Principle** prior to the lesson **Permutations**. Furthermore, the lessons can be added to or altered in the future as needed. The unit also includes additional activities and problems that can be used to edit the unit either for a younger grade or for an older grade. Finally, the unit includes the author’s thoughts and comments concerning the teaching of selected lessons.
Counting (grades 1-12)

"The art of counting should form a continuous thread throughout each child's education. It includes sequential counting, careful systematic counting, grouping and using multiplication to count arrays or grouped collections - often in two ways, formulating and applying the addition and multiplication rules and the standard trick of double counting. At each stage there should be lots of combinatorial world problems with the emphasis on thinking and common sense rather than on standard methods of solution."

--Excerpted from *NCTM Yearbook, 1991*
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Overview and Rationale

This thesis is a culminating project which reflects the knowledge and skills I have acquired during my four years at Ball State University. It combines concepts learned from both my major course of study, Elementary Education, and from courses in my endorsement, Junior High Mathematics. I have chosen to create a unit of lessons and activities for sixth grade, which can then easily be adapted for either upper-elementary classes or the middle school grades.

The topic for this unit, counting techniques, is an important component of discrete mathematics, or the study of those problems which have distinct solutions. Because counting is the main focus of these types of problems, I have decided to center this unit on the different techniques which can be used to count in order to find solutions. I feel it is important for students to own a multitude of such techniques so that, when faced with such problems in the real world, they will be well-equipped to analyze the situation and determine the best solution.

The lessons in this unit are all based on a hands-on, discovery learning approach. Students will be given the opportunity to work with concrete manipulatives while solving given problems. At the same time, they will be discovering the basic theories which make up the counting techniques, i.e. the fundamental counting principle, the pigeonhole principle, and so on. By linking these concepts to hands-on materials, the students will better understand these abstract concepts. In turn, they will be able to utilize these techniques when problem-solving in the real world.
Goals

1. Students will be able to derive a number of counting techniques through activities and experiments involving manipulatives.

2. Students will be able to analyze problems and determine the solutions to those problems using counting techniques.

3. Students will be able to apply counting techniques to a number of different real-world situations.
Evaluation Procedures

In order to evaluate the level of understanding that sixth grade students have concerning the topic of counting techniques, a pretest should be administered. This pretest gives students the opportunity to solve problems involving combinations, permutations, the fundamental counting principle, and the pigeonhole principle, each of which is included in this unit. An attitude inventory should also be given in order to evaluate the students' feelings toward the subject. After the pretest and attitude inventory have been given to the students, the teacher can then evaluate the students' present levels of understanding. From there, the teacher can choose which of the lesson plans, activities, and problems he or she wishes to teach. A sample pretest and attitude inventory, which were administered to a sample group of sixth graders, follows this page. The results of this test are discussed in section V.

After the lessons have been taught, the teacher can give the students a post-test. This test should directly assess those concepts that were taught during the lessons. An attitude inventory similar to the first one could also be administered if desired. The results of these evaluations can then be compared to the results obtained from the pretest. The teacher can easily observe any progress that was made by the students. A sample post-test, which was administered to the same group of sixth graders, also follows. The questions found in the post-test were selected from the pretest and assessed the concepts that were taught during a three-day period in a sixth grade mathematics classroom at Burris Laboratory School. The results of the post-test, as well as a discussion of the methods involved in the teaching of the selected lessons, can also be found in section V.
1. You have four different-colored pieces of candy. You want to give one piece to each of four friends: Amy, John, Mike, and Sue. How many different ways could the four pieces of candy be given to your friends?

2. A local contest was held to see who could guess the correct number of beans in a glass jar. Harry, Brandy, and Christy all guess exactly right. The judges for the contest decided to have the three winners pick numbers out of a hat; this would then decide who would win first, second, and third place. How many different ways are there for the contest to end?
3. There are ten dimes and ten quarters in a bag. Without looking, how many coins MUST you remove from the bag so that you are sure that you have six coins of one kind (either six dimes or six quarters)?

4. You are in the cafeteria line at school, and you need to decide what you want for lunch. You MUST choose one meat, one vegetable, one fruit, and one dessert. The menu for today is:

<table>
<thead>
<tr>
<th>Meats</th>
<th>Vegetables</th>
<th>Fruits</th>
<th>Desserts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamburger</td>
<td>Potatoes</td>
<td>Peaches</td>
<td>Brownie</td>
</tr>
<tr>
<td>Meatloaf</td>
<td>Green Beans</td>
<td>Pears</td>
<td>Cookie</td>
</tr>
<tr>
<td>Pork Chop</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

How many different lunches could you order?
5. Suppose that license plate numbers for a certain state consist of one letter followed by a five-digit number. Any number can be used, and repeats are allowed. If the state has a population of 3,000,000, would there be enough license plates to assign one per person?
Attitude Inventory

Name:__________________________

Date:__________________________

Please rate the following statements.

1. I enjoy doing problem-solving activities.
   
   1  2  3  4  5
   Not at all Somewhat Don't know A little Very much

2. I feel I am a good problem-solver.
   
   1  2  3  4  5
   Never Seldom Don't know Sometimes Always

3. When I come across a problem I don't understand, I usually give up.
   
   1  2  3  4  5
   Never Seldom Don't know Sometimes Always

4. I have worked with counting techniques before.
   
   1  2  3  4  5
   Never Seldom Don't know Sometimes Always

5. I do not like solving story problems because I never know where to begin.
   
   1  2  3  4  5
   Never Seldom Don't know Sometimes Always
Please finish the following statements.

6. I would enjoy mathematics more if.....

7. My best subject is ____________ because.....

8. The best part about math is.....

9. The worst part about math is.....

10. If I don't understand how to do a problem, I.....
Post-Test

Name: __________________

Date: __________________

1. You have four different-colored pieces of candy. You want to give one piece to each of four friends: Amy, John, Mike, and Sue. How many different ways could the four pieces of candy be given to your friends?

2. There are ten dimes and ten quarters in a bag. Without looking, how many coins MUST you remove from the bag so that you are sure that you have ten coins of one kind (either ten dimes or ten quarters)?
3. You are in the cafeteria line at school, and you need to decide what you want for lunch. You MUST choose one meat, one vegetable, one fruit, and one dessert. The menu for today is:

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</tr>
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<td>Pork Chop</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

How many different lunches could you order?
Lesson Plans

A total of five lesson plans are included in the following section. These lessons were not designed to be taught during a specific time period. Rather, the teacher can use the results of the pretest to determine how much time he or she feels should be spent on each lesson. Furthermore, the lessons were not designed to be taught in a specific order; this is also to be decided by the teacher. However, the author does suggest that the lesson *Fundamental Counting Principle* proceed the lesson *Permutations*, since the background knowledge obtained from the former lesson may be helpful in the latter. The materials suggested by the author for use in each lesson are also listed; however, other manipulatives could be substituted as needed. Finally, the author suggests using additional activities and problems to supplement each lesson. A suggested list for each of these can be found in sections VI and VII.
Fundamental Counting Principle

I. Objectives

1. Given a variety of manipulatives, the students will be able to solve problems involving counting techniques.
2. Students will be able to derive the fundamental counting principle by observing the patterns found during the solving of the problems.

II. Materials

1. Number tiles
2. Pattern blocks
3. Color tiles
4. "Code symbols"
5. Markers
6. Paper
7. Sack
8. Student work pages

III. Procedure

A. Introduction/Direct Teaching

1. The teacher will introduce the lesson by presenting a problem scenario: "We all work for the Secret Service on a special mission. They've decided to give us secret code symbols to use so that we won't be recognized by anyone. They have given us three symbols we can choose from in order to create secret code names:"

![Symbols]

(The teacher will have these three symbols on cards, and will use them as manipulatives during this time). "But we all have to have the same number of symbols: Either we all have to have one symbol, or we all have two symbols, or we all have three symbols. Each one
of our code names HAVE to be different. Should we use just one symbol, two symbols, or three symbols?"

2. Start with just one symbol. "What are the code names we could have if we used just one symbol, only once?" The teacher will list the symbols on the board as the students answer. "Would that be enough for our whole class?" (No, there would only be three code names possible).

3. "What choices would we have with two symbols?" As the students answer, the teacher will again make a list, then show the students how the list could have been made using a tree diagram. "Now do we have enough symbols for our whole class?" (No, there are only nine code names possible).

4. "What about three symbols?" Allow the students to give answers; again, write down the symbols in both list fashion and in a tree diagram. The students will now note that there are twenty-seven possible code names, which should be enough for a class.

B. Guided Practice

1. The teacher will hand out packets to groups of three. A few of the packets will be the same as others, but the students will not be told this. The teacher will tell the students that these are their problems that need to be solved as part of their first assignment. The students' packets will contain problem pages (see attached pages) and manipulatives. The students will be allowed to work on their problems using these manipulatives. They will also be instructed to either make a list or use a tree diagram.

2. After the students are through with their problems, each group will have a representative explain the problem and solution to the rest of the class. The teacher will write down important information (the number of choices involved in each problem) on the board as they speak.

3. When all groups are finished, the groups will examine the information the teacher has written on the board and hypothesize a pattern. The teacher will guide them toward the pattern if needed. (The pattern is that the variables can be multiplied together).

C. Independent Practice

1. The students will test their hypotheses by checking the answer
found to the problem addressed in the introductory section. The students should be able to discover that the solution they found, that there are twenty-seven choices, is right because:

\[
\begin{align*}
3 \text{ choices for first symbol} \\
\times 3 \text{ choices for second symbol} \\
\times 3 \text{ choices for third symbol} \\
= 27 \text{ different choices altogether}
\end{align*}
\]

IV. Evaluation

1. Were the students able to use the manipulatives to solve problems involving counting techniques?
2. Were the students able to derive the fundamental counting principle by observing the patterns found in the solving of the problems?

NOTE: The following materials are to be placed in the packet number indicated.

GROUP #1: Number tiles
GROUP #2: Pattern blocks, specifically hexagons, triangles, and squares
GROUP #3: Color tiles and a sack
GROUP #4: Color tiles and Pattern blocks
GROUP #5: Color tiles and a sack
GROUP #6: Number tiles
GROUP #7: Pattern blocks, specifically hexagons, triangles, and squares
GROUP #1

You are to pick a three-digit locker combination using only the numerals 0, 1, and 2. You are allowed to use a numeral more than once in the same combination. How many different locker combinations could you make?

Note: Make sure you either use a list or a tree diagram when explaining how you got your answer. The materials which you can use are also included in the packet.
GROUP #2

You are trying to pick a design as a logo for your company. You are to use at least one of the following shapes in your logo: a hexagon, a triangle, and a square. Your boss wants the logo to include three of any of these shapes, and they are to be situated so that they are in a straight line across the top of a page. However, he does not care in what order the three shapes are aligned, and you are allowed to use more than one of each shape (i.e., you could have a design that has two hexagons, then a triangle). How many different logos could you create?

Note: Make sure to use a list or a tree diagram when discussing your answer. Also, the materials which you can use are included in your packet.
GROUP #3

There are four color tiles in the sack: one each of red, blue, green, and yellow. How many different ways can the color tiles be chosen out of the sack? (You do not put a tile back into the sack once it has been removed).

Note: Make sure to use a list or a tree diagram when discussing your answer. Also, the materials which you can use are included in your packet.
GROUP #4

You have packed 3 pairs of jeans and 4 shirts to take with you on vacation. How many different outfits can you make?

Note: Make sure to use a list or a tree diagram when discussing your answer. Also, the materials which you can use are included in your packet. You might want to let color tiles represent jeans and pattern blocks represent shirts, etc.
GROUP #5

There are four color tiles in the sack: one each of red, blue, green, and yellow. How many different ways can the color tiles be chosen out of the sack? (You do not put a tile back into the sack once it has been removed).

Note: Make sure to use a list or a tree diagram when discussing your answer. Also, the materials which you can use are included in your packet.
GROUP #6

You are to pick a three-digit locker combination using only the numerals 0, 1, and 2. You are allowed to use a numeral more than once in the same combination. How many different locker combinations could you make?

Note: Make sure you either use a list or a tree diagram when explaining how you got your answer. The materials which you can use are also included in the packet.
GROUP #7

You are trying to pick a design as a logo for your company. You are to use at least one of the following shapes in your logo: a hexagon, a triangle, and a square. Your boss wants the logo to include three of any of these shapes, and they are to be situated so that they are in a straight line across the top of a page. However, he does not care in what order the three shapes are aligned, and you are allowed to use more than one of each shape (i.e., you could have a design that has two hexagons, then a triangle). How many different logos could you create?

Note: Make sure to use a list or a tree diagram when discussing your answer. Also, the materials which you can use are included in your packet.
I. Objectives

1. Given egg cartons marked with the months of the year and trading chips, the students will be able to solve problems involving counting techniques.
2. The students will be able to derive the patterns involved with the pigeonhole principle by examining the patterns found during the solving of the problems.

II. Materials

1. Egg cartons, marked with the months of the year
2. Trading chips
3. Overhead, transparency

III. Procedure

A. Introduction/Direct Teaching

1. The teacher will begin by posing a question to the students: “If I put everyone’s name from this class in a hat, how many times would I have to draw out a name before I could be certain that two of the people whose names have been drawn were born in the same month?”
2. The teacher will allow the students to give their guesses, as well as reasons for their answers. The teacher should make sure to point out that we cannot control the names that are chosen; the selection must be totally random. “We need to know how many names we would have to pick so that no matter whose names we get, we will be sure that at least two of the persons were born in the same month.”
3. Each group of students will then receive an egg carton and some trading chips.
4. The teacher will explain: “When a person’s name is drawn, a chip could be put in the cup that identifies his birth month. What we want is to have two chips in the same cup--how many names would I have to draw before this must be true?”
5. The teacher will allow the students to then work on the problem
in their groups. When all of the students have their answers, the teacher will discuss their solutions while using an overhead.

6. The teacher will also ask, “Does this mean that I would have to draw exactly thirteen names before I would get two people with the same birth month?” The teacher will use a transparency to begin a table showing the number of people with the same birth month and the number of draws required to be sure of that outcome (see attached page).

B. Guided Practice

1. The students will then work on the following problems:

a. How many draws would I have to make to be sure that there are 3 people with the same birth month?

b. How many draws would I have to make to be sure that there are 4 people with the same birth month?

2. When the students have their answers, the teacher will continue to fill in the table on the transparency. “Do you see any patterns?” In their own words, the students should be able to realize that to guarantee that there are \( n \) people with the same birth month, we need to draw \( 12(n - 1) + 1 \) names.

C. Independent Practice

1. The students will work on a final problem, using the pattern they just discovered in order to solve it: “How many names would we have to draw until we were sure that two people were born on the same day of the week?”

IV. Evaluation

1. Were the students able to solve problems involving counting techniques?
2. Were the students able to derive the patterns involved with the pigeonhole principle by examining the solutions to their problems?
### Pigeonhole Principle Patterns

<table>
<thead>
<tr>
<th>Number of people with same birth month</th>
<th>Number of draws required</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(13)</td>
</tr>
<tr>
<td>3</td>
<td>(25)</td>
</tr>
<tr>
<td>4</td>
<td>(37)</td>
</tr>
<tr>
<td>5</td>
<td>(49)</td>
</tr>
<tr>
<td>6</td>
<td>(61)</td>
</tr>
</tbody>
</table>
Explorations with
Socrates and the Three Little Pigs
by Mitsumasa Anno

I. Objectives

1. Students will be able to solve problems involving permutations and combinations which are found within the book Socrates and the Three Little Pigs.
2. Students will be able to use manipulatives in order to solve these problems.

II. Materials

1. Book: Socrates and the Three Little Pigs by Mitsumasa Anno
2. Various transparencies
3. Various student record sheets
4. Trading chips

III. Procedure

A. Introduction

1. The teacher will begin by introducing the book. The title, author, and illustrator will be discussed. The teacher will also mention that the story is about Socrates, who is a wolf; his friend Pythagoras, the frog; his wife, Xanthippe; and the three little pigs.
2. The teacher will ask if anyone knows who Socrates and Pythagoras were in real life (they were famous mathematicians and philosophers). “Does that give you a clue to what this story might be about?” (It involves mathematics).

B. Direct Teaching/Guided Practice

The teacher will then proceed to read the book. The teacher will stop at certain points during the book to do various activities and to hold discussions. The stopping points, dialogue, and activities are as follows:

1. Page 6: Stop before the pictures of where the pig could be. The
teacher will show the students a transparency illustrating the five houses where the three little pigs could live. The teacher will ask the students where they think the first little pig, the red pig, might live. Using their sheets, chips, and markers, the students will first illustrate and then mark on their sheets the five choices. The teacher will do the same on the transparency. The students can then check their answers by listening to the teacher read the rest of page 6.

2. Page 7: The teacher will do the same for this page as was done for page 6.

3. Pages 8 & 9: The teacher will ask the students why Socrates thinks "it is getting very confusing." (Because there are so many possibilities to consider). The teacher will also discuss with the students Pythagoras' idea of thinking of the problem as a tree. The teacher will ask if anyone knows what such a diagram is called (a tree diagram). Also, it will be important to note Pythagoras' last comment on page 9 ("There are 5 times 5 times 5 choices, don't you see?"). The teacher will then have the students form five groups. Each group will find, using the trading chips, and draw all of the ways the pigs could be situated in the houses when the red pig is in a specific house. For example, group #1 will find all of the ways the pigs could be found when the red pig is in the first house; group #2 will find all the ways possible when the red pig is in the second house; etc. After each group has finished, they will tape their illustrations to the board in order to be examined by the whole class. The teacher will ask how many total ways were found (125).

4. Page 11: The teacher will ask the students what would change about the number of ways the pigs could be situated if it did not matter what color each pig was (there would be many patterns that would be the same, so there would be a fewer number of ways).

5. Pages 12 & 13: The teacher will stop and point to the appropriate combinations on the board which are referred to in these pages. The teacher will ask why "supposing there are no roommates" would be easier (because there would be less ways to have to consider).

6. Page 14: The teacher will show the students the picture on this
page by using the overhead projector to create a diagram. The teacher will ask, "What will the choices for the second pig be this time?" The teacher will show the choices for the yellow (second) pig if the red (first) pig were in the first house. The students, in the same five groups as before, will work on this new problem as discussed in step #3. After the students have found all of the choices, the teacher will proceed to read pages 15, 16, and 17.

7. Page 17: The teacher will ask the students what is different about this tree diagram as compared to the last one (there are fewer choices for the second and third pigs). The teacher will ask each group if there was also something different about how many ways each group found for the pigs to be situated (there were only 12 ways per group this time, as compared to 25 ways per group last time).

8. Page 19: The teacher will ask the students to compare the new diagrams they have placed on the board: Are there any repeated patterns? Is Xanthippe right?

9. Page 21: The teacher will point out one group of six like patterns. The teacher will ask the students to decide how many different patterns there are if you do not count any repeating patterns (10).

10. Pages 23, 24, 25, & 26: The teacher will again use the overhead to create diagrams that illustrate the drawings in the book.

11. Page 27: The teacher will have the students come back up to the board and rearrange their patterns in the same order. The teacher will point out that each of the ten patterns is repeated six times. Therefore, if it matters what color the pig is, there would be sixty different choices. However, if it does not matter which pig is caught, there would be only ten different choices. "Does this really solve the problem, though? Why not?" (You still have to consider having more than one pig in a house).

12. Pages 30, 31, 32, & 33: Again, the teacher will use the overhead to illustrate the drawings found in the book. The teacher will ask the students to figure out how many different choices there are
based on Pythagoras' comment that there are "5 times 6 times 7" choices (210).

13. Page 35: The teacher will stop and have the students predict how many choices there will be after the pigs are all colored gray; i.e., how many choices will there be if no repeating patterns are counted? The students might need to be reminded that patterns are repeated six times each. Therefore, there are 35 choices in all (210/6 = 35). The teacher can then read through the rest of the story so that the students can check their final answer.

C. Independent Practice

1. The teacher will first begin a discussion about the different ways Socrates and Pythagoras tried to solve the problem. "Which way gave a fewer number of choices: when the order mattered, or when the order did not matter?" (When the order did not matter).
2. The teacher will tell the students that when you are trying to find how many ways something can happen and the order does matter, then you are finding permutations. However, when the order does not matter, then you are finding combinations. "Can anyone think of a time in real life when the order of something does matter? How about when the order does not matter?"
3. The students will then work on a final problem: If the three little pigs only had four houses, and roommates were not allowed, how many different ways could there be for the pigs to be found? (24 ways).

IV. Evaluation

1. Were the students able to solve problems involving permutations and combinations which were found in the book Socrates and the Three Little Pigs?
2. Were the students able to use manipulatives in order to solve these problems?
Where Are the Three Little Pigs?
Combinations

I. Objectives

1. Students will be able to use geometric shapes and patterns to discover different combinations of objects taken zero, one, two, three, four, and five at a time.
2. Students will be able to see the pattern of Pascal’s Triangle in the number of combinations they discover.

II. Materials

1. String
2. Large sheets of paper
3. Markers
4. Transparencies
5. Colored transparency pieces, cut into triangles and trapezoids

III. Procedure

A. Introduction/Direct Teaching

1. The teacher will introduce the lesson by posing the students a question: “Let’s pretend that your class was holding an election for student council. Five of your classmates have decided to run, but only two of them will be chosen to be on the council. How many different pairs of candidates could win?”
2. The teacher will then give groups of five students ten pieces of string (the number of pieces will not be told to the students, however). The students will be allowed to go out into the hall as needed; they will stand in a fairly small circle. The students will be told to work together to come up with different pairs of possible winners. Each pair of people will then hold the ends of piece of string. When they have come up with all of the different combinations, they will gently lay the string down and draw the figure they made on a large sheet of paper.
3. The students will also note the number of pairs they found (10).
4. The students will then compare the figures they drew to the one on the transparency shown to them by the teacher. “What shape was
made?" (a pentagon with a star in it; see figure 1 following this lesson). "How many different combinations did you come up with?" (10).

5. The teacher will say, "We showed the fact that we were choosing only two students at a time by drawing a straight line between them. How could we show choosing three students to be on the council? What shape might we use?" (Triangle).

B. Guided Practice

1. The students will then work at their seats in their table groups with the transparencies and the colored transparency pieces to solve the following problems:

a. There are five people running, but no one will win. How many different ways could the election turn out? (1).
b. There are five people running, but only one person can win. How many different ways could the election turn out? (5).
c. There are five people running, and two people can win. How many different ways could the election turn out? (10).
d. There are five people running, and three people can win. How many different ways could the election turn out? (10).
e. There are five people running, and four people can win. How many different ways could the election turn out? (5).
f. There are five people running, and five people will win. How many different ways could the election turn out? (1).

The students will use the pages following this lesson to keep a table of their answers. They will also use the transparency pieces to solve a few of these problems. (See figures 2 & 3 following this lesson).

2. When the students are finished, the teacher will ask the students if they see any patterns between the number of people chosen for the council and the number of different ways the election could turn out (the patterns can be found as a line in Pascal's Triangle).

C. Independent Practice

1. The students will work alone on the following problems:
a. What if you had four contestants; how many ways could the election end if two people could win?
b. If you had six candidates, how many ways could the election end if three people could be elected?

IV. Evaluation

1. Were the students able to use geometric shapes and patterns to discover different combinations of objects taken zero, one, two, three, four, and five at a time?
2. Were the students able to see the pattern of Pascal's Triangle in the number of combinations of objects they discovered?
### Combinations

<table>
<thead>
<tr>
<th># of people running</th>
<th># of people chosen</th>
<th># of ways election could end</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0</td>
<td>(1)</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>(5)</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(10)</td>
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<tr>
<td>5</td>
<td>3</td>
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<td>4</td>
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<td>5</td>
<td>5</td>
<td>(1)</td>
</tr>
</tbody>
</table>

**Extension Problems:**

1. If you had four contestants, how many different ways could the elections end if two people were chosen?

2. If you had six contestants, how many different ways could the election end if three people were chosen?
Figure 1
Figure 3: Templates for Transparency Pieces
Permutations

I. Objectives

1. Students will be able to solve problems involving permutations.
2. Students will be able to apply this technique to real-world situations.

I. Materials

1. Word Scramble List
2. Letters of the word “pencil” on 5 x 7 cards
3. Tape or magnets for back of letters
4. Paper

III. Procedure

A. Introduction

1. The teacher will begin by passing out the Word Scramble List to each student.
2. The teacher will say, “Here is a page of scrambled words. You have three minutes to unscramble as many of the words as you can.”
3. The teacher will allow the students to work on the list for three minutes (time may vary). Then the teacher will stop the students and discuss their findings (see World Scramble List following this lesson).
4. The teacher will ask, “What did you notice about the words?” (They were all the same word scrambled in a different way).
5. The teacher will ask, “Do you think this is all the different ways to rearrange the letters in this word?” The teacher will allow the students to examine the page for a moment to see if all the ways are there. (No).

B. Direct Teaching

1. The teacher will place the letters of the word “pencil” on the board. The teacher will say, “Let’s try to find out how many
different ways we could rearrange the letters in this word.”

2. The teacher will draw six small lines side by side. “Each of these lines represents a letter. Let’s look at the very first slot. How many choices do we have for the first slot?” (Six, because there are six letters in the word “pencil”).

3. The teacher will place one letter in the first slot. “Let’s say we chose the letter “I” for our first letter slot. How many choices does that leave us for the second slot?” (Five).

4. The teacher will say, “We had six choices for the first slot, and five choices for the second slot.” The teacher will write “6” below the first slot, then “5” below the first slot. “How many choices does that leave us for the next letter?” (4).

4. The teacher will continue in the same fashion for each of the remaining slots. Then the teacher will say, “We have six choices for the first slot, five for the second slot, four for the third slot, three for the fourth slot, two for the fifth slot, and only one for the sixth slot. So, how do we know how many different ways we can arrange the letters in this word?” (Use the Fundamental Counting Principle and multiply).

5. The teacher will write the multiplication symbol (x) in between the numbers already on the board so that it will read:

\[ 6 \times 5 \times 4 \times 3 \times 2 \times 1 \]

6. The teacher will then ask the students to figure out how many different arrangements there are (720). The teacher will say, “Now you know why I didn’t make a list of all of the different arrangements!”

C. Guided Practice

1. The teacher will then have the students get into groups (they could work in their table groups if the room is already situated in such a fashion). The teacher will say, “Usually when you turn in a group paper, you list all of the group members’ names at the top of the page. I want you to figure up how many different ways there are to list your group members’ names at the top of your paper.”

2. The students will then work on this problem. When all of the groups are finished, the teacher will ask, “How many different ways are there?” (Answers will vary according to the number of people in
each group).

3. The teacher will then ask the students to each pick a number between one and twenty. The teacher will tell the students the number of which she was thinking. Then she will tell the students to find out which member of each group was closest. The teacher will say, “Let’s pretend the person who was closest to my number gets to have his or her name listed first. Now how many different ways are there to list the names of the people in your group?”

4. The teacher will again allow the students to work on the problem. When all the groups have finished, the teacher will discuss the results with the class. The teacher will ask, “How does the number of ways to list the names change when you have to have a certain name in the first slot?” (The number of ways decreases).

C. Independent Practice

1. The students will work alone on the following problem, using the information they have obtained about permutations: “How many different ways could I list everyone’s name in our whole class?” (Answers will vary according to the number of students in the class; the teacher may allow students to use calculators).

IV. Evaluation

1. Were the students able to solve problems involving permutations?
Word Scramble List

1. lenpic
2. cepiln
3. ienlpc
4. epncli
5. pielnc
6. neicpl
7. eclnip
8. clipen
9. licnep
10. nepicl
Pilot Testing of Selected Lessons:
A Discussion of Methods and Results

As part of my course of study at Ball State University, I was able to participate in a Junior High Mathematics Methods course in the Spring of 1995. This course involved observing a sixth-grade mathematics class at Burris Laboratory School for several weeks, as well as teaching the class for a three-day period. I had been hoping for an opportunity to test a few of my lessons in a real sixth grade class, and after discussing my ideas with the Burris teacher I decided to do so.

I first administered a pretest to seven sixth-grade students, six of whom would be in the Burris class with which I was to work. The topics covered in the pretest included the fundamental counting principle, the pigeonhole principle, combinations, and permutations. Along with the pretest, I gave the students an attitude inventory which allowed the students to explain their feelings toward both mathematics and themselves as mathematicians and students.

After reviewing both the pretest and the attitude inventory, I decided to teach the lessons on the fundamental counting principle, combinations, and the pigeonhole principle. The students did not seem to understand the basic idea of the fundamental counting principle, nor were they able to solve the problems dealing with the pigeonhole principle and combinations. However, I found that some of the students were able to list many of the choices involved in the problem dealing with permutations (see Pretest, problem #2). Overall, the students seemed fairly confident in themselves as problem-solvers, and many of them wrote that they enjoyed math when it involved "physical objects" or "hands-on materials". Therefore, I decided to proceed with the lesson plans entitled Fundamental Counting Principle, Combinations, and Pigeonhole Principle.

The first lesson, Fundamental Counting Principle, was taught on Wednesday, April 19, 1995. I proceeded as stated in the lesson, and the students seemed to be enjoying it. They worked on their group problems, then gave their answers as I wrote the key numbers on the board. I was very pleased that the students were able to discover that you could just multiply the factors together. The students were also able to create tree diagrams with very little difficulty. Overall, I felt the lesson went quite well.

The second lesson, Combinations, was taught on Thursday, April 20, 1995. Again, the students were very active and seemed to enjoy the
hands-on activities involving the string and the transparency pieces. They were very excited when they kept finding the star patterns after placing the pieces on the pentagon! (See Figure 2 following the lesson). They were also amazed that the pattern they found when working on the problem was a line in Pascal's Triangle. The students had studied Pascal's Triangle earlier in the year, and they were very pleased to have found yet another place in mathematics where Pascal's Triangle can be discovered.

The final lesson, Pigeonhole Principle, was taught on Friday, April 21, 1995. The students enjoyed working with the egg cartons and chips, and they seemed to grasp the concept rather quickly. I was able to give the students more difficult problems on which to work, such as: "How many people would there have to be in a room before we could be certain that two of them had the exact same birthday?" They really liked working on these problems, since they involved problem-solving strategies, real-world situations, and manipulatives.

In order to assess the impact of the lessons on the students' understanding of counting techniques, I administered a post-test to the same sample group as before. I used the same questions that were on the pretest, but I only included those questions that dealt with the fundamental counting principle, the pigeonhole principle, or combinations. I found that overall, the students did much better on the post-test than they had on the pretest. Not only did they answer more of the questions right on the post-test, but they were also able to give a more thorough explanation as to how and why they got that answer.

I really enjoyed being able to teach these lessons to the Burris sixth-grade class. I had a wonderful time working with them, and I think they truly enjoyed and benefited from the lessons as well.
Additional Activities

1. Use mobiles to teach the concept of tree diagrams. Allow students to create their own mobile showing the different combination choices for a particular problem. For example, consider the following problem:

You have brought three different shirts, a pair of jeans, and a skirt with you to your friend's house. How many different outfits could you make?

The students can use construction paper to cut out shirts, pants, and skirts. Allow the students to name the mobile "Clothing" and use that as the top of their mobile. String can be used to attach the parts. (See figure 4 following this section for an example). By turning the mobile sideways and taping it to the chalkboard, you are providing the students with a concrete example of the tree diagram that illustrates the above problem.

2. Use an array model in order to link the concept of the fundamental counting principle with multiplication. Consider the following problem:

You walk into the local ice cream shop for a sundae. Your choices for flavor of ice cream are vanilla, strawberry, and chocolate. Your topping choices are chocolate fudge or strawberry. How many different sundaes could you make?

Set the variables up in an array (see figure 5 following this section for an example). Allow the students to give combinations as you place them in the appropriate spots, or the students themselves can paste the sundae choices in the appropriate boxes. When finished, ask the students if the figure reminds them of a model used for one of the four operations of numbers. If the students have learned multiplication using an array model, they will easily see the connection.

3. As an extension for combinations, a study on various lotteries can provide a wonderful real-world model. The main focus of such an extension should be the analysis and discussion of the following question: Is it really in our best interest to play the lottery? In other words, how many different number combinations could be made in various state lottery games, and what would be our chance of winning? The outcome for
which the teacher would hope is that the students would see how little their chance is of winning the lottery, and that if they do decide to play someday they should take this fact into consideration and play wisely.

In beginning such a study, the teacher should begin by allowing students to examine simple, fictional lotteries or other game situations. For example, you could begin by allowing the students to play a game with a six-faced die. Have the students play in pairs. One person is the die thrower, and the other is the "number guesser." Before the thrower rolls the die, the other student guesses what the number on the die will be. The thrower then rolls the die. If the guesser is right, he wins ten points. If the guesser is wrong, the thrower wins two points. The students can keep track of the score on a piece of paper. Allow the students to play fifty or so rounds, then have the students switch "jobs" and play again, keeping track of their new scores. After the students have played another fifty rounds, ask the class if they feel this game is fair or not, and why. The students may feel that the game is not fair either because the thrower can only win two points at a time while the guesser can win ten points at a time, or because it is easier for the thrower to win than the guesser. Have the students then list all of the different numbers which could be rolled. Then say, "Let's pretend I'm the guesser. I'm going to pick one number out of these six, and that will be my guess. That's the only way I can win. But how many ways are there for the thrower to win?" (5). "So there are more ways for the thrower to win than the guesser. So, if we only look at the chances of winning, then this game is unfair for the guesser." Then proceed to talk about the points that can be won. "The thrower could win two points, but the guesser could win ten points. Do you feel this is fair?" Allow the students to discuss. "Maybe we should try looking at both the chance of winning and the points that can be won at the same time." Make a chart, with the numbers which can be rolled on one side and the thrower and guesser at the top. "Pretend the guesser has predicted that the number 2 will be rolled. If the number 1 is then rolled by the thrower, how many points will be won by the thrower?" (2). The teacher can then mark the chart appropriately. Continue in this manner for each of the rolls, then tally the possible points for each player. The students will then see that each person has the possibility of winning the same points (10) if the game was played six times.

Further extension activities could include an examination of various state lotteries. For example, in the Michigan lottery, a person chooses 6 numbers between 1 and 45. Students could decide how many different combinations of 6 numbers could be chosen. Then, they could decide
whether they feel this lottery would be fair if a ticket costs $1 and the jackpot is $1.5 million. This activity could be extended to lotteries in Indiana and Ohio as well.
Additional Problems

1. How many different license plates are available if each plate contains a sequence of two letters followed by 4 digits?

   (Answer: 6,760,000 license plates)

2. How many ways are there to select 3 of your 6 best friends to go with you on vacation?

   (Answer: 20 ways)

3. A race is being held at your high school. There are nine runners competing. The first place winner will receive a blue ribbon, the second place winner will receive a red ribbon, and the third place winner will receive a white ribbon. How many different ways could the ribbons be awarded, if everyone is equally likely to win the race?

   (Answer: 504 ways)

4. How many students must be in a class in order to guarantee that at least two students received the exact same score on a test, if the test was worth 30 points?

   (Answer: 62 students)

5. How many ways are there to choose 4 one-digit numbers, from 0-9?

   (Answer: 210 ways)

6. You have 10 different birthday cards that you have bought to send to your friends for their birthdays. This month, 4 of your friends have birthdays. How many different ways are there to send birthday cards to your friends this month?

   (Answer: 5,040 ways)
7. You are going on a vacation. While you are there, you want to visit the beach, the local museum, the zoo, and a water park. How many possible orders are there for you to visit these places?

   (Answer: 24 different orders)

8. A drawer contains 6 black socks, 4 red socks, and 8 white socks.

   a) How many socks would you have to take out to be sure you had 2 of the same color?

      (Answer: 4 socks)

   b) How many socks would you have to take out to be sure you had 2 red socks?

      (Answer: 16 socks)

9. There are 10 people in the room. If each person shakes hands with every other person exactly once, how many total handshakes will be made?

   (Answer: 45 handshakes)

10. The telephone system in the United States consists of ten digits: a three-digit area code, a three-digit office code, and a four-digit station code. There are certain restrictions which are placed on a telephone number. Neither the area code nor the office code can begin with the numeral 0 or 1. Considering this restriction, how many different telephone numbers are possible in the United States?

    (Answer: 6,400,000,000)
Glossary

1. **Combination**: an unordered arrangement of a distinct set of objects.

2. **Combinatorics**: the study of arrangements of objects.

3. **Discrete Mathematics**: the portion of mathematics which involves the study of discrete objects. In general, this portion of mathematics is used whenever the counting of distinct elements is needed.

4. **Fundamental Counting Principle**: a basic counting principle which states that if there are $n$ ways to do one task and $m$ ways to do another, then there are $nm$ ways to do both.

5. **Pascal's Triangle**: a special triangular array of integers formed as shown below:

   ![Pascal's Triangle Diagram]

   In the array, the first and last integers are always one.

6. **Permutation**: an ordered arrangement of a distinct set of objects.

7. **Pigeonhole Principle**: a counting principle which states that if there are more pigeons than available pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it. In algebraic terms, this means that if $k + 1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects. Also known as the Dirichlet box principle.

8. **Tree Diagram**: a diagram which can be used to solve counting problems. The diagram consists of a root, a number of branches protruding from the root, and possibly more branches leaving the endpoints of other branches.
Bibliography


