Asset/Liability Management:
An Overview of Immunization Theory

An Honors Thesis (ID 499)
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Preface

The purpose of this paper is to provide an overview of asset/liability management through immunization theory. Redington's theory of immunization, as well as McCutcheon's theory are explained in detail. The investment options that can be used to immunize a fund are also discussed. For a more detailed explanation of how a particular investment technique may be utilized, it is suggested that the reader reference the cited papers. Also included are the management considerations of asset/liability management and a synopsis of the existing software packages available to assist in the immunization process.
Introduction

Immunization and the process of asset/liability matching have a history that dates back to the winter of 1951. One morning Frank Redington, an English actuary, decided that it was simply too cold to get up and work in the garden so he remained in bed and created the concept of immunization. His work was presented to the Institute of Actuaries in February, 1952, in London in a paper titled "Review of the Principles of Life-Office Valuations."

In this paper he addressed the following question: "Is the actuary's certification of the solvency of a life insurance company dependent on a certain level of interest rates being maintained, or can the actuary give a more absolute certification of solvency?" ("Asset/liability...", Shewan, 1984:1252). In an effort to answer this question, Redington provided the definitions to two phrases, "matching" and "immunization."

Matching is defined as "the distribution of the term of assets in relation to the term of the liabilities in such a way as to reduce the possibility of loss arising from a change in interest rates" (Redington, 1952:289). Immunization is the process of making investments in a manner which will protect the existing business from a change in the interest rates.

In his paper, Redington concluded that the results of the actuary can be absolute under certain conditions, with
the primary condition being that the duration of the payments going out of the company be equal to the duration of the cash being received by the company. It was his opinion that if this criterion was met, the assets and the liabilities would then be equally effected by changes in the market interest rate.

Prior to 1914, the world economies were in a period of stable economic and social growth and, consequently, there was little need for the practice of immunization. The financial results of the World Wars changed this, however. Items such as interest risk and tax implications now had to be included in the cost of products along with the standard mortality uncertainties (Redington, 1952:286). During the 1970's, under the Carter administration, the forceful impact of interest rates and inflation became apparent as they began to skyrocket and interest rate risk took its toll.

James Tilley, a vice president at Morgan Stanley, and David Jacob, a research analyst, attribute the increased awareness of asset/liability management to two events: increased volatility of interest rates and the changes in the financial services marketplace. As a result of the "Saturday Massacre" of October, 1979, when the Federal Reserve adopted a new policy direction and shifted the prime rate on a Saturday, several life insurance companies found themselves adversely exposed to interest rate risk. Investment policies of the late seventies called for investing in long maturities
to match the stable market in order to obtain high yields. The federal government also regulated the policy loan requirements forcing antiselection upon the insurers to the policyholder's benefit (Tilley & Jacob, 1983:1).

Changes in the investments offered in the marketplace have also become visible. Mortgage-backed securities and interest rate swaps have appeared while the abundance of private placement bonds have disappeared. High-quality intermediate-term investments have increased in demand as have fixed-income securities forcing investors to search elsewhere for better values. Together, the changes in the marketplace and the volatility of interest rates has forced insurers to begin to segment their portfolio in an attempt at immunizing against these risks (Tilley & Jacob, 1983:1).

In spite of the elevated awareness of immunization, it appears that many companies have not yet incorporated the actuary into their financial planning. In a survey conducted of plan sponsors from the Fortune 1000 list supplemented by 50 retail, bank, utility, and transportation companies, it was discovered that 54% of those responding "either saw no value in such meetings, or had not considered it, or thought it was 'a good idea, but with too many problems'" ("Asset/liability...", Shewan, 1984:1252). Andrew Shewan believes that the reason for this is the difficulty in applying Redington's theory to practical, everyday usage
"other than through a loose statement of intent" ("Asset/liability...", Shewan, 1984:1252).

James Tilley offers several reasons for the slow move to asset/liability matching theory. He attributes this to the fact that the investment officers and the actuaries have different backgrounds, and therefore, neither understands the other's professional jargon, causing a snag in the communication process. He also feels that the organizational structure of most companies separates the investment functions from those of the actuaries. He also associates immunization theory itself with the problem. Because it is not well understood by both the actuaries and the investment officers, it tends to be difficult to put into practice. Also, immunization theory puts several constraints on the investment opportunities removing several policy decisions. Finally, the theory is simply difficult to apply to practical situations (Tilley, 1980:264). With all of these strikes against immunization theory, it is easy to see why several companies have been slow to incorporate the actuary into the investment environment. Nevertheless, immunization is rapidly becoming an important tool to insure a company's survival.

The ultimate goal of partitioning a company's portfolio is to recognize the differences in investment strategies in the different business lines with respect to the duration that the assets are held. For example, the time periods for
which health insurance premiums are held differ significantly from those of a pension plan. The optimal portfolio should be based on the needs and resources of the entire company.

In order to understand how the techniques of asset/liability matching can be applied to new and existing business, it is necessary to fully understand interest rate risk, duration, and the various insurance products that they affect.

Interest Rate Risk

At the base of asset/liability matching lies the fundamental idea of interest rate risk and, therefore, immunization. Interest rate risk can be divided into two categories: correlated and direct risk. Correlated, or parallel, interest rate risk such as bond default and mortgage delinquency occurs when interest rates rise. This is the result of incorrectly measuring the risk attributed to interest sensitive products. Direct risk is the result of mismatched asset and liabilities and can be subdivided into three categories: reinvestment risk, price/liquidity risks, and investment antiselection.

Reinvestment risk is defined by Tilley as "having to reinvest funds when market yield is below levels guaranteed to policy holders and contract holders" (Tilley in Platt, 1986:229). This form of risk usually stems from a net cash inflow that exists when the duration of the assets is less
than the duration of the liabilities. For example, policy and contract holders are guaranteed a return of 7%, while the current market yield is only 5.25%. Another form of reinvestment risk becomes apparent when call options on bonds are exercised by the issuer. By calling a bond, its duration is shortened as the issuer takes advantage of lower interest rates. This forces the company to reinvest its funds at the lower rate.

Price/liquidity risk, conversely, is defined as "having to liquidate assets when the market yields are below levels at which the assets are purchased" (Tilley in Platt, 1986: 230). This arises out of a net cash outflow as the liability duration is shorter than that of the assets. For example, the company owns bonds with a yield of 11%, but due to a cash shortage, must sell these bonds when the market is only yielding 7%.

Investment antiselection is a result of the options that are built into the policy which contribute to interest rate risk. When the market interest rates become such that it is advantageous for the insured to deposit or withdraw money from the contract, a cash inflow or outflow will result that creates investment antiselection against the company.

In order to control interest rate risk, care must be taken to separate risk from uncertainty. Uncertainty is a result of random events, and a specific probability cannot be assigned to the event. A risk-taking event is one that
yields odds. Several asset/liability models are deficient in that many executives assume that interest rate risk is equal to uncertainty which in turn limits the number of events (Toevs & Haney in Platt, 1986:256-7).

**Interest Sensitive Products**

There are several products available in today's market which are sensitive to interest rate risk. The most common products are universal life products, single premium deferred annuities (SPDA), flexible premium annuities, structured settlement annuities, guaranteed interest contracts (GIC) and pension plan closeout products.

A universal life product is characterized as having a recurring premium payment. The interest income is credited to the policy, either at a constant interest rate or at a variable rate that reflects the interest rate environment at the time of the premium payment. Universal life products also have a cash surrender value and usually contain provisions that allow the insured to take out a loan from the cash value or the policy.

Single premium deferred annuities or SPDA's are primarily used for the basis of a retirement fund. In most cases the income tax on the policy is deferred until payments from the annuity are received. This allows the insured to receive the income at a later age when he or she is usually in a lower tax bracket. SPDA's are also distinguished by
being a no-load policy and having a decreasing surrender charge. In most cases the interest rate used to credit interest income is subject to change at the anniversary dates. A flexible premium annuity is essentially a single premium deferred annuity except it features a recurring premium instead of a single premium payment.

A structured settlement annuity is usually used as a means to settle a judgement in a court case. Instead of a lump sum payment, the annuity is constructed to provide a steady income for the plaintiff over a period of years, usually twenty to eighty years or more.

Guaranteed interest contracts (GIC) are single premium group annuities that are usually sold to pension or profit sharing plans. In the case of profit sharing plans, GIC's usually appear as recurring premium products. Withdrawals from the product are generally restricted to benefit payments only and are not subject to loans from the surrender value. As is apparent from its name, the GIC guarantees a specific interest rate over a specific period of time. Benefits are usually paid in one lump sum, therefore the GIC closely resembles a zero-coupon bond.

The pension plan closeout product is usually a complex single premium GIC with a long term liability. It provides death, survivor, and disability benefits along with early retirement and varied payment benefits (Tilley in Platt, 1986:226-229).
Pricing Methodology

In the following example (J. Tilley in Platt, 1986:230-235) James Tilley has used immunization theory to develop the price of a single premium deferred annuity. It is assumed that the immunized product consists of matched asset and liability durations in order to neutralize the value of the redemption option. In addition, the ideal product would be free from the effects of adverse interest rate fluctuations.

The SPDA that is being priced has the following features:

1) No front-end load or administrative charges

2) Surrender charges of 7%, 6%, 5%, 4%, 3%, 2%, and 1% in years one to seven, respectively, and none thereafter

3) A bailout rate 50 base points below the interest rate offered at issue of the policy

4) A "money back guarantee" that forgives the surrender charges to the extent they would otherwise invade the policyholder's original deposit, and

5) A non-cumulative, once-per-year withdrawal of 10% of the cash surrender value free of surrender charges

The following expenses apply to the insurer's writing of SPDA business:

1) 5% of the single premium as a commission to the selling agent, fully recoverable if the policy is redeemed within the first six months from issue and 50% recoverable if the policy is redeemed in the second six months.
2) $200 to cover the costs of issuing a policy

3) $25 annually after the first year to maintain policy records and to prepare and mail a report to the policyholder, and

4) 0.25% of the original single premium each year as the expense of managing and accounting for the investments made by the insurer to support the SPDA liability.

Recurring expenses are assumed to grow at a rate of 7.5% annually. The insurer also has the right to reset the interest rate on policy anniversaries, however the rate between anniversaries is guaranteed.

For simplicity the right to reset the interest rate at anniversary dates is ignored due to the strict bail-out provision afforded by the policy contract. In reality, however, this can be a useful provision and should be evaluated in real-life situations. Taxes have been eliminated from the example for simplicity. It is also assumed that acquisition costs will be amortized over a 7-year period. This creates an investment horizon of seven years. Over this period the present value of the profits is equal to 2.5% of the $25,000 average single premium or $625 per policy.

The redemption option is valued by assuming that the policyholder will decide to redeem the SPDA, paying the surrender charge, and purchase a new SPDA at current interest rates whenever the policyholder feels that they can break even. It is assumed that this period will be no longer than three years creating a 3-year investment horizon. This is
equivalent to an American put option on a zero coupon bond that matures three years from the exercise date. The strike price at any time is equivalent to the redemption value of the SPDA which is dependent on the interest rate of the SPDA, the size of the single premium, the surrender charge scale, the money-back guarantee, and the free withdrawal provision.

The risk to the insurer for including the redemption option is determined to be the gain or loss that is incurred when the option is exercised. The investment strategy of the insurer is assumed to involve paying out the commission and other acquisition expenses, deducting the present value of the profits from the premium payment, and using the balance of the single premium to buy:

1) Assets to fund the cash flow stream associated with all recurring expenses

2) A zero coupon bond maturing at the 7-year investment horizon for a par amount equal to the policyholder's single premium accrued with interest for seven years at the SPDA interest rate.

3) The particular put option that exactly neutralized the redemption right of the policyholder.

In the event that the redemption option is exercised, the insurer must sell the zero coupon bond and the assets purchased to pay the stream of recurring expenses. The "unearned" commissions are then collected from the selling agent or broker.

The actual price of the SPDA is usually derived through an iterative algorithm that converges to the selling price.
The SPDA is properly priced when the single premium payment is equal to the sum of the following:

1) The commission paid to the selling agent

2) The acquisition expense incurred in issuing the policy.

3) The present value (at issue) of the recurring expenses.

4) The market value of the zero coupon bond

5) The market value of the particular put required to neutralize the policyholder's redemption right.

6) The desired present value (at issue) of profits.

This ideal investment strategy is not practical in a real-world situation for several reasons: 1) No insurer would dedicate an entire block of assets against a stream of expenses; 2) The exact put option would not be available in any market, and if it were, insurance laws prohibit its purchase; 3) The zero-coupon bonds needed would not be available in large enough quantities, with exception of stripped U.S. Treasuries, to be of successful use.

Bearing this in mind, Tilley surmises that the practical strategy is to create a portfolio of fixed-income assets that have the same duration and convexity as the theoretical portfolio. These fixed-income assets can include bonds, mortgages, mortgage-backed securities, options, futures, and interest rate swaps. The portfolio will also have to be rebalanced over time as the interest rates change in order to
prevent duration drift, a concept that will be discussed later.

**Duration Analysis**

Immunization is defined by Redington as "the investment of the assets in such a way that the existing business is immune to a general change in the rate of interest" (Redington, 1952:289). One method of immunizing a fund, or determining if the fund is immunized, is through duration analysis. Mathematically, Redington defines duration in the following manner.

Let $V_L(\delta) = \Sigma v^t L_t$.

This is the sum of the present values of the out-going liabilities at force of interest $\delta$. Likewise,

$V_A(\delta) = \Sigma v^t A_t$,

is the sum of the present values of the in-coming assets at force of interest $\delta$. Let it be further assumed that at interest rate $\delta$,

$V_A(\delta) = V_L(\delta)$.

The difference in these two values is the surplus and is defined as

$S(\delta) = V_A(\delta) - V_L(\delta)$.

Now suppose the interest rate shifts from $\delta$ to $\delta+\epsilon$, with a shock factor of $\epsilon$. This will change $V_A(\delta)$ and $V_L(\delta)$ to $V_A(\delta+\epsilon)$ and $V_L(\delta+\epsilon)$, keeping in mind that both $\delta$ and $\delta+\epsilon$ are functions of time. Taylor's theorem with remainder is
defined as:

\[ f(t) = f(0) + tf'(0) + \int_0^t (t-w)f''(w)\,dw. \]

Applying this to \( S(\delta+\epsilon) \) gives:

\[
S(\delta+\epsilon) = V_A(\delta+\epsilon) - V_L(\delta+\epsilon) = \frac{(V_A(\delta) - V_L(\delta)) + \epsilon d(V_A(\delta) - V_L(\delta)) + \epsilon^2 (V_A(\delta) - V_L(\delta))}{d\delta} + \frac{\epsilon^2}{2!} \frac{d^2(V_A(\delta) - V_L(\delta))}{d\delta^2}.
\]

By definition the first term vanishes since \( V_A(\delta) = V_L(\delta) \). Further, if the assets and liabilities are matched such that there will be no profit or loss from the change in the interest rate, then all of the derivatives must also vanish. An immunized fund is then defined as having the first derivative, \( \frac{d(V_A(\delta) - V_L(\delta))}{d\delta} \), equal to zero.

If the second derivative is positive, then any change in the interest rate will result in a profit since \( \epsilon^2/2! \) is always positive. It is advantageous then to have \( \frac{d^2(V_A(\delta) - V_L(\delta))}{d\delta^2} \) positive. The definition of an immunized fund can then be defined to have the first derivative equal to zero, and the immunization as being stable or unstable based on the positive or negative value of the second derivative.

The mathematical definition for immunization, as provided by Redington, is then:

\[
\frac{d(V_A(\delta) - V_L(\delta))}{d\delta} = 0 \quad \text{or} \quad \Sigma tv^t A_t = \Sigma tv^t L_t
\]

\[
\frac{d^2(V_A(\delta) - V_L(\delta))}{d\delta^2} > 0 \quad \text{or} \quad \Sigma t^2 v^t A_t > \Sigma t^2 v^t L_t
\]

(Redington, 1952:290-1).

The mathematical definition of "duration" first appeared in 1938 in "Some Theoretical Problems suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856," written by F.R. Macaulay for the National Bureau of Economics Research. G. Bierwag, G. Kaufman, and A. Toevs expanded the fundamental insights supplied by Macaulay and introduced the Taylor expansion to estimate a measurement of duration for small changes in interest rates expressed as changes in the force of interest. Simplified in other articles, the definition can be expressed mathematically as:

\[
\text{Duration} = -\frac{d\text{PRICE}}{\text{PRICE} \, d\text{i}}
\]

This states that "duration is a measure of price sensitivity and is computed by finding out how much the price will change (d\text{PRICE}) as interest rates change a small amount (d\text{i}). It is turned into a measurement index by taking the ratio of this price change to the beginning price" (Norris & Epstein, 1988:2). A negative sign is added so that positive durations will result with those instruments whose price has an inverse relation with interest rates. By applying this equation to several different portfolios, one can easily
identify those portfolios that are extremely sensitive to market shifts: the higher the duration, the more price sensitive it will be in comparison to lower duration portfolios (Toevs in Platt:1986,61).

Using notation supplied by Peter Ho and Irwin Vanderhoof:

\[ \text{Duration} = \sum_{t} \frac{c_t v^t}{p} \]

where \( c_t \) = prospective cash flow and \( p \) = price to purchase those cash flows. Further, if \( p \) is made a unit investment, then:

\[ \text{Duration} = \sum_{t} c_t v^t \]

(Ho, 1988:13).

Because interest rates do change over a period of time, the duration of a particular portfolio will change also. This event is called duration drift. In order to prevent duration drift from adversely affecting the immunized position of a fund, the portfolio has to be rebalanced periodically.

**Duration Analysis Example**

Consider a $1000 bond with annual coupon payments of $80 at the end of each of the next ten years. The call price is $1000 and the interest rate is 7.0%
If \( p \) is the price of the bond, then using Makeham's formula,

\[
p = K + (g/i) \cdot (C-K)
\]

where:
- \( K = Cv^n \)
- \( g = Fr/C \)
- \( C \) = redemption value,
- \( n \) = number of interest conversion periods
- \( Fr \) = the coupon amount.

Substituting,

\[
p = 1000 \cdot (1.07)^{-10} + \frac{80}{1000} \cdot (1000 - 1000 \cdot (1.07)^{-10})
\]

\[
= 508.35 + (1.14286) \cdot (1000 - 508.35)
\]

\[
= 1070.24
\]

Therefore,

\[
\text{Duration} = \left( \sum_{t=1}^{2} t \cdot v^t \cdot 80 + 1080 \cdot (10) \cdot v^{10} \right) / p
\]

\[
= (2372.45 + 5490.17) / 1070.24
\]

\[
= 7.35 \text{ or 7 years, 4 months, 6 days}
\]

which implies that this "bond portfolio" with a duration of 7.35 years can be matched with a liability schedule to immunize against interest rate changes. It should be noted that duration is additive so therefore the duration of a portfolio of bonds is the price-weighted average of the durations of the bonds in the portfolio.

Although duration analysis is somewhat limited as an approach to immunizing a fund, it is still a very effective means of quickly identifying the interest rate risk exposure of a cash flow. It can also be easily used to determine...
which portfolio is less sensitive to market changes if several portfolios are compared. Duration can also be used to construct a portfolio that has a duration that matches the remaining holding period for a liability and subsequently locking in a particular return. If a portfolio is rebalanced periodically to counter duration drift, a portfolio will yield the promised return, regardless of the number or size of the interest rate fluctuations (Toevs in Platt, 1986:61).

Full Immunization

There is another theory of immunization, termed 'full immunization', that allows the investor to make a profit whenever the interest changes. Remember, under Redington's theory a profit is earned only when the immunization is stable which occurs when the second derivative of the surplus, \( S(\delta + \varepsilon) \), is positive. This theory is derived by J. McCutcheon in An Introduction to the Math of Finance (McCutcheon, 1986:247-248).

Consider an investor who has a known liability \( S \) due at time \( t_1 \). He also has two receipts of income, assets, defined as A and B, which will be received at times \( t_1-a \) and \( t_1+b \). This sets up the following time line:

\[
\begin{array}{ccc}
| \text{A} & \text{S} & \text{B} \mid \\
0 & t_1-a & t_1 & t_1+b
\end{array}
\]

It is assumed that \( a \) and \( b \) are positive, although not necessarily equal, and for practical use, \( a \leq t_1 \).
If two of the four values of a, b, A, and B are known, then the other two values can be determined in a manner which satisfies the following equations:

\[ A \cdot e^{\delta'a} + B \cdot e^{-\delta'b} = S \]
\[ A \cdot e^{\delta'a} = B \cdot b \cdot e^{-\delta'b}. \]

The four known quantities are (i) a,b (ii) B,b (iii) A,a or (iv) A,b and remaining two quantities are solved below.

(i) If a and b are known then \( A \cdot e^{\delta'a} = B \cdot b \cdot e^{-\delta'b} \) and \( B \cdot e^{-\delta'b} = S - A \cdot e^{\delta'a} \). Multiplying both sides of the equation by b yields \( b \cdot B \cdot e^{-\delta'b} = b(S - A \cdot e^{\delta'a}) \). This can be substituted into \( A \cdot e^{\delta'a} = B \cdot b \cdot e^{-\delta'b} \) giving 
\[ A \cdot e^{\delta'a} = b(S - A \cdot e^{\delta'a}). \]
Collecting like terms gives
\[ A \cdot e^{\delta'a}(a+b) = bS \]
and solving for A gives
\[ A = \frac{bS}{(a+b) \cdot e^{\delta'a}}. \]

To solve for B, begin with \( A \cdot e^{\delta'a} = B \cdot b \cdot e^{-\delta'b} \) and substitute for the value of A giving the following:
\[ b \cdot A \cdot e^{\delta'a} = \frac{abS}{(a+b) \cdot e^{\delta'a}} \]
and solving for B gives
\[ B = \frac{aS}{(a+b) \cdot e^{-\delta'b}}. \]

(ii) If B and b are known (with \( B \cdot e^{-\delta'b} < S \)) then a and A are easily found. Given that \( A \cdot e^{\delta'a} + B \cdot e^{-\delta'b} = S \), you can solve for \( A \cdot e^{\delta'a} \) giving \( A \cdot e^{\delta'a} = S - B \cdot e^{-\delta'b} \). Multiplying both sides of the equation by a and

\[ a \cdot A \cdot e^{\delta'a} = (S - B \cdot e^{-\delta'b}) \cdot a \]

and solving for A gives
\[ A = \frac{aS}{(a+b) \cdot e^{-\delta'b}}. \]
substituting into \(Bbe^{-\delta b} = Aae^{\delta a}\) yields \(Bbe^{-\delta b} = a(S-Bbe^{-\delta b})\). Solving for \(a\) produces the following:

\[
a = \frac{Bbe^{-\delta b}}{S-Bbe^{-\delta b}}
\]

Substituting \(a\) into \(Bbe^{-\delta b} = Aae^{\delta a}\) gives \(Bbe^{-\delta b} = Ae^{\delta a}Bbe^{-\delta b}/(S-Bbe^{-\delta b})\). Finally, solving for \(B\) and simplifying produces

\[
A = \frac{S-Bbe^{-\delta b}}{e^{\delta a}}
\]

(iii) If \(A\) and \(a\) are known with \(Ae^{\delta a} < S\), then \(b\) and \(B\) can be calculated. Beginning with \(Ae^{\delta a} + Be^{-\delta b} = S\), solving for \(Be^{-\delta b}\) and multiplying both sides of the equation by \(b\) produces \(bBe^{-\delta b} = b(S-Ae^{\delta a})\). This can be substituted into \(Aae^{\delta a} = Bbe^{-\delta b}\) resulting in \(Aae^{\delta a} = b(S-Ae^{\delta a})\). Solving for \(b\) gives the following:

\[
b = \frac{Aae^{\delta a}}{S-Ae^{\delta a}}
\]

This can be substituted into \(Aae^{\delta a} = Bbe^{-\delta b}\) giving \(Aae^{\delta a} = Be^{-\delta b}[S-Ae^{\delta a}/Aae^{\delta a}]\). Solving for \(B\) and simplifying yields the following:

\[
B = \frac{S-Ae^{\delta a}}{e^{-\delta b}}
\]

(iv) If \(A\) and \(b\) are known and \(A < S\), from the
assumptions then a single unique solution can be found for a and B. Beginning with \( S = A e^{\delta a} + B e^{-\delta b} \), then \( S - A e^{\delta a} = B e^{-\delta b} \). From \( B e^{-\delta b} = A e^{\delta a} \), it can be shown that \( B e^{-\delta b} = A e^{\delta a} / b \). Substituting into the previous equation gives \( S - A e^{\delta a} = A e^{\delta a} / b \), which is equivalent to \( bS - A e^{\delta a} = A e^{\delta a} \).

Collecting like terms yields \( bS = A e^{\delta a} (a + b) \), and dividing by A gives \( bS/a = e^{\delta a} (a + b) \).

From this it can be said \( f(a) = bS/A \), where \( f(x) = (x + b) e^{\delta x} \). Note that \( f(x) \) is an increasing function tending towards infinity. Also note that \( f(0) = b < bS/A \). Therefore, there is one unique positive value for \( x \) such that \( f(x) = bS/A \).

B can then be calculated from \( B e^{-\delta b} = A e^{\delta a} \) giving

\[
B = \frac{A e^{\delta a}}{b e^{-\delta b}}
\]

(McCutcheon, 1986:410-11)

Let \( V(\delta) \) denote the present value of the total assets minus the present value of the total liabilities at the force of interest \( \delta \). \( V(\delta) = e^{-\delta t_1} (A e^{\delta a} + B e^{-\delta b} - S) \). \( V(\delta') = 0 \) is equivalent to \( A e^{\delta' a} + B e^{-\delta' b} = S \).

**Proof:**

\[
V(\delta') = 0 \quad \Rightarrow \quad A e^{\delta' a} + B e^{-\delta' b} = S
\]

Setting \( V(\delta') = 0 \) yields
\[ V(\delta') = e^{-\delta't_1}(Ae^{\delta'a} + Be^{-\delta'b} - S) = 0. \]

Dividing by \( e^{-\delta't_1} \) gives
\[
(Ae^{\delta'a} + Be^{-\delta'b} - S) = 0,
\]

and solving for \( S \) results in
\[
Ae^{\delta'a} + Be^{-\delta'b} = S
\]

Also, \( V'(\delta') = 0 \) is equivalent to \( Aae^{\delta'a} = Bbe^{-\delta'b} \).

**Proof:**

Given
\[
\frac{d[V'(\delta')]}{d\delta'} = 0
\]

yields
\[
(Ae^{\delta'a} + Be^{-\delta'b} - S)(-t_1 e^{-\delta't_1}) + e^{-\delta't_1}(Aae^{\delta'a} - Bbe^{-\delta'b}) = 0
\]

and therefore
\[
(Ae^{\delta'a} + Be^{-\delta'b} - S)(-t_1 e^{-\delta't_1}) = -e^{-\delta't_1}(Aae^{\delta'a} - Bbe^{-\delta'b}) .
\]

Dividing both sides by \( (-t_1) \) gives
\[
(Ae^{\delta'a} + Be^{-\delta'b} - S)(e^{-\delta't_1}) = \frac{e^{-\delta't_1}(Aae^{\delta'a} - Bbe^{-\delta'b})}{t_1} .
\]

Now, \( (Ae^{\delta'a} + Be^{-\delta'b} - S)(e^{-\delta't_1}) = V(\delta) = 0 \) from the previous proof. This results in
\[
\frac{e^{-\delta't_1}(Aae^{\delta'a} - Bbe^{-\delta'b})}{t_1} = 0
\]

and multiplying by \( t_1/e^{-\delta't_1} \) yields
and finally
\[ Aae^\delta'a = Bbe^{-\delta'b} . \]

Using the fact that \( V(\delta') = 0 \) and \( V'(\delta') = 0 \), \( V(\delta) \) can be manipulated to remove the liability, \( S \), from the equation in the following manner:

\[ V(\delta) = e^{-\delta't1}(Ae^\delta'a + Be^{-\delta'b} - S) \]

Multiplying \( Ae^\delta'a \) and \( Be^{-\delta'b} \) by judicious choices for 1 gives

\[ V(\delta) = e^{-\delta't1}[Ae^\delta'a(e^{\delta'a}/e^{\delta'a}) + Be^{-\delta'b}(e^{-\delta'b}/e^{-\delta'b}) - S] \]

and by collecting like terms

\[ V(\delta') = e^{-\delta't1}[Ae^\delta'a(e^{(\delta-\delta')a} + Be^{-\delta'b}e^{-(\delta-\delta')b} - S] . \]

Factoring out \( Ae^\delta'a \) results in

\[ V(\delta') = e^{-\delta't1}Ae^\delta'a[e^{(\delta-\delta')a} + \frac{Be^{-\delta'b}e^{-(\delta-\delta')b}}{Ae^\delta'a} - S] . \]

Since \( Aae^\delta'a = Bbe^{-\delta'b} \), both sides of the equation can be divided by \( b \) giving \( Aae^\delta'a/b = Be^{-\delta'b} \). If both sides of the equation are then divided by \( Ae^\delta'a \) we get

\[ \frac{Be^{-\delta'b}}{Ae^\delta'a} = \frac{a}{b} . \]

Likewise, if \( Aae^\delta'a/b \) is substituted for \( Be^{-\delta'b} \) in

\[ Ae^\delta'a + Be^{-\delta'b} = S \],

the result is

\[ Ae^\delta'a + Aae^\delta'a/b = S \cdot \]

Factoring out \( Ae^\delta'a \) yields

\[ Ae^\delta'a (1 + \frac{a}{b}) = S \]

and
Substituting these results back into $V(\delta)$ results in

$$V(\delta) = e^{-\delta' t} A e^{\delta'a} [e^{(\delta-\delta')a} + A e^{-(\delta-\delta')b} - (1 + a)]$$

Now consider the sign of the function

$$f(x) = e^{ax} + A e^{-bx} - (1 + a)^x$$

First, note that $f(0) = 0$. The derivative of $f(x)$ is

$$f'(x) = a(e^{ax} - e^{-bx})$$

This means that

$$f'(x) > 0 \text{ for } x > 0,$$

$$f'(x) = 0 \text{ for } x = 0,$$ and

$$f'(x) < 0 \text{ for } x < 0.$$ Therefore $f(x)$ has a minimum value for $x = 0$. If $x = \delta - \delta'$, then

$$V(\delta) = e^{-\delta' t} A e^{\delta'a} [e^{xa} + A e^{-xb} - (1 + a)]$$

and $V(\delta) > 0$ for all $\delta < \delta'$. This means that a profit is earned whenever the interest rate changes. For this to be applied, however, it means that the liabilities must be split up and paired with two items of total assets. Also, the earlier asset must be received before the due date of the liability. As an observation, McCutcheon reaches the same conclusion about $f(x)$, however he states that $f'(x) > 0$ for $x < 0$ which is apparently a typographical error.
The following example is used to illustrate the application of 'full immunization':

Consider a $10,000, 20-pay whole life policy for a life age 30 with interest at 6%. Assume the premium now due has been received. This gives the following

\[ A = \frac{10,000 \cdot 20P_{30}}{M_{30}} = \frac{10,000 \cdot M_{30}}{N_{30} - N_{50}}. \]

Using the Illustrative Life Tables with interest at 6% from Actuarial Mathematics (Bowers et al, 1986) yields

\[ A = 10,000 \cdot \frac{1695.3711}{197839.26} = 85.69 \]

You have 19 premium (asset) receipts. Dividing the Total Liability of $10,000 gives you an individual liability, \( S \), of

\[ S = \frac{10,000}{19} = 526.32 \]

Now consider the item of liability linked to the premium-due at time \( r \) years. The two resulting equations are:

\[ Ae^{\delta' a} + Be^{-\delta' b} = S \]
\[ 85.69(1.06)^{20-r} + B(1.06)^{-b} = 526.32 \]

and

\[ Aae^{\delta' a} = Bbe^{-\delta' b}. \]
\[ 85.69(20-r)(1.06)^{20-r} = Bb(1.06)^{-b} \]

The values of \( B \) and \( b \) are the unknowns and are easily found.

\[ B = [526.32 - 85.69(1.06)^{20-r}](1.06)^{-b} \]

= the proceeds of a zero-coupon bond due at time 20+b years
A portfolio of zero-coupon bonds can be constructed so that any immediate change in interest is immunized against. The nineteen bonds would have the following amounts and maturities:

<table>
<thead>
<tr>
<th>Amount of zero-coupon bond ($)</th>
<th>Time to Maturity (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>38.45</td>
</tr>
<tr>
<td>2</td>
<td>35.63</td>
</tr>
<tr>
<td>3</td>
<td>33.27</td>
</tr>
<tr>
<td>4</td>
<td>31.28</td>
</tr>
<tr>
<td>5</td>
<td>29.60</td>
</tr>
<tr>
<td>6</td>
<td>28.16</td>
</tr>
<tr>
<td>7</td>
<td>26.92</td>
</tr>
<tr>
<td>8</td>
<td>25.85</td>
</tr>
<tr>
<td>9</td>
<td>24.92</td>
</tr>
<tr>
<td>10</td>
<td>24.12</td>
</tr>
<tr>
<td>11</td>
<td>23.41</td>
</tr>
<tr>
<td>12</td>
<td>22.80</td>
</tr>
<tr>
<td>13</td>
<td>22.27</td>
</tr>
<tr>
<td>14</td>
<td>21.80</td>
</tr>
<tr>
<td>15</td>
<td>21.39</td>
</tr>
<tr>
<td>16</td>
<td>21.03</td>
</tr>
<tr>
<td>17</td>
<td>20.72</td>
</tr>
<tr>
<td>18</td>
<td>20.45</td>
</tr>
<tr>
<td>19</td>
<td>20.21</td>
</tr>
</tbody>
</table>

Another example can be constructed that shows that if the interest rates change on a fully immunized fund, a profit will be earned: An investor, who must provide $1 million in ten years, possesses exactly enough cash to meet this debt if the current interest rate, \( \delta \), is maintained. It can be shown that, by purchasing suitable quantities of five-year and
fifteen-year zero-coupon bonds, the investor can be sure of making a profit on any immediate change in the interest rates.

Assume that the current interest rate is $\delta = 0.05$. From the given information $a = 5$, $b = 5$, and $t = 10$. The liability $S$ is $1,000,000$. Substituting into the appropriate equations

\[ Ae^{25} + Be^{-25} = 1,000,000 \]
\[ 5Ae^{25} = 5Be^{-25} \]

and solving for $A$ and $B$ gives

\[ A = 389,400 \]
\[ B = 642,013. \]

This means that the investor presently has $606,530. If the profit, or surplus, is defined as the asset minus the liability then,

\[ \text{Profit} = 389,400e^{-5\delta'} + 642,013e^{-15\delta'} - 1,000,000e^{-10\delta} \]

at time $t = 0$. If $\delta'$ remains at the original interest rate of 5%, then the profit is zero, however if $\delta'$ becomes 0.07, the profit earned will be $2,485. If the interest rate were to drop to 3%, then a profit of $3,707 is earned. If the interest rate changes at all, a profit is earned.

Although zero-coupon bonds were used in the example above, there are several different investment tools that the actuary has available to him or her to use to immunize the existing and future business of a particular line. His
alternatives include options and futures, interest rate swaps, put bonds, and mortgage-backed securities.

**Options and Futures**

The use of options and futures by the insurance industry has been strictly limited by state regulations which have providing a narrow definition of legitimate hedging techniques and which have restricted the percentage of assets that may be used for options and futures. In recent years these constraints have been eased, although many statutory accounting practices still exist (Tilley & Jacob, 1983:2).

Options and futures can be used in two ways to hedge a portfolio. By calculating hedge ratios, futures and options can be purchased that change the duration and convexity of fixed-income portfolios. The second use of futures and options is to cover temporary naked positions to support a temporary mismatch in assets and liabilities until the planned balance can be restored. Dr. Norman E. Mains, the First Vice President and Director of Research of Drexel Burnham Lambert, offers an excellent description of futures and options and their application to immunizing a fund in volume thirteen of the *Record Society of Actuaries* (Practical Aspects..., 1987:1252).
Interest Rate Swaps

Interest rate swaps can be defined as the "swapping" of payment streams between two parties. One party makes floating-rate payments while the other makes fixed rate payments. By swapping payments, the two companies are able to achieve the desired return scenarios they need. The floating-rate portion of the swap can be based on LIBOR, prime rates, bank CD rates, commercial paper rates, or T-bill rates (Tilley, 1983:29). LIBOR is the "rate paid in London on short-term dollar deposits from other banks, and is used as a base rate in international lending" (Tilley in Platt, 1986:250). Interest rate swaps first appeared in 1981 and have become quite popular since then.

Tilley also points out four reasons that swaps are more popular than futures:

1) Their financial structure appears easier to comprehend.

2) They are over-the-counter instruments and can be customized to the needs of the insurer.

3) They conform to hedging horizons better because they have longer maturities.

4) Their regulatory status is much more favorable as they are legal in almost all states.

(Tilley in Platt, 1986:250). Additional advantages include
retention of control over the principal investment, superior yield with less risk, and shorter time necessary to close the deal (Tilley, 1983:31).

Adjustable-Rate Debt Instruments

Adjustable-rate debts offer two features that fixed rate investments do not: "(1) the transformation of a 'portfolio rate' into a 'new-money rate,' and (2) price support near par" (Tilley, 1983:31). This makes them ideal for matching with traditional products. Adjustable-rates are created by shorting interest rate futures against a fixed rate instrument yielding a short-term earned rate. It allows for the investor to index the rate to some point several years in the future, and by updating the rate periodically to keep the return up to date (Tilley, 1983:31).

Coupon Stripping

Guaranteed Interest Contracts have prompted greater attention to coupon stripping as a method of hedging against interest rate risk. The basic thrust behind coupon stripping calls for the liabilities to be designed to match a particular asset, rather than the assets to be found to match an existing liability. This allows for a much better asset/liability matching than the standard duration analysis approach. The best application of coupon stripping is found
when the liabilities are created across product lines. For example, a GIC, deferred annuity, and whole life insurance product are all designed to match a particular asset together allowing the cash flows to be adjusted to one extreme or the other.

**Put Bonds**

Put bonds appeared in the market in 1984. They are corporate bonds that allow the investor to put the bonds back to the issuer at some point in the future for a specific strike price. They resemble over-the-counter put options, yet are not restricted by regulations (Tilley in Platt, 1986:254-55).

**Simulation Models**

Simulation models are very useful tools for asset/liability management. Their effectiveness is highlighted in two ways. Several investment scenarios contain so many fluctuating variables that evaluating the necessary equations can be very tedious and time consuming. Models, through their simplicity, can quickly evaluate these equations. This also allows the investment manager to view the impact of changes in the investment environment as well as changes in the business environment. Difficulties arise in simulation models when the interest scenarios are created.
There are two methods that are used to create these interest scenarios, either manual creation or through a random generation of numbers. By incorporating the interest scenario into the simulation model, actuaries can then generate model office or asset share projections. Model office projections are used to evaluate a writing of business over an extended period of time, whereas asset share projections are used for studying a single block of business written at a specific point in time.

Cash inflows for the model include money generated from the asset portfolio and premium payments by policyholders. Outflows are identified as "death benefits, cash surrenders, policy loans, commissions, investment and insurance expenses, and federal income taxes" (Tilley in Platt, 1987:243). The net cash flow, inflows minus outflows, is either invested or borrowed. In some instances, asset liquidation is used as a means of generating cash.

Through the use of simulation models the investment manager is able to create a duration-based strategy that is derived from either static or dynamic investment strategies. Static strategies are those methods in which the assets are divided among various instruments whose maturity dates are found on several points on the yield curve. Dynamic strategies, on the other hand, entail the placements of assets in such a way that timing and the shape of the yield curve are taken into account.
Management Involvement

The senior management of the insurance company has an important role in the optimization of the company's investments for asset/liability management. These decisions can be used at several different levels. They can be used to aid in the design, pricing, and investment strategy of new products or it can be applied to the existing lines of business. James A. Tilley and David Jacob point out three basic components of the financial strategy of a company. These are the investment strategy, the dividend strategy, and the tax strategy. The investment strategy is defined as how the cash flow surplus will be invested or divested. Controlling how much money will be paid to the stockholders and holders of participating policies is the dividend strategy. Finally, the tax strategy decides how much will be paid in federal and state taxes each year of the projection period (Tilley & Jacob, 1983:5-6).

In order to optimize the investment portfolio, Tilley and Jacob also offer six stages that should be progressed through. The first stage is deciding what should be optimized. These can include the net cash flow pattern, operating earnings, and surplus, statutory or GAAP earnings, and the time period of earnings which will be optimized (Tilley & Jacob, 1983:7).
Deciding what objective function will be used is the second stage. Tilley and Jacob offer the following choices:

(1) Maximize earnings for the "most likely" scenario; (2) Maximize "average" earnings over all scenarios; (3) Minimize the variability of earnings over the entire set of scenarios considered; (4) Optimize a combination of some of the above; or (5) Maximize earnings for the "worst" of the scenarios considered (Tilley & Jacob, 1983:7).

The third stage of the process is choosing the decision variables. These variables include the volume and mix of the assets to be purchased or liquidated, the amount of new business that will be solicited and how much of it will be renewed, and what forms of financing will be used.

The fourth stage consists of choosing the constraints on the optimization. Tilley and Jacob suggest the following:

(1) Maximum tolerable net cash flow imbalances by year (on a dynamic projection basis); (2) Maximum/minimum amounts to be invested in or liquidated from various asset categories; (3) Maximum net capital gain or loss from asset liquidations; (4) Maximum/minimum volume of new business and renewals per year by product line; and, (5) Maximum cumulative borrowing by year (Tilley & Jacob, 1983:8).
The fifth stage of optimization is solving the mathematical equations necessary to immunize the fund. The sixth and final step is then to evaluate the results for reasonability and practicality (Tilley & Jacob, 1983:8).

**Software Tools**

When an insurance company has decided to begin a policy of asset/liability management, the use of a software package is a logical decision because of the tedious and complex calculations necessary for the construction of an immunized portfolio. There are four choices left open to the company: 1) Build your own system; 2) Rent a system from a consulting firm; 3) Buy an existing system; or 4) Construct a system out of a combination of the previous three choices (Smith in "Software Tools...," 1987:1668). There are several software systems available in today's market that handle asset/liability management. An excellent comparative analysis can be found in Volume 13:3 of the *Record, Society of Actuaries*. Some of the features of each system will be highlighted in the next several paragraphs.

Bambrough and Associates Asset/Liability Matching System is designed to operate on the existing business of a company. Present values of projected cash flows are calculated as well as Macaulay durations. The system was created in 1983 for a major East Coast insurance company and then redeveloped and
sold to several insurance companies. It is currently being used by Western Life Insurance Company, Berkshire Life, Capital Holding Corporation, Connecticut General, Farm Family Life of Albany, Knights of Columbus, National Liberty, Phoenix Mutual, Union Labor Life and Unity Mutual. It can be used to calculate single cell scenarios using bonds, equities, and mortgages or a complete scenario can be generated (Bambrough in "Software Tools...," 1987:1668-9).

CALMS, marketed by Tillinghast/TPF&C, is a micro-computer based system that was designed to handle interest sensitive products as well as fixed cash flow products. Projects on the system are done on a quarter by quarter basis. The package allows for user-defined variables as well as providing a random interest rate scenario. The system can be used to aid in the design of future products as it allows the user to evaluate the impact of different product features such as no-cost policy loans, bail-out provisions, and surrender scales (Carr in "Software Tools...," 1670-1).

Another system available for purchase is PALLM PRO which works on an annual basis optimizing each year's performance and expected profits. It can also be used for product development. The PALLM PRO system is available in both a main frame and a PC version and is marketed by PALLM, Inc. (Deakins in "Software Tools...," 1987:1671).

Milliman and Robertson offers a system called PCAPS system primarily on a consulting basis, although some firms
have purchased the system. It is very similar to the PALM PRO system. The PCAPS system will support product development and pricing, financial pricing, and financial analysis of a company. It handles dynamic projects and has a huge database that interfaces between the user and the system (Stanley in "Software Tools...," 1987:1680).

Another PC system available in the market today is called PTS and is sold by Shane A. Chalke, Inc. This system features a pricing and profit projection model as well as a decision theory model. The decision theory model allows the user to take the results from several different scenarios and present their fixed-dollar equivalency in a three-dimensional graph highlighting the optimal combination of interest and investment strategies across many different scenarios (Chalke in "Software Tools...," 1987:1672-3). This system is used by several companies including AMEV Holding Company's insurance companies.

Morgan Stanley offers a option pricing model on a consulting basis that can be used to protect from interest rate risk. It has not been used for valuation work for assets, nor does it produce GAAP or statutory statements. It includes separate models for callable bonds, mortgage-backed securities and individual options and futures (Epstein in "Software Tools...," 1987:1673-4).

Hawley Actuarial Software markets a system, HAS AM88, that allows you to calculate the market value of your
investments. It will also make ten to forty year projections of your cash flows from bonds and other assets to determine the impact of interest rate risks. It is able to handle such investment techniques as investment swaps, and has features that provide optimization, and decision theory (Hawley in "Software Tools...," 1987:1676).

Sibigtroth & Consultants, Inc. offers a system, SIBCO FIT, that has evolved from several consulting contracts. It will create estimates of cash investments and financial expectations in new business. It will also estimate investment risk and can be used with the financial projections portion of the program. This portion handles the GAAP and cash flow statements. The consolidation module permits the user to take several different product lines and do a single analysis on them. The option composition analysis module lets the user choose several different investment opportunities and use them in one portfolio (Sibigtroth in "Software Tools...," 1987:1677-8).

DELPHI has been marketed by PolySystems since 1986 as an asset/liability tool. This system combines several actuarial tasks such as product development; regular, mutual and purchase GAAP; valuation; forecasting and experience analysis. The projections are done on a month by month basis. Projections can be made using assets such as bonds, stocks, mortgages, real estate, and cash. The system is flexible enough to allow the user to make many assumptions,

Instant Forecast is a relatively new entry in the asset/liability software market, and is touted as "radically different" by its producer, Forecast Consultants, Inc. It is available for the PC or main frame and is set up for on-line capability. Although it was not designed for daily investment decision making, it can be used by several people at once. In addition to interest rate risk it can create scenarios based on econometrics. The system is also available on a time-share basis for consulting rather than purchase (Stein in "Software Tools...," 1987:1681-2).

Conclusion

Although immunization has been slow to take hold in the investment management of the insurance industry, it has a very promising future. It appears that the largest hurdle facing immunization theory is simply the understanding of it. The general idea, matching assets to liabilities in order to lessen interest rate risk, is easy to grasp; however, the detailed understanding necessary to actually apply the theory is not so easily mastered.

In time, this hurdle will be cleared. Several articles and books have been written that attempt to explain the theory and how to apply it. Papers and seminars are
beginning to appear that deal specifically with the application of the theory. Three such examples include "Finding the Immunizing Investment for Insurance Liabilities: The Case of the SPDA," by Peter Noris and Sheldon Epstein; "A Case Study in Asset/Liability Management," moderated by Gregory D. Jacobs; and "A Practical Approach to Applying Immunization Theory," by A.D. Shedden. Each of these presentations attempt to make a difficult topic easily understood. Also, many college curriculums have been expanded to include a discussion of Redington's theory. All of these steps will slowly make immunization theory a more easily grasped concept.

It is hard to believe that a theory that had a birth as a result of a cold winter's day in England in 1952, could play such an important role in the 1980's and beyond. Fred Carr said "risk is your enemy" (Practical Aspects..., 1987:1249) at the Society of Actuaries meeting in 1987; however, Dale Wolf provides a better picture: "We must all believe that risk is our opportunity. If it really is the enemy, we would just try to eliminate it, and that obviously would not make any money" (Practical Aspects..., 1987:1250). Immunization is the tool in today's economic environment for manipulating risk.
REFERENCES


