INTEREST RATE FLUCTUATIONS:
WHY THEY OCCUR AND HOW TO MINIMIZE YOUR RISK

An Honors Thesis (HONRS 499)
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Purpose of Thesis

In our complex financial world, interest is a common thread that binds everything together. This thread of interest may also be the unravelling of your particular fabric if the mechanism controlling its fluctuations is not properly understood. This discussion of interest rates has two major thrusts: what causes these fluctuations to occur and what you can do to minimize your risk associated with these fluctuations. This discussion is of particular interest to actuaries and accountants since both of these must deal with certain and uncertain flows of money while valuing these flows in the present for decision making and financial reporting. For these purposes, the study of interest rates is essential to the life of the business.
Interest Rate Fluctuations:

Why They Occur and How to Minimize Your Associated Risk

Interest. We find interest everywhere—it permeates our everyday lives. We pay interest on credit cards and mortgages while, at the same time, we collect interest on bonds and on savings. What is interest and why is it needed? What is its purpose in our society? Some would say that interest is the price we must pay to use someone else’s money—like renting the money just as we might rent an apartment or house. Others refer to interest as a premium required to overcome an owner’s time preference of having the money available for use now. Interest may also be viewed as a capital expenditure necessary to achieve the goal of a borrower—to reproduce more capital.

With these definitions of interest and the rationale for its necessity, we can begin to take a closer look at the interest rates that exist in our society and try to answer some basic questions about these rates. What determines the interest rates we pay and receive? What are the mechanisms driving these interest rates? Why do these rates fluctuate almost daily? After exploring these questions, we come upon a bigger question, one of relatively more significance: namely, what can I (or my firm) do to protect myself against the ravages of fluctuating interest rates? Some practical theory concerning this subject will be studied as well as attempting to explain why interest rates fluctuate like they do. Tackling interest rates may seem like a formidable task, but by taking it one step at a time, we hope to be able to tame the wild beast we call interest rates.
According to basic economic theory, interest rates should be set by the interaction of supply and demand. A high supply and low demand should tend to drive interest rates down, while low supply and high demand would drive the rates up. Supply and demand theory sounds logical enough, but in the case of interest rates, this theory is not enough to explain the actual fluctuations we find. Kellison [7] gives this "list of major factors" that affect the current level of interest rates on pages 297 and 298:

1. The underlying "pure" rate of interest.
2. Inflation.
3. Risk and uncertainty.
4. Length of investment.
5. Quality of information.
6. Legal restrictions.

When these factors are added to the general supply and demand theory, the interest mechanism is made much more complex, indeed! While a short discussion of each of these eight factors follows, some are covered in greater detail due to their complexity or relative importance. For those factors that mathematical examples will assist the fuller comprehension, these examples are usually deferred to the end of that factor's discussion. Many of these examples are adaptations of examples or exercises that appear in Kellison's [7] book.

Of these factors, the first to be discussed is the "pure"
underlying rate of interest. Kellison [7] calls this the rate that could be found on an investment in the absence of all the other factors (no risk, no inflation, etc.). He contends that this rate has been relatively stable and, in the United States, has hovered around the 2% to 3% range. This "pure" underlying rate can be looked upon as the opportunity cost of holding or borrowing money. With regards to the definitions of interest stated earlier, this "pure" rate can be thought of as the basic rent due on a dwelling when location is not considered (certainly, an apartment or house located closer to the necessities would command a higher rent, just as borrowing money in a more dynamic market should command higher interest rates), the premium required to overcome the time preference of money, or the cost of attempting to accumulate capital. This underlying rate is the basis upon which the other factors will build--adding to or subtracting from this rate, their contribution to the overall rate.

The effect of inflation on interest rates is intuitively simple. The lender of the dollars will demand a higher interest rate in the face of inflation since the dollars he receives in return will have less purchasing power due to the inflation and the dollars are therefore worth less to the lender. The borrower, likewise, should be able to afford the higher interest rate since his repayment dollars will be "cheaper" to him. The inequity of this transaction lies in the fact that the interest rate used is based upon the expected rate of inflation, not the actual inflation rate since the loan will be repaid in the future and, of course,
the future rate of inflation is unknown. Furthermore, the lender and the borrower will probably expect different rates of inflation, leading to different expected interest rates.

This relationship can be expressed mathematically. We shall call the interest rate charged on the loan, or "nominal rate of interest," \( i \), the "real rate of interest" (the rate that would prevail without inflation) \( i' \), and the inflation rate \( r \). These three rates are related in the following formula:

\[
(1 + i) = (1 + i') (1 + r).
\]  
(Eq. 1)

Solving equation 1 for \( i \), we obtain

\[ i = i' + r + i'r. \]  
(Eq. 2)

This can be interpreted by noting that the "nominal" interest rate equals the "real" interest rate plus the inflation rate plus the product of the two. This last term is usually ignored due to its usual minuscule magnitude.

Equation 1 also yields the following interpretation of the "real" interest rate, \( i' \).

\[ i' = \frac{i - r}{1 + r}. \]  
(Eq. 3)

This equation implies that the "real" rate will remain fairly steady, as long as the "nominal" interest rate and inflation rate fluctuate similarly.

Next, we shall consider the effect of inflation on the present value and the accumulated value of an asset. Consider a stream of annual payments for \( n \) years where the first payment is \( S \) dollars, and all subsequent payments are increased with respect to
inflation. Using the symbols previously defined, the present value of such a stream of payments is

\[ S \left[ \left( \frac{1+r}{1+i} \right)^2 + \left( \frac{1+r}{1+i} \right)^3 + \cdots + \left( \frac{1+r}{1+i} \right)^n \right] = S \left( \frac{1 - \left( \frac{1+r}{1+i} \right)^n}{\frac{1}{1+i^r} - \frac{1}{1+r}} \right). \]  \hspace{1cm} (eq 4)

Substituting the results from equations 1 and 3, it can be shown that the present value of this stream of payments (as in equation 4) is equivalent to an n-year annuity certain computed at the "real" rate of interest i'.

\[ S \left[ \frac{1}{(1+i')^2} + \frac{1}{(1+i')^3} + \cdots + \frac{1}{(1+i')^n} \right] = S \left( \frac{1 - \left( \frac{1+i'}{1+i} \right)^n}{\frac{1}{1+i} - \frac{1}{1+i'}} \right). \]  \hspace{1cm} (eq 5)

Accumulation of value is similarly affected by inflation. Suppose an investor makes an investment of B dollars at an interest rate of i for n years. At the end of the n years, the investment is worth

\[ B \left( 1 + \frac{r}{i} \right)^n \]

dollars. Since inflation has affected the value of the dollar, the "real" value of the investment is

\[ B \left( \frac{1+i'}{1+i} \right)^n = B \left( 1 + \frac{i'}{i} \right)^n. \]  \hspace{1cm} (eq 6)

Example 1. Compute the present value of 15 annual payments where the first payment is $17,000 and subsequent payments are increased to reflect inflation. All payments will be made at the end of the year. Assume the rate of interest is 7% and the rate of inflation is 3%. By equation 4, we find the present value to be

\[ 17 \times \frac{1 - \left( \frac{0.03}{1.07} \right)^{15}}{-0.03 + 0.07} = 190,561.73 \]
Using equation 5 with the $i'$ resulting from equation 3, we obtain

$$i' = \frac{.07 - .03}{1.03} = .03835$$

$17,000 \cdot \frac{1}{.03835} = 17,000 \cdot \frac{1 - (1.03835)^{-15}}{.03835} = 170,561.73.

This example illustrates the equivalence of equations 4 and 5.

Another source of interest rate fluctuation is risk and uncertainty. This category includes risks and uncertainties such as default (bonds), prepayment, refinancing (loans), reinvestment rates, and uncertain redemption dates on callable bonds [7, pages 302-303]. The presence of risk or uncertainty forces a lender to demand a higher interest rate to compensate for the additional risk assumed.

How can we determine the present value of an uncertain stream of payments? If we know the amounts of the possible payments ($R_1, R_2, \ldots, R_n$) and we assume the probability that such payments will be made ($p_1, p_2, \ldots, p_n$), then the expected present value is

$$EPV = \sum_{t=1}^{n} R_t \frac{(1+i)^{-t}}{p_t}.$$  \hspace{1cm} (eq 7)

In this equation, $i$ is not the investment's stated interest rate (e.g. 6% on a 6% coupon bond), but rather, it is this stated rate plus a "risk premium" to reflect the amount and magnitude of the risk involved (e.g. a "risk premium" of 2.5% on the previously mentioned bond).

This formula is insufficient when we are unsure of what these
probabilities of payments should be. Sometimes these probabilities are readily available, other times they are not. If we assume the probability of payment to be a constant, $p$, over the life of the stream, and that each future payment is contingent upon the previous payments (i.e. for the $k^{th}$ payment to be made, all prior payments (at times 1 through $k-1$) must have been made), then equation 7 may be simplified to

$$E_{\overline{R}} = \sum_{t=1}^{n} R_{t} \left( \frac{p}{1+i} \right)^{t}.$$  \hfill (eq. 8)

The constant $p$ makes equation 8 less computationally intensive and, therefore, easier to apply than the awkward equation 7 from above. This simplification makes the computation of expected present values much easier and faster.

Example 2. On January 1, 1976, an investment of $25,000 in a business venture was made. On December 31 of each year, $3000 was returned. These proceeds were reinvested in an account paying 6% effective. On January 1, 1993, the business venture failed and the $25,000 was lost. What is the realized rate of return on the original investment? We can solve this problem by equating the accumulated value of the original investment at the unknown interest rate to the accumulated value of a 17-year annuity certain of $3000 payments at 6% interest.

\[ 25,000 \times (1+i)^{17} = 3000 \times \bar{s}_{17}, \nu = 4,638.64 \]
\[ (1+i)^{17} = 3.385546 \]
\[ i = 0.07437 \text{ or } 7.437\%. \]

As this example shows, the risk of the investment has a definite
impact on the realized interest rate. The realized rate is less than 25% of the reinvestment rate which was only half of the rate of return on the original investment.

Example 3. What is the expected value of a $1000 15-year bond with 6% annual coupons bought to yield 9% effective with the following assumptions? The probability of coupons being paid on time is assumed to be 98% (i.e. a 2% risk of default). The chance that the principal will be repaid at the end of the fifteenth year is 65%. The reason this risk is much greater is due to the magnitude of the premium repayment ($1000 as compared to the $60 annual coupons). Using equation 7 with the last coupon and the principal computed separately, we obtain

$$E_{PV} = \sum_{t=1}^{15} \left( \frac{.98}{1.09} \right)^t + \frac{1000 \cdot .05}{1.09^{15}}$$

$$= \$60 \left( \frac{1 - \left( \frac{.98}{1.09} \right)^{15}}{.01} \right) + \$1000 \cdot \frac{.05}{1.09^{15}} = \$652.44.$$ 

The risk of default obviously reduced the price an investor was willing to pay for this bond.

The length of an investment is the next factor affecting the interest rates that we will examine. In general, short-term and long-term interest rates are different. Usually long-term rates are higher than short-term rates for these three reasons: the time preference of money, the expectation that interest rates will rise, and the uncertainty of future inflation.

The time preference of money will push interest rates up for longer termed investments since most firms and individuals prefer
to invest for short periods of time to allow for greater access to their assets. To entice investors to commit their assets for longer periods, borrowers must offer higher interest rates. The difference in the short-term and long-term rates is the premium it takes to overcome this preference for liquidity.

The expectation factor gives higher long-term rates if there is a greater number of investors that believe that interest rates will rise than those that believe that rates will decline. The difference between the short-term and long-term rates can then be assumed to follow basic supply and demand theory.

The uncertainty of future inflation will tend to drive interest rates up as investors demand higher interest rates on long-term investments as a hedge against possible inflation. As time passes and the inflation rate becomes known, investors may be required to accept lower interest rates if their expectation of inflation was too high. This reduction of short-term rates is acceptable in lieu of less uncertainty in the future inflation rates.

A phrase used to describe the level of interest rates is the yield curve. The yield curve can be illustrated graphically by plotting the interest rate against the length of the investment. The specific interest rate for a particular length of investment is called a spot rate.

When long-term rates are greater than short-term rates, the yield curve has a positive slope. This is generally the case, but exceptions do exist. An "inverted yield curve" would be
encountered when current interest and inflation rates are high and are expected to drop dramatically (or at least significantly) in the future. A yield curve relatively more common than the inverted yield curve is the "flat yield curve." This might occur in a period of relatively stable markets and interest rates, and if investors do not expect any major changes in the markets, interest rates, or inflation rates.

It has been argued that to find the expected present value of a stream of payments, one should discount the payments by the associated spot rate from the yield curve. Using this idea, the expected present value would be

\[
EPV = \sum_{t=0}^{n} R_t (1 + i_t)^{-t},
\]

where \( i_t \) is the spot rate for an investment of length \( t \). Discounting by the associated spot rate should give a more accurate picture of the economic output that will be realized. This idea will be further expanded when duration is discussed.

Another related interest rate that we see in practice is called a forward rate. The forward rate is the expected value of a future spot rate. For example, a 3-year spot rate and a 1-year deferred 2-year spot rate are not the same. In this case, the forward rate is the expected 2-year spot rate that will be effective in one year.

These spot rates and forward rates have become the basis of our modern financial dealings. These rates allow us to analyze investment alternatives more realistically than we could have in
the past. For these reasons, the spot and forward rates are of great interest to our markets.

Example 4. A business firm has $1,000,000 to invest for five years. The balance may be reinvested at either the end of the second or third years (not both), or not at all. The following spot rates are currently effective: 2-year--8%, 3-year--8.25%, and 5-year--8.75%. The 2-year deferred 3-year spot rate is expected to be 9.3%, while the 3-year deferred 2-year spot rate is expected to be 9%. Assuming these forward rates hold true, what is the value of the investment if reinvestment occurs at the end of two years?

\[ \$ 1,000,000 \left(1.08\right)^2 \left(1.093\right)^3 = \$ 1,523,028.38. \]

At the end of three years?

\[ \$ 1,000,000 \left(1.0825\right)^3 \left(1.09\right)^2 = \$ 1,567,081.40. \]

Not at all?

\[ \$ 1,000,000 \left(1.0875\right)^5 = \$ 1,521,059.94. \]

In this example, the best return occurs if the reinvestment occurs after the second year (8.778%), but the others (not at all--8.75% and after the third year--8.549%) do not lag far behind.

The quality of information can affect the interest rate a lender is willing to accept and a borrower is willing to give. The information in question could be the probability of default of a bond, the expected inflation rate, the availability of other endeavors in which to invest, or one of a plethora of other
matters. It is elementary how the discrepancies in the quality (and timeliness) of this information could have an effect on the interest rates because a more informed party would have greater bargaining leverage. This has become less of a problem in our modern society due to the current abilities of global communications, sophisticated computers, as well as the quality and quantity of financial advice available to almost anyone that wants it. This is not trying to imply that inequities no longer exist in our markets (I am sure they do) but the chances of large scale inequity has certainly diminished.

Sometimes, interest rates are even regulated by the law. Some federal and state laws specify minimum or maximum interest rates. For example, it is illegal in the state of Indiana to charge more than 1.75% per month (21% annually, compounded monthly) on a revolving charge account such as a credit card account [1, 24-4.5-2-207]. A similar example of a maximum interest rate regulated by law is usury. Usury is defined as an "excessive" rate of interest and has been declared illegal. "Truth in Lending" laws have been enacted to deter the use of hidden charges and penalties that were sometimes used in the past to increase the interest rate collected on certain types of transactions. Although there has been a recent trend towards less governmental control, interest rates are still under some governmental regulation.

Very similar to the legal restrictions placed on interest rates are government policies that influence these interest rates. While these factors in interest rate fluctuations are similar in
the fact that they both stem from the government, the monetary and fiscal policies differ from the other laws in many ways. Monetary and fiscal policies are attempts to regulate the money supply and level of government expenditures while the other laws, such as usury, usually place limits on the nominal rate of interest. The monetary and fiscal policies in any particular time frame can have a drastic influence on the level of current interest rates.

For example, let us suppose that the Federal Reserve Board decides on an expansionary monetary policy--to increase the amount of money in circulation. The short-term effects of this one-time shift in the money stock are higher output, higher prices, and lower interest rates. This works to stimulate the economy in the short-term, but in the long-term it is no help. In the long-term, there is no change in output or in the interest rate, but prices will be proportionally higher to match the increase in the money stock [5]. Of course, a contractionary monetary policy would have the opposite effect. Monetary policy has no long-term effect on interest rates but it does in the short-term.

On the other hand, fiscal policy can affect interest rates in the long-term. Fiscal policy is the policy that sets the level of government purchases, taxes, and transfer payments. In an expansionary fiscal policy (e.g. a tax cut or a spending increase), the government is attempting to get more money into the hands of the people. In the short-term, this policy will lead to higher output, higher prices, and higher interest rates. However, in the long-term, output will return to its initial level while prices and
interest rates will rise even more [5]. As with monetary policy, contractionary fiscal policy will have the reverse effect.

Monetary and fiscal policies are very powerful tools for the economy and obviously affect the interest rate. A combination of these policies is used to get the desired mix of economic results, and realistic expectations of future interest rates will take the goals of these policies into account. From these relations, it is obvious that substantial governmental control of interest rates may result.

Of the major factors influencing interest rates, the only one we have not yet discussed is the possibility of random fluctuations. These are changes in the interest rates that have no apparent reason for existing. Kellison [7] deems this a worthy subject and he dedicates an entire chapter to the pursuit of this idea through stochastic processes. Random fluctuations seem to act as a catch-all; if a fluctuation occurs that is not caused by another factor, then it must have been caused by a random fluctuation.

Now that we have a basic understanding of where our interest rates come from and why they may fluctuate, there is only one item left to discuss before we begin to concentrate on the process used to minimize risk. This discussion centers on the debate of what interest rate to use for valuation of future payments and receipts. This subject is very important in present value analysis used in both the actuarial and accounting fields. These present value flows are often used in decision making and financial reporting.
To decide upon what rate to use for discounting these asset flows, we must study some fundamental questions that will direct us towards the right choice. Kellison [7] addresses these points on page 314.

1. How will inflation be recognized in our rate?

2. Should we use a risk-free or risk-adjusted rate?

3. Should our rate be based on the predominant yield curve, or will a level interest curve suit our purposes?

4. Should we base our rate on "best estimates" or be conservative? If conservative, how much so?

5. Should we apply a rate on individual transactions, or assume a common rate for an entire portfolio?

6. Should the rate be a "before tax" or an "after tax" rate? Keep in mind that taxes on interest income can severely diminish an investment's rate of return.

The following are five common rates that can be used based upon your particular needs. A brief description of each follows.

1. New transaction rate. The rate that would be effective on a new transaction of the same type. For borrowing, the rate payable for a new liability. For lending, the rate collectible on a newly invested asset.

2. Average transaction rate. The average rate being earned by similar transactions.

3. Settlement rate. The rate for a series of payments if it were bought or sold today.

4. Rate that associates assets and liabilities. The rate
considers related assets and liabilities, not just one by itself.

5. Specified rate. The rate is specified by an exterior entity such as legislation, regulation, the prime rate, etc.

We now have the tools and basic understanding of what drives our interest rates. We have considered the underlying factors and discussed the rates we may want to use in valuating future outlays of assets. With this information, we should be more able to control our own destiny, instead of allowing ourselves to fall into the clutches of the interest rates.

Knowing what we must now bear in mind when considering future flows of money, we will begin analyzing how an individual or firm can attempt to minimize the risk associated with fluctuating interest rates. We will use the techniques of duration, immunization, and some other ideas related to the matching of assets and liabilities.

Crucial to the process of minimizing risk of interest rate fluctuations is the timing of payments of a transaction. This timing can greatly affect the value of the transaction. One example of this is seen in a common home mortgage. By shifting the timing of the payments forward, one can greatly reduce the amount of interest paid on the loan.

Several methods of measuring the timing of future payments exist. The most basic of these is term to maturity. This is defined as the time until the financial obligation of the transaction has ended. For example, a 20-year mortgage has a term to maturity of 20 years and a 15-year bond has a 15 year term to
maturity at the time of issue. The problem with using term to maturity is that this method cannot distinguish between two obligations of the same length that have different interest rates. While it is clear that a 20-year mortgage at 9.5% apr is a transaction of greater value than a 20-year mortgage at 6.5% apr, the method of term to maturity would not distinguish between the two. This trait of term to maturity leads us to seek a better method of comparing the financial obligations of a transaction. Ideally, it should be one that considers the timing of the payments more realistically.

One method that fits the requirements is the method of equated time. This index of payment timing is computed as the weighted average of the time of payments using payment amounts for weights. Expressed mathematically, this is

\[
\bar{t} = \frac{\sum_{t=1}^{n} t \cdot R_t}{\sum_{t=1}^{n} R_t}
\]

\(\text{(eq.10)}\)

where we have a series of payments of \(R_1, R_2, \ldots, R_n\) made at times 1, 2, \ldots, \(n\) and \(\bar{t}\) is the value of the method of equated time. This may also be thought of as the average term to maturity.

Example 5. Consider a $10,000 15-year bond with 7% annual coupons. The average term to maturity of this bond would be

\[
\bar{t} = \frac{1 \cdot 700 + 2 \cdot 700 + \cdots + 14 \cdot 700 + 15 \cdot 700 + 15 \cdot 10,000}{700 + 700 + \cdots + 700 + 700 + 10,000} = 11.415 \text{ years}.
\]

This is less than the term to maturity discussed earlier because the method of equated time takes into account the fact that
payments are not just made at bond maturity, but at earlier times as well. The earlier payments lower the burden of the remaining financial obligation.

While the method of equated time is better than term to maturity for analyzing financial transactions, an even better method exists in duration. Duration is very similar to the method of equated time—the only difference in calculation is the weights being used. The method of equated time used payment amounts for the weights while duration uses the present value of those payment amounts. Using the stream of payments defined for equation 10, duration, called \( \bar{d} \), is defined as

\[
\bar{d} = \frac{\sum_{t=1}^{n} t v^t R_t}{\sum_{t=1}^{n} v^t R_t}.
\]

This could be called the average present value term to maturity.

Example 6. Using equation 11, calculate the duration for the bond in example 5 assuming the interest rate is 4%.

\[
\bar{d} = \frac{1(1.04)^{-700} + 2(1.04)^{-700} + \cdots + 14(1.04)^{-700} + 15(1.04)^{-700} + 15(1.04)^{-700}}{1(1.04)^{-700} + 2(1.04)^{-700} + \cdots + 14(1.04)^{-700} + 15(1.04)^{-700} + 15(1.04)^{-700}} = 10.490
\]

The duration is shorter than the other two measures we examined due to these facts: we value a dollar in the future as somewhat less than a dollar today, and we consider that the payments are made earlier.

These methods are fine if interest rates stay stable over time, but what happens to the present value of a transaction if interest rates change? How rapidly does the present value change?
Obviously, the answer is going to be different for nearly every transaction, so we must find a simple way to quantify this value. This is accomplished by what is known as volatility. Volatility is the rate of change in the present value as the interest rate changes. It is measured in units independent of the present value's size. If we let $PV(i)$ denote the present value of a stream of payments, then the volatility of that stream of payments is

$$V = - \frac{PV'(i)}{PV(i)} = \frac{\sum_{t=1}^{n} t \cdot v^{t-i} \cdot R_t}{\sum_{t=1}^{n} v^{t} \cdot R_t} = \frac{\overline{d}}{1+i}.$$  \hspace{1cm} (eq. 12)

Due to its close relationship to duration, volatility is also known as modified duration. With this definition, volatility may be called a measure of the instability of duration due to interest rate changes.

Example 7. Calculate the volatility of the bond used in example 6. From equation 12, we see that

$$V = \frac{\overline{d}}{1+i} = \frac{10.490 \text{ years}}{1.04} = 10.087 \text{ years}.$$

Volatility and duration both play an important role in immunization theory as well as in other types of financial analysis. These quantities provide an index of the average length of an investment which is very useful when assessing risk due to uncertain reinsurance rates.

Now we turn our attention to immunization theory. Where duration was merely concerned with analyzing an investment, immunization uses similar concepts applied to a full range of investments--specifically a portfolio of assets and liabilities.
Since liabilities are usually structured by the lending entity, the main focus of immunization is to help structure the assets to the most advantageous position we possibly can. Immunization is an attempt to achieve mathematical equilibrium between cash inflows and outflows.

Assume cash flows occur at times $1, 2, \ldots, n$. Cash inflows (from assets) are denoted $A_1, A_2, \ldots, A_n$ while cash outflows (from liabilities) are $L_1, L_2, \ldots, L_n$. The receipts, defined as $R_k=A_k-L_k$, are then the same type of series as defined in the discussion of duration (similar to above). Armed with these definitions, we can begin to attack the problem.

Since assets and liabilities are defined as cash inflows and outflows, all transactions of an entity are classified as one of the two. With this the case, the present values of these assets and liabilities will be assumed to be zero.

The present value function, $PV(i)$, will have a local minimum at $i$ if the first derivative is zero and the second derivative is greater than zero. If we have a local minimum, then a small change in the interest rate will result in an increase in the present value of the receipts. The direction of the interest rate change makes no difference! If this result is achievable, it is highly desired; it could even be called our ultimate goal in immunization.

We use the second derivative of $PV(i)$ to define another property of a transaction (or a portfolio of them), convexity.

$$\tilde{C} = \frac{PV''(i)}{PV'(i)}.$$

(eq.13)
Convexity can be seen as a rate of change of the volatility caused by an interest rate change, or as a measure of a portfolio's propensity to affect the present value of the net receipts when the underlying interest rate changes. The size of the present value itself does not matter.

Immunization of a portfolio of assets and liabilities will be achieved if we can meet the following three criteria. The present values of the assets must equal the present values of the liabilities to assure that all financial obligations will be met. The modified durations of the combined assets and of the combined liabilities must be equal to assure that the values have similar sensitivities to interest rate fluctuations. The convexity of the assets must exceed that of the liabilities to insure an increase in present value of receipts if interest rates change. A decrease in interest rates will lead the present value of the assets to increase more than liabilities while an increase in the interest rates will cause both present values to fall, but assets will not fall as much, also leading to a better financial position.

If these three criteria are consistently met, then our portfolio should be successfully immunized, or protected, against changes in the interest rate. However, immunization has some limitations and drawbacks and these should certainly be kept in mind when performing this process.

Immunization is an active process, not a static one. We cannot perform the immunization technique and claim that we are free from interest risk; rather, we should continually (or at least
periodically) monitor our portfolio and our immunization strategy because of the limitations and drawbacks of this theory. One such problem is that the interest rate that should be used in valuating the portfolios is rarely certain. Great care should be taken in selecting an accurate, reasonable interest rate since significantly different strategies may result from using different rates.

The process of immunization is designed to inoculate an entity against small changes in the interest rate; large changes may produce effects quite different from the intended results. Along with this, the theory described here assumes simultaneous shift of all relevant interest rates (i.e. the entire yield curve), but this often is not the case. This fact is evident if you follow the interest markets; you will undoubtedly notice that short-term rates are much more volatile than long-term rates.

The duration of the portfolios can also have adverse effects on implementation of immunization. Knowing that duration does not decrease by one year for every passing year, it is obvious that frequent rebalancing, or at least checking, of portfolio duration is required to remain properly immunized. Also, assets of the duration desired may not be available, leaving you in the position of being inaccurately or only partially immunized. Uncertain cash flow amounts may also contribute to the problems duration can cause for immunization.

Immunization is far from an exact science when you consider all of the associated uncertainties, but when you immunize an entity’s portfolios, the risk of adverse effects from interest rate
fluctuations is diminished which partially achieves the desired results.

Example 8. C owes D $10,600 at the end of one year and must set up a fund of investments to meet this obligation. C has two choices of investments available: a money market fund currently earning 6% and two-year zero coupon bonds earning 6%. Assume 6% effective interest rate for all calculations and set up an investment program. Also find the volatility and convexity of the assets and liabilities.

Solution. Let x be the amount invested in the money market fund and y be the amount invested in the bonds.

\[
P_V(i) = x + 1.1236 y (1 + i)^2 - 10,600 (1 + i)^{-1}.
\]

\[
P'_V(i) = -2.2472 y (1 + i)^{-3} + 10,600 (1 + i)^{-2}.
\]

\[
P''_V(i) = 6.7416 y (1 + i)^{-4} - 21,200 (1 + i)^{-3}.
\]

To find the amounts that will lead to the best strategy, the present value function and its derivative must both equal zero. Using the 6% interest rate, we can then solve for x and y.

\[
P_V(0.06) = x + y - 10,600 = 0.
\]

\[
P'_V(0.06) = -2.2472 y (1.06)^{-3} + 10,600 (1.06)^{-2} = \frac{25}{1.06} + \frac{10,600}{1.06} = 0.
\]

From these, \(x = 5000\) and \(y = 5000\). A local minimum exists at \(i = 6\%\) if the second derivative of the present value function is positive.

\[
P''_V(0.06) = \frac{6.7416 (5/1.06^2)}{1.06^4} - \frac{21,200}{1.06^3} = 8899.9644 > 0.
\]

Through this result, we are certain that we are at a local minimum and small interest rate changes will only increase the present
value of our portfolio.

Volatility and convexity of assets:

\[
\nu = -\frac{PV'(i)}{PV(i)} = -\frac{-2.2472 \times (1.050)(1.050)^{-3}}{10,000 + 1.023(1.050)(1.050)^{-2}} = 0.9434.
\]

\[
\bar{C} = \frac{PV''(i)}{PV(i)} = \frac{\frac{6416(1.050)(1.050)^{-4}}{10,000 + 1.023(1.050)(1.050)^{-2}}}{2.700} = 2.700.
\]

Volatility and convexity of liability:

\[
\nu = -\frac{PV'(i)}{PV(i)} = -\frac{-10,600(1.050)^{-2}}{10,600(1.050)^{-1}} = 0.9434.
\]

\[
\bar{C} = \frac{PV''(i)}{PV(i)} = \frac{21,200(1.050)^{-3}}{10,600(1.050)^{-1}} = 1.7800.
\]

These results are exactly as we want. The volatilities are equal meaning that for a small universal change in the interest rate, the duration of the assets and of the liabilities will both change, but by the same amount and, therefore, remain equal. Asset convexity is greater than liability convexity assuring us that the present value of the net receipts will increase as previously outlined if there is any change in the interest rate.

Although immunization is a very useful tool in minimizing investment risks, there are other methods. One such method is dedication. In dedication, an entity would structure assets to perfectly match the liabilities which would eliminate all risk. Some problems, not unlike those encountered in immunization, do exist. Uncertain cash flows that lead to unmatched obligations is one. Reinvestment risk, especially on long term assets, is another. Dedication's propensity towards a lower rate of return
than other portfolios that are not entirely matched is a third significant risk. These problems may make dedication undesirable or even impossible.

For our last topic of discussion, we will require extensive use of an example. For this purpose, let us consider a bank which is offering a two-year certificate of deposit at a guaranteed 5% interest rate. Funds may be withdrawn at the end of the first or second year without penalty. The bank has a choice of investments to place the deposits in: one-year notes yielding 5% effective or two-year notes yielding 5.25% effective. The main question centers upon how much of the total should the bank invest in each? Keep in mind that if interest rates are higher after one year, depositors will tend to withdraw their money to invest at the higher rate. If interest rates are down, depositors will tend to leave their deposits alone to keep the guaranteed rate. Trying to reduce the risk of interest rate fluctuations in such a condition can be very difficult, especially since you are still trying to make a profit.

Let \( s_1 \) and \( s_2 \) be the amount of funds withdrawn by the depositors at the end of years 1 and 2, respectively. Since \( s_1 \) and \( s_2 \) are amounts withdrawn, \( s_1 + s_2 > 1 \). In fact, \( 1.05 < s_1 + s_2 < (1.05)^2 \). By equating the present values of deposits and withdrawals, we get

\[
1 = (1.05)^1 s_1 + (1.05)^2 s_2. \tag{eq.14}
\]

Let \( p_1 \) and \( p_2 \) be the proportion of the deposited funds invested in the one-year and two-year notes. Since these are proportions, \( p_1 + p_2 = 1 \). If \( f \) is the forward rate on one-year notes at the end of the first year (for reinvestment) and \( A_2 \) is the
accumulated value of the funds at time 2, then
\[ A_2 = \left[ p_1 (1.05) - S_1 \right] (1+f) + \left[ p_2 (1.0525)^2 - S_2 \right]. \] (eq.15)

Substituting for \( S_2 \) and \( p_2 \), we obtain
\[ A_2 = \left[ p_1 (1.05) - S_1 \right] (1+f) + \left[ (1-p_1) (1.0525)^2 - (1.05)^2 + (1.05) S_1 \right] \]
\[ = \left[ (1.05)(1+f) - (1.0525)^2 \right] p_1 + S_1 (1.05^2 - (1.05)^2). \] (eq.16)

To determine \( p_1 \) (and, therefore, \( p_2 \)), we must make some assumptions for values of \( f \) and \( s_1 \).

We will first consider the case of interest rates rising. Suppose that \( f \) is 5.75%. High rates of withdrawal will occur and we will assume \( s_1 \) to be 85 cents per original dollar deposited. We then find \( A_2 = 0.00261875 p_1 - 0.00111875 \). Since we want \( A_2 > 0 \), we require that \( p_1 > 0.4272 \).

Next consider the case of lower interest rates. Suppose \( f \) is 4.5%. Low rates of withdrawal are likely and we will assume \( s_1 \) to be 10 cents per original dollar deposited. Using these values, we obtain \( A_2 = -0.01575625 p_1 + 0.00625625 \). Keeping in mind our desire for profits, we require \( p_1 < 0.5479 \).

With the assumptions we made, the bank should choose a value of \( 0.4272 < p_1 < 0.5479 \). Since \( p_2 = 1-p_1 \), the allocation of deposited funds is complete.

This technique is very sensitive to the choices of the forward rate and withdrawal amount. If these quantities can be forecasted with a fair degree of certainty, then this method of risk reduction can be quite successful. When this is not the case, use of this method can be dangerous indeed.

This completes our very basic study of the concepts of
interest rate fluctuations and how to minimize the risk associated with these changes. We looked at what the interest rate is and what causes it to change. Inflation, risk and uncertainty, and length of investment seemed to be the factors that contributed the most to the changes in the nominal interest rate. We discussed choosing an interest rate for financial valuation and noted the importance of these valuations to the accountants and actuaries. We then studied practical methods of reducing interest rate risk. This formidable task was made less complicated with the concepts of duration, volatility, and convexity. While this was merely a scratch of the surface of these subjects (many books have been written on these subjects), hopefully a greater insight was gained into the world of interest rates. Armed with these new tools, conquering interest rates means more than just monetary gain, it means success.
Bibliography


