COMBINATORIAL LEGENDRIAN KNOT INVARIANTS: REPRESENTATION NUMBERS

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Abstract

THESIS: Combinatorial Legendrian Knot Invariants: Representation Numbers

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The study of Legendrian knots lies within the larger fields of contact geometry and knot theory. The requirements for Legendrian invariance is strictly stronger than its topological analog, as there are Legendrian knots that are not Legendrian isotopic, but are isotopic as topological knots. As with topological knot theory, the classification problem, i.e. classify all knots up to Legendrian isotopy, is still a main problem in Legendrian knot theory.

We consider Legendrians lying within the standard contact structure $(\mathbb{R}^3, \xi_{std})$. One of the most powerful Legendrian knot invariants is a differential graded algebra, $(\mathcal{A}, \partial)$, introduced by Chekanov and Eliashberg. It has been shown that representation numbers, a normalized count of representations from $(\mathcal{A}, \partial)$, are a Legendrian knot invariant. This project addresses the Chekanov-Eliashberg differential graded algebra and representation numbers, and provides a definition for the 1-graded 2-colored ruling polynomials $R^1_{2,K}(q)$. We then show that $R^1_{2,K}(q)$ recovers the 1-graded total 2-dimensional representation number.

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Contents

Chapter 1

Legendrian Knots and Classical Invariants

We will assume the reader has a working understanding of algebra, some introductory knot theory, and rudimentary combinatorics. For those unfamiliar with these subjects we suggest [10] and [14]. We also adopt standard notations from these texts. Any additional information or notations will be introduced as we need it. All knots will presumed to be smooth and lying in the standard contact structure (see 1.1).

Knot theory has developed into a fruitful field of mathematical research ever since the 19th century. The field is not exclusive to only mathematicians, as many notable scientists (e.g. Lord Kelvin and Peter Tait) have been driven to the subject by applications and have contributed to the development of knot theory. For more information on the history of knot theory and its applications to DNA, physics and quantum computing see [2], [10]

1.1 Legendrian Knots

The study of Legendrian knots can be viewed as a subfield of topological knot theory and contact geometry. We follow the conventions of [5], [11], and [16].

Definition 1.1.1. The **standard contact structure** on $\mathbb{R}^3$ (in Cartesian coordinates) is the non-integrable plane field given by

$$\xi_{std} = \text{Span}\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right\}$$

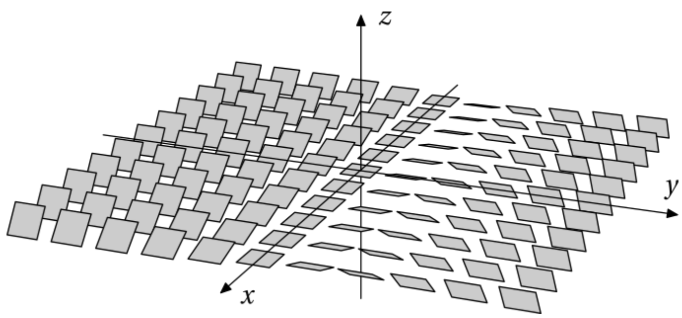


Figure 1.1: The Standard Contact Structure

Equivalently $\xi_{std} = \ker(dz - y\ dx)$. The non-integrability condition, is that $\xi_{std} \wedge d\xi_{std} \neq 0$ everywhere on $\mathbb{R}^3$. This implies that there is no surface that is everywhere tangent to ξ_{std}. More generally, one can define a contact structure on an $(2n+1)$-manifold M as the kernel of a (locally) defined 1-form α with $\alpha \wedge (d\alpha)^n$ nowhere zero [8],[16]. In the field of contact geometry on a 3-manifold, it is well known that any contact structure is locally isomorphic to the standard contact structure on $\mathbb{R}^3$ (Darboux's Theorem) [5].

Definition 1.1.2. (Legendrian Knot) A knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is said to be **Legendrian** if K is always tangent to the standard contact structure i.e. $K'(t) \in \xi_{std}(K(t))$ for all t.

1.2 Projections

As in usual knot theory, many invariants are defined using projections (which we require to produce generic diagrams) for the knot K. This is also the case for Legendrian knots. There are two primary projection types that we review. The front projection and the Lagrangian projection.

The **Lagrangian projection** is obtained from the mapping $\pi_{xy} : (x, y, z) \mapsto (x, y)$. We will now always assume that all knots are generic with respect to this projection, in the sense that the only singularities that appear in $\pi_{xy}(K)$ are precisely a finite number of transverse double points. We encode the information of the crossing diagrammatically as in the case of smooth knots. It is standard to use the term "Reeb chords" for the crossings in $\pi_{xy}(K)$ in the place of the term "crossing". This is in fact a slight abuse of terminology, since Reeb chords are the vertical chords obtained by connecting the lower strand of a crossing to its upper strand. One nice result associated with xy-projections of a Legendrian is that one can recover the original embedding of K up to vertical translation by $z(t) = \int_{t_0}^{t} y(s)x'(s)\, ds$, whereas in the case of smooth knots one can only recover an embedding that is ambient isotopic to the original projection [16].

The **front projection** of a Legendrian K is obtained analogously by projecting the knot to the xz-plane. The resulting diagram we appropriately denote by $\pi_{xz}(K)$. The contact condition imposed on K allows one to recover the y-coordinate of K by examining the slope of the resulting projection $y = \frac{dz}{dx}$. Therefore we do not encode the crossings of K in such a way. Notice that since the contact planes are never vertical, $\pi_{xz}(K)$ contains no vertical tangents. In place of vertical tangencies we have semi-cubical *cusps*.

There is a very standard way to go from a front diagram of K to an xy-diagram for a knot that is Legendrian isotopic to K. The procedure amounts to smoothing out the left cusps in $\pi_{xz}(K)$, encoding the downward sloping strand in $\pi_{xz}(K)$ as the overstrand of the resulting crossing in $\pi_{xy}(K)$, and introducing a right-handed twist at right cusps. This procedure is called the Ng-resolution and is illustrated in Figure 1.2 [13]. This yields an xy-diagram that is planar isotopic to $\pi_{xy}(K)$.

Remark. There are analogs of the Riedemiester type I, II, and III moves for front diagrams for K. However, there is no analog of a type I Riedemiester move for Lagrangian projections.

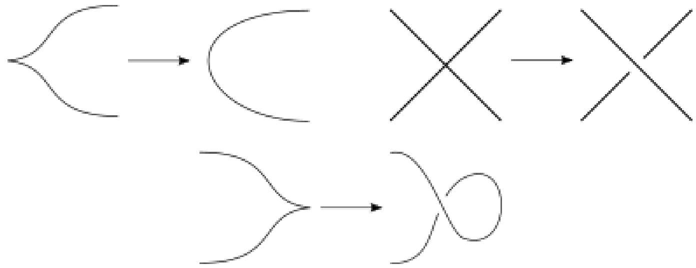


Figure 1.2: The Ng-resolution Procedure.

1.3 Some Classical Invariants

It is clear that if $K_1 \cong K_2$ as Legendrian knots then $K_1 \cong K_2$ as topological knots since the contact structure can be ignored for topological knots[1]. Legendrian knot invariants can be classified into two categories: classical invariants, and combinatorial invariants. The classical invariants are easily defined.

Definition 1.3.1. The **Thurston-Bennequin number** of a Legendrian K is defined by $tb(K) = w(K) - c_{\rightarrow}(K)$ where $w(K)$ is the writhe of $\pi_{xz}(K)$ and $c_{\rightarrow}(K)$ is the number of right cusps appearing in the front of K.

Definition 1.3.2. The **rotation number**, for an oriented Legendrian K, is $r(K) = \frac{1}{2}(c_{\downarrow} - c_{\uparrow})$ where $c_{\downarrow}$ (resp. $c_{\uparrow}$) is the number of downward (resp. upward) cusps appearing in $\pi_{xz}(K)$ as depicted in Figure 1.3.

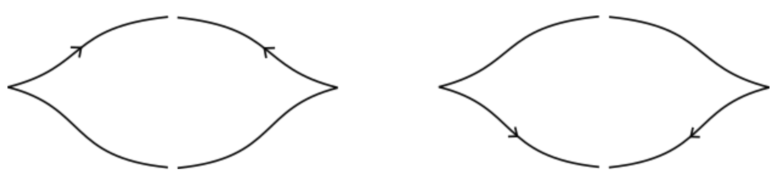


Figure 1.3: Upward cusps (left) and downward cusps (right).

These classical invariants are extremely powerful for certain Legendrian knot isotopy classes. In particular Legendrian isotopy classes for unknots (cf. [4]), torus knots, and

[1] The converse of the previous statement is not true. There are Legendrian knots that are isotopic as smooth knots, but are not Legendrian isotopic. The most famous example arises from Chekanov's *two* $m(5_2)$ knots.

figure-8 knots (cf. [6]), are completely classified by these two classical invariants. Such knot types that are classified by these classical invariants are called *Legendrian simple*.

Chapter 2

The Chekanov-Eliashberg DGA

2.1 Differential Graded Algebras

For the remainder of this text we shall stray away from the classical Legendrian invariants and introduce a "combinatorial" invariant. Perhaps the most powerful such invariant is the Chekanov-Eliashberg differential graded algebra (DGA). The algebra of K can be defined from a front projection or xy-projection of K. We adopt the latter convention and introduce the DGA following the sign conventions of [11] in Section 2.2.

We introduce differential graded algebras in general. Let R be a commutative ring. We say A is an **associative R-algebra** if A is an associative ring, A is an R-module, and the multiplication operation satisfies $r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$ for all $r \in R$ and $x, y \in A$. An algebra is said to be G-**graded** if $A = \bigoplus_{j \in G} A_j$ such that for all i, j in the index monoid[1] G, $A_i A_j \subseteq A_{i+j}$. The monoids we consider are abelian groups of the form $\mathbb{Z}/m\mathbb{Z}$ where $m \geq 0$. For the remainder of this document we adopt the notation $\mathbb{Z}/m := \mathbb{Z}/m\mathbb{Z}$.

A **differential $\mathbb{Z}/m$-graded algebra** $(\mathcal{A}, \partial)$ is a graded algebra equipped with a linear $\partial : \mathcal{A}_i \to \mathcal{A}_{i-1}$ that satisfies $\partial^2 = 0$, giving $(\mathcal{A}, \partial)$ the structure of a chain complex, and obeys the signed Liebnez rule: $\partial(xy) = \partial(x)y + (-1)^{|x|}x\partial(y)$, where $|x|$ is the degree of x in the grading.

[1]Recall: A monoid is a semigroup with identity [14].

2.1.1 Maslov Potential

Let $C(K)$ denote the set of cusp points appearing in $\pi_{xz}(K)$. A **Maslov potential** for K is a function $\mu : \pi_{xz}(K) \setminus C(K) \to \mathbb{Z}/2r(K)$ that is locally constant and increments by $+1$ as we travel up through a cusp as depicted in Figure 2.1. Maslov potentials are often used for grading purposes.

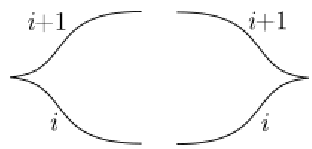


Figure 2.1: The behavior of Maslov potentials near cusps.

2.2 Chekanov-Eliashberg DGA

Convention. All of the DGA's discussed henceforth are unitial and graded by $\mathbb{Z}/m$ ($m > 0$). In addition, we assume the coefficient ring has characteristic 2 in the case that m is odd [11].

Let K be an oriented Legendrian knot. Further, let $\{a_1, a_2, \ldots, a_n\}$ be the set of transverse double points appearing in $\pi_{xy}(K)$, and t be a basepoint on K. The algebra is the unitial associative algebra $\mathcal{A}(K) = \mathbb{Z}\langle a_1, a_2, \ldots, a_n, t^{\pm 1}\rangle$ with non-invertible generators $a_1, a_2, \ldots, a_n$ and an invertible generator $t^{\pm 1}$. The only relation is that $t^{-1} \cdot t = t \cdot t^{-1} = 1$. Elements of $\mathcal{A}(K)$ are $\mathbb{Z}$-linear combinations of noncommutative words in $a_1, a_2, \ldots, a_n, t^{\pm 1}$. Assigning some $(\mathbb{Z}/2r(K))$-valued Maslov potential leads to a $\mathbb{Z}/2r(K)$ grading on $\mathcal{A}(K)$.

Definition 2.2.1. For each a_i generator appearing as a double point in $\pi_{xy}(K)$ we define the degree of a_i to be

$$|a_i| := \begin{cases} \mu(U_i) - \mu(L_i) & \text{if } z|_{U_i} - z|_{L_i} \text{ has a max at } x_i \\ \mu(U_i) - \mu(L_i) - 1 & \text{if } z|_{U_i} - z|_{L_i} \text{ has a min at } x_i \end{cases}$$

where U_i (resp. L_i) is the upper (resp. lower) strand of a_i and x_i is the x-coordinate of a_i,

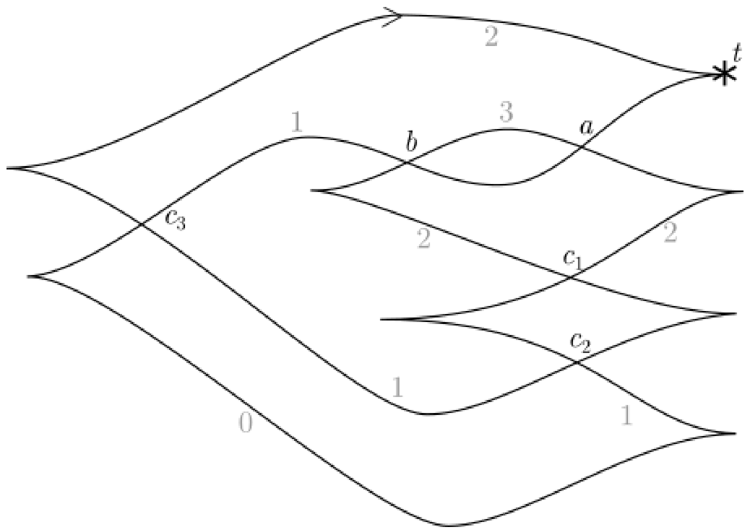


Figure 2.2: A front for a $m(5_2)$ knot equipped with a $\mathbb{Z}/2r(K)$-valued Maslov potential indicated in gray. Degree distribution: $|b| = -2$, $|a| = 2$, and $|c_i| = 0$ for $i = 1, 2, 3$.

see Figure 2.2. The degree of the t generator we define to be $|t| = 0$. For any generators r, s we compute $|rs| = |r| + |s|$.

In particular,

$$\mathcal{A}(K) = \bigoplus_{k \in \mathbb{Z}/2r(K)} \mathcal{A}_k$$

where $\mathcal{A}_k = \{a \in \mathcal{A}(K) : |a| = k \mod 2r(K)\}$.

Remark. Whenever $\pi_{xy}(K)$ is obtained from Ng's resolution the degree of each a_i generator is $\mu(U_i) - \mu(L_i)$.

The differential for the Chekanov-Eliashberg DGA is obtained via a signed count of admissable disks. Formally, let $\mathcal{M}(a; b_1, b_2, \dots, b_j)$ be the set of orientation preserving immersed boundary punctured disks (up to reparametrization) in $\mathbb{R}^2$ whose boundary lies on $\pi_{xy}(K)$ and such that the images of neighborhoods of the boundry punctures cover a positive convex corner at a and negative corners at $b_1, b_2, \dots, b_j$ appearing in a counterclockwise order initiating from the overstrand at a to the understrand. (See Figure 2.3). Then

$$\partial a := \sum_{k \geq 0} \sum_{b_1, b_2, \dots, b_k} \sum_{[u] \in \mathcal{M}(a; b_1, b_2, \dots, b_k)} \iota(u) \cdot w(u).$$

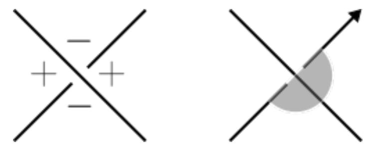


Figure 2.3: Reeb decorations (left) and orientation signs (right) where the shaded region indicates a negative orientation sign

Here $\iota(u) = \iota' \iota_0 \iota_1 \ldots \iota_k$ where $\iota' = +1$ if the orientation of the initial arc (the part of the overpassing strand at the positive corner) of the boundary of u agrees with the orientation of K, and $\iota' = -1$ if the orientation disagrees. The ι_j are the "orientation signs" of the quadrants covered by the disk u depicted in Figure 2.3. For each u, $w(u)$ is the product of the generators corresponding to negative corners of u and intersections of u with the basepoint as they appear counter-clockwise from a, where at a basepoint we assign the generator t if the orientation of K agrees with that of u, otherwise we assign t^{-1} to the basepoint.

2.3 Example

We admit the definition of the Chekanov-Eliashberg DGA (especially the differential) is rather complicated so we illustrate an example. Let us consider the $m(5_2)$ in Figure 2.4. The grading computations carry over from Figure 2.2. Note that the additional e_i generators all have degree 1.

Next we compute the differential on the generators. It is clear that the differential on the c_i generators is 0. An exhaustion argument can easily show that there are no admissible disks with the positive corner at a or b. We provide such an argument for the case of a positive corner at b to illustrate a standard way to construct admissible disks. Traveling on the overstrand of b, the first vertex we encounter is c_3 (resp.a) in a positive quadrant so we must pass through it. The next vertex we come to is e_4 (resp. e_1), again in a positive quadrant. After passing through the first two vertices, we see that it is no longer possible for the region to be bounded, and so we see that there are no admissible disks with the positive

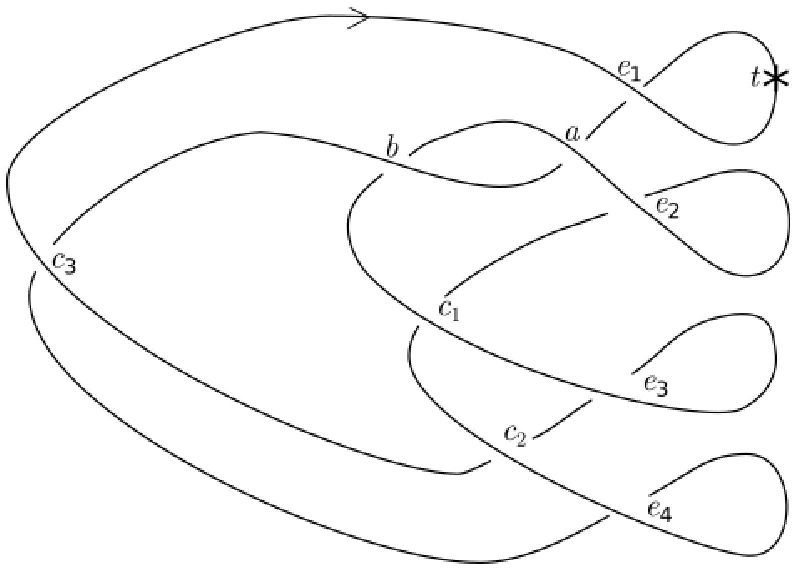


Figure 2.4: The knot in Figure 2.2 after the Ng-resolution

corner at b. We now compute the differential for the other generators as

$$\partial e_1 = t - c_3 - c_3 ba$$

$$\partial e_2 = 1 + abc_1 + c_1$$

$$\partial e_3 = 1 + c_1 c_2$$

$$\partial e_4 = 1 + c_2 c_3$$

where the negative signs appearing in ∂e_1 is a result of the ι' factor.

Chapter 3

Representation Numbers

The goal of this next section is to introduce a fairly new invariant for Legendrian knots.

3.1 Representation Numbers

Representation numbers were introduced in [11]. We introduce them following their conventions: Suppose $(\mathcal{A}, \partial)$ is a $\mathbb{Z}$-graded DGA and $(\mathcal{B}, \delta)$ is a $\mathbb{Z}/M$ graded DGA. For $m|M$ let

$$\mathcal{B}_k^m = \bigoplus_{\substack{l \in \mathbb{Z}/M \\ l = k \mod m}} \mathcal{B}_l$$

for each $k \in \mathbb{Z}$. That is, $\mathcal{B}_k^m$ is the algebra $\mathcal{B}$ with grading collapsed mod m.

Definition 3.1.1. An m**-graded DGA representation** $f : (\mathcal{A}, \partial) \to (\mathcal{B}, \delta)$ is a unital algebra homomorphism that satisfying the representation equation, $f \circ \partial = \delta \circ f$, and the the grading relation, $f(\mathcal{A}_k) \subseteq \mathcal{B}_k^m$.

In order to define m-graded representation numbers there are specific requirements that are imposed on the knot K and the algebras $(\mathcal{A}, \partial)$ and $(\mathcal{B}, \delta)$. Let K be an oriented Legendrian knot equipped with a single basepoint[1]. If m is even we impose that $r(K) = 0$,

[1]In [11] they use a collection of basepoints.

and that K is equipped with a $\mathbb{Z}$-valued Maslov potential and the degree of the invertible basepoint t is 0. On the other hand, when m is odd we impose that μ is $\mathbb{Z}/m$ valued and $2m\big||t|$. In the case when m is odd, K is equipped with an additional $\mathbb{Z}$-valued Maslov potential $\tilde{\mu}$ which is also discontinuous at t that is equivalent to μ when collapsed mod $2m$. The DGA $(\mathcal{B},\delta)$ we require to be finite, with $|\mathcal{B}^m_k| = |\mathcal{B}^m_{-k}|$. In addition we choose a subset $\mathbf{T} \subseteq (\mathcal{B}^m_0)^*$ to send the invertible t generator.

Definition 3.1.2 (shifted Euler characteristic)**.** Let r_n be the number of Reeb chords that have degree n. Then the **shifted Euler characteristic** of K centered at $k \in \mathbb{Z}$ is defined by

$$\chi^k = \sum_{l\geq 0}(-1)^l r_{l+k} + \sum_{l<0}(-1)^{l+1} r_{k+l}$$

Definition 3.1.3. Let $(\mathcal{B},\delta)$ be a DGA with the requirements prescribed above. Then the m**-graded representation number** for K into $(\mathcal{B},\delta)$ is

$$Rep_m(K,(\mathcal{B},\delta),\mathbf{T}) = \left(\lim_{N\to\infty} \prod_{\substack{k\in\mathbb{Z}\\|k|\leq N}} |\mathcal{B}^m_k|^{-\chi^k/2}\right) \cdot |\ker\delta \cap (\mathcal{B}^m_0)^*|^{-1} \cdot |\overline{Rep}_m(K,(\mathcal{B},\delta),\mathbf{T})|$$

where $\overline{Rep}_m(K,(\mathcal{B},\delta),\mathbf{T})$ is the set of all m-graded representations from the Chekanov-Eliashberg DGA $(\mathcal{A},\partial)$ into (B,δ) mapping t into $\mathbf{T}$.

In addition when (V,d) is a differential $\mathbb{Z}$-graded vector space with differential d having degree $+1$ mod m, we denote $Rep_m(K,(V,d),\mathbf{T}) = Rep_m(K,(-End(V),\delta),\mathbf{T})$. Here $-End(V)$ denotes the set of endomorphisms of V where the grading is the negative of its standard grading (i.e $|T| = k \mod m$ if $T(V_j) \subseteq V_{j-k}$ for all $j \in \mathbb{Z}$) and is then collapsed mod m, that is

$$(-End(V))_k = \bigoplus_{j\in\mathbb{Z}} Hom(V_{j+k},V_j) \qquad -End(V) = \bigoplus_{k\in\mathbb{Z}/m} (-End(V))^m_k$$

The differential δ is then defined by $\delta(T) = d\circ T - (-1)^{|T|}T\circ d$ (graded commutator) and we assume that V is a vector space over a field of characteristic 2 when m is odd.

Definition 3.1.4. The m**-graded reduced representation number** is

$$\widetilde{Rep}_m(K, (\mathcal{B}, \delta), \mathbf{T}) = |\mathcal{B}_0^m|^{-1/2} \cdot |(\mathcal{B}_0^m)^* \cap \ker \delta| \cdot Rep_m(K, (\mathcal{B}, \delta), \mathbf{T})$$

Definition 3.1.5. The m**-graded total** n**-dimensional representation number** is

$$Rep_m(K, \mathbb{F}_q^n) := Rep_m(K, (\mathbb{F}_q^n, 0), GL(n, \mathbb{F}_q))$$

3.2 $Rep_2(m(5_2), \mathbb{F}_q^3)$

We now compute the 2-graded total 3-dimensional representation number for $m(5_2)$ knot given in Figure 2.4. For this we record a result from [11] which states that there is a bijection $|\overline{Rep_2}(m(5_2), \mathbb{F}_q^n)| \leftrightarrow \{(A, B) \in \mathrm{Mat}(n, \mathbb{F}_q) \times \mathrm{Mat}(n, \mathbb{F}_q) : E_{-1}(AB) = \{0\}\}$ where $E_{-1}(AB)$ denotes the (-1)-eigenspace of AB (a detailed proof of this is provided in [11]). In addition, we use the q-analog of the binomial coefficient adopting a similar notation as in [12].

Definition 3.2.1. The q**-binomial coefficient** is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q) \dots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \dots (q^k - q^{k-1})}$$

Proposition 3.2.1 ([12])**.** *The number of k-dimensional subspaces of $\mathbb{F}_q^n$ is given by $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Remark. We appropriately denote the set of k-dimensional linear subspaces of $\mathbb{F}_q^n$ by the grassmannian $Gr(k, \mathbb{F}_q^n)$. Also note that $|GL(n, \mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$ (The proof is straightforward and can be found in [12]).

Proposition 3.2.2. *The total number of 2-graded representations for* $m(5_2)$ *on* $(\mathbb{F}_q^3, 0)$ *is given by*

$$|\overline{Rep}_2(m(5_2), \mathbb{F}_q^3)| = q^{18} - q^{17} - q^{16} + 2q^{14} + q^{13} - 2q^{12} - q^{11} + q^{10} + q^9 - q^7 + q^6$$

Proof. Let $X_i = |\{C \in \mathrm{Mat}(3, \mathbb{F}_q) : \mathrm{rank}(C) = i \text{ and } E_{-1}(C) = \{0\}\}|$ and Y_i be the number of ways to factor a given $C \in \mathrm{Mat}(3, \mathbb{F}_q)$ having rank i into a product of matrices $AB = C$. Then, $|\overline{\mathrm{Rep}_2}(m(5_2), \mathbb{F}_q^n)| = \sum_{i=1}^3 X_i Y_i$. We now compute the X_i's and Y_i's.

X_3: Clearly, $X_3 = |GL(3, \mathbb{F}_q)| - \sum_{i=1}^{3} |W_i|$ where $W_i = \{D \in GL(3, \mathbb{F}_q) : \dim E_{-1}(D) = i\}$. W_3 is just the singleton $\{-I\}$, so $|W_3| = 1$. Now for each $P \in Gr(2, \mathbb{F}_q^3)$ fix a complementary vector v_P such that $P \oplus \mathrm{Span}\{V_P\} = \mathbb{F}_q^3$. There is a bijection $W_2 \leftrightarrow \{(P, \alpha) \in Gr(2, \mathbb{F}_q^3) \times \mathbb{F}_q^3 : \alpha \notin P \text{ and } \alpha \neq -v_P\}$ obtained from mapping the pair (P, α) to $D \in W_2$ so that $E_{-1}(D) = P$ and $D(v_P) = \alpha$. There are $|Gr(2, \mathbb{F}_q^3)|$ choices for P and $q^3 - q^2 - 1$ choices for α given P. Hence, $|W_2| = |Gr(2, \mathbb{F}_q^3)| \cdot (q^3 - q^2 - 1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q (q^3 - q^2 - 1)$.

To compute $|W_1|$ it is useful to fix two complementary vectors v_ℓ and w_ℓ for each line $\ell \in Gr(1, \mathbb{F}_q^3)$ such that $\ell \oplus \mathrm{Span}\{v_\ell\} \oplus \mathrm{Span}\{w_\ell\} = \mathbb{F}_q^3$. We now show that there is a bijection between W_1 and $\{(\ell, \alpha, \beta) \in Gr(1, \mathbb{F}_q^3) \times \mathbb{F}_q^3 \times \mathbb{F}_q^3 : \alpha \notin \ell, \beta \notin \mathrm{Span}(\ell \cup \{\alpha\}), \text{ and } \alpha + v_\ell, \beta + w_\ell \text{ are linearly independent}\}$. To see this, map (ℓ, α, β) to the matrix D such that $D|_\ell = -\mathrm{id}_\ell, D(v_\ell) = \alpha$, and $D(w_\ell) = \beta$. We claim that the conditions on α and β are necessary and sufficient for $\dim E_{-1}(D) = 1$. Indeed, $\dim E_{-1}(D) \geq 2$ iff there exists c_1, c_2 not both zero such that $c_1 v_\ell + c_2 w_\ell \in E_{-1}(D)$. Therefore, $\dim E_{-1}(D) = 1$ exactly when $c_1 v_\ell + c_2 w_\ell \in E_{-1}(D)$ only for $c_1 = c_2 = 0$. Consequently, $\dim E_{-1}(D) = 1$ exactly when $c_1(\alpha + v_\ell) + c_2(\beta + w_\ell) = 0$ only for $c_1 = c_2 = 0$. The claim now follows by definition.

Now observe that $|W_1| = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q (A_1 B_1 + A_2 B_2)$ where the subscripts denote the cases when $\mathrm{Span}\{\alpha + v_\ell\} \subset \mathrm{Span}(\ell \cup \{\alpha\})$ and $\mathrm{Span}\{\alpha + v_\ell\} \not\subset \mathrm{Span}(\ell \cup \{\alpha\})$ respectively, and A_i , B_i correspond to the number of choices for α and β respectively for case i.

Since $v_\ell \notin \ell$, then $\mathrm{Span}\{\alpha + v_\ell\} \subset \mathrm{Span}(\ell \cup \{\alpha\})$ iff $v_\ell = c\alpha + x$ (i.e. $\alpha = \frac{1}{c}(v_\ell - x)$) for some $x \in \ell$. Note here that $c \neq 0$ since $v_\ell \notin l$. There are $q-1$ choices for c and q choices for x and the only case that is not permitted is when $c = -1$ and $x = 0$. Thus $A_1 = (q-1)q - 1$. Futhermore since there are only $q^3 - q - 1$ total choices for α, $A_2 = q^3 - q - 1 - (q-1)q + 1$. Lastly, notice B_1 is the number of vectors not in $\mathrm{Span}(\ell \cup \{\alpha\})$ and avoiding the affine line $\{x - w_\ell : x \in \mathrm{Span}\{\alpha + v_\ell\}\}$ so $B_1 = q^3 - q^2 - q$. Similarly, $B_2 = q^3 - q^2 - q + 1$. Putting this together and simplifying one has $X_3 = q^9 - 2q^8 - q^7 + 2q^6 + 2q^5 + q^4 - 4q^3$.

Y_3: Choose any $A \in GL(3, \mathbb{F}_q)$, then B must be given by $A^{-1}C$ in order for $C = AB$. Hence $Y_3 = |GL(3, \mathbb{F}_q)|$.

X_2: Let R_2 be the set of rank 2 matrices with entries in $\mathbb{F}_q$ and $K_i = \{C \in R_2 : \dim E_{-1}(C) = i\}$. Then $X_2 = R_2 - |K_2| - |K_1|$. There is a bijection $K_2 \leftrightarrow \{(P, \alpha) : P \in Gr(2, \mathbb{F}_q^3) \text{ and } \alpha \in P\}$. The bijection arises from the map $(P, \alpha) \mapsto D$ where D in K_2 such that $P = E_{-1}(D) = \mathrm{im}\, A$ and $\alpha = Dv_P$ where v_P is as before. There are $\begin{bmatrix}3\\2\end{bmatrix}_q$ such 2-dimensional subspaces, and for each of these q^2 choices for α, so $|K_2| = \begin{bmatrix}3\\2\end{bmatrix}_q q^2$.

In order to compute $|K_1|$, for each line $\ell \subset P$ with $P \in Gr(2, \mathbb{F}_q^3)$ fix complementary vectors $v_{\ell,P} \in P \setminus \ell$ and $w_{\ell,P} \notin P$. Note $P = \mathrm{Span}\{\ell \cup \{v_{\ell,P}\}\}$ and $\mathbb{F}_q^3 = \mathrm{Span}\{\ell \cup \{v_{\ell,P}\} \cup \{w_{\ell,P}\}\}$. There is a bijection between K_1 and

$$\{(P, \ell, \alpha, \beta) \colon P = \mathrm{Span}(\ell \cup \{\alpha\} \cup \{\beta\}) \in Gr(2, \mathbb{F}_q^3), \ell \in \mathbb{P}^2, \ell \subset P, \alpha, \beta \in P, \alpha \neq -v_{\ell,P}\}.$$

The mapping $(P, \ell, \alpha, \beta) \mapsto C$ such that $\mathrm{im}\, C = P$, $C|_\ell = -\mathrm{id}_\ell$, $Cv_{\ell,P} = \alpha$ and $Cw_{\ell,P} = \beta$ gives the required bijection. Given P and ℓ, we consider the case when $\alpha \notin \ell$, and $\alpha \in \ell$. In the former $\beta \in P$ can be chosen arbitrarily, which results in a total of $(q^2 - q - 1)q^2$ choices for α and β. For the latter, choose $\beta \in P \setminus \ell$ and conclude there are $q(q^2 - q)$ choices for α and β. Therefore, $|K_1| = \begin{bmatrix}3\\2\end{bmatrix}_q \begin{bmatrix}2\\1\end{bmatrix}_q ((q^2 - q - 1)q^2 - q(q^2 - q))$. Lastly $|R_2| = \begin{bmatrix}3\\2\end{bmatrix}_q (q^3 - 1)(q^3 - q)$. Therefore,

$X_2 = \begin{bmatrix}3\\2\end{bmatrix}_q((q^3-1)(q^3-q) - \begin{bmatrix}2\\1\end{bmatrix}_q((q^2-q-1)q^2 + q(q^2-q)) - q^2).$

Y_2: There are three disjoint cases to consider for factoring a given rank 2 matrix as a product $C = AB$.

Case 1: $A \in GL(3, \mathbb{F}_q), B \in R_2$

Case 2: $A \in R_2, B \in GL(3, \mathbb{F}_q)$

Case 3: $A, B \in R_2$.

Cases 1 and 2 are similar: the rank 2 matrix is uniquely determined by the invertible matrix appearing in the factorization (i.e. if $B \in GL(3, \mathbb{F}_q)$, $A = CB^{-1}$) and so the first two cases contribute $2|GL(3, \mathbb{F}_q)|$ to the total count. For the last case, $\ker C = \ker B$ since $\ker B \subset \ker C$ and $\dim \ker B = \dim \ker C = 1$. Let $\ell = \ker C$. Then the matrix B is uniquely determined by Bv_ℓ and Bw_ℓ. There are $|\mathbb{F}_q^3 \backslash \{0\}| = q^3 - 1$ choices for Bv_ℓ and $|\mathbb{F}_q^3 \setminus \mathrm{Span}\{Bv_\ell\}| = q^3 - q$ choices for Bw_ℓ. Now for each matrix B, the matrix A must be chosen such that $A(Bv_\ell) = Cv_\ell$, $A(Bw_\ell) = Cw_\ell$ and $\dim \ker A = 1$. No non-trivial linear combination of Bv_ℓ, Bw_ℓ can be in the kernel of A otherwise $\dim \ker A > 1$. Thus there are $\begin{bmatrix}3\\1\end{bmatrix}_q - \begin{bmatrix}2\\1\end{bmatrix}_q = q^2$ choices for A. Therefore, $Y_2 = 2|GL(3, \mathbb{F}_q)| + q^2(q^3-1)(q^3-q)$.

X_1: Note that $X_1 = |R_1| - |W_{1,1}|$ where R_1 denotes the set of rank 1 matrices with entries in $\mathbb{F}_q$ and $W_{1,1} = \{C \in R_1 : \dim E_{-1}(C) = 1\}$. It is standard that $|R_1| = \begin{bmatrix}3\\1\end{bmatrix}_q(q^3-1)$. Next observe that there is a bijection $W_{1,1} \leftrightarrow \{(\ell, \alpha, \beta); \ell \in Gr(1, \mathbb{F}_q^3), \alpha, \beta \in \ell\}$. This bijection is arises from mapping $C \mapsto (\ell, \alpha, \beta)$ where $\ell = \mathrm{im}\, C = E_{-1}(C)$, $C(v_\ell) = \alpha$, and $Cw_\ell = \beta$. It follows that $|W_{1,1}| = \begin{bmatrix}3\\1\end{bmatrix}_q q^2$ so $X_1 = \begin{bmatrix}3\\1\end{bmatrix}_q(q^3 - q^2 - 1)$.

Y_1: There are 6 disjoint cases to consider when counting the number of ways to factor a given rank 1 matrix as a product $C = AB$.

Case 1: $A \in GL(3, \mathbb{F}_q)$, $B \in R_1$: This is similar to cases 1 and 2 in the computation of Y_2: The rank 1 matrix B is uniquely determined by the invertible matrix A. Case 1 contributes $|GL(3, \mathbb{F}_q)|$ to Y_1.

Case 2: $A \in R_1$, $B \in GL(3, \mathbb{F}_q)$: This is similar to Case 1, and contributes $|GL(3, \mathbb{F}_q)|$ to Y_1.

Case 3: $A, B \in R_2$. Let $P = \ker C$ ($P \in Gr(2, \mathbb{F}_q^3)$ and note that $\ker B \subsetneq P$. Now for each 1-dimensional $\ell \subset P$ fix complementary vectors $v_{\ell,P} \in P \setminus \ell$ and $w_{\ell,P} \notin P$. The number of acceptable matrices for B is $\begin{bmatrix}2\\1\end{bmatrix}_q (q^3-1)(q^3-q)$ where the first factor is the number of choices for $\ell = \ker B$ and the remaining factors are the number of choices for $Bv_{\ell,P}$ and $Bw_{\ell,P}$. Now for any given B, A must bechosen such that $ABv_{\ell,P} = Cv_{\ell,P}$ and $\ker A = \text{Span}\{Bv_{\ell,P}\}$. Fix a vector k_B such that $\text{Span}\{k_B, Bv_{\ell,P}, Bw_{\ell,P}\} = \mathbb{F}_q^3$. The matrix A is determined by its action on basis vectors $Bv_{\ell,P}$, $Bw_{\ell,P}$ and k_B. Since $A \in R_2$, we see that A is uniquely determined by Ak_B (since $ABv_{\ell,P} = 0$ and $ABw_{\ell,P} = Cw_{\ell,P}$). Since there are $q^3 - q$ choices for $Aw_{\ell,P}$, we see that case 3 accounts for $\begin{bmatrix}2\\1\end{bmatrix}_q (q^3-1)(q^3-q)^2$ additional factorizations.

Case 4: $A \in R_2$, $B \in R_1$. Observe $\ker B = \ker C = P$. Fix a complementary vector v_P such that $\text{Span}\{P \cup \{v_P\}\} = \mathbb{F}_q^3$. Now, B is uniquely determined (with the given kernel P) by Bv_P. There are $q^3 - 1$ possibilities for Bv_P, and therefore $q^3 - 1$ possible matrices for B. For each $\ell' \neq \text{im}\, B = \ell$, fix r'_ℓ such that $\text{Span}\{\ell \cup \ell' \cup \{r'_\ell\}\} = \mathbb{F}_q^3$. Here ℓ' is our choice of the kernel of A. The only such restriction on $\ker A$ is that it is in $Gr(1, \mathbb{F}_q^3)$ and $\ker A \neq \text{Span}\{Bv_P\} = \text{im}\, B$. Hence there are $\begin{bmatrix}3\\1\end{bmatrix}_q - 1$ choices for $\ker A$. Furthermore A must send Bv_P to Cv_P and send r'_ℓ to any vector avoiding the span of Cv_P. Therefore case 4 has a contribution of $(q^3-1)(q^2+q)(q^3-q)$.

Case 5: $A \in R_1$, $B \in R_2$: This has the same count as case 4. To see this, just notice that the number of ways to factor a given matrix C as $C = AB$ is the same as factoring its transpose $C^T = (AB)^T = B^T A^T$. Now $B^T \in R_2$ and $A^T \in R_1$ and so one just applies case 4.

Case 6: $A, B \in R_1$. As seen in case 4, then number of possible matrices for B is $q^3 - 1$. The matrix A is then chosen such that $ABv_P = Cv_P$ (where v_P is the same as in case 4) and so A has 2-dimensional kernel. Notice $\ker A$ cannot contain $\operatorname{im} B = \operatorname{Span}\{Bv_{\ell,P}\}$. The number of choices for the $\ker A$ is $\frac{(q^3-q)(q^3-q^2)}{(q^2-q)(q^2-1)}$. The numerator is the number of choices for $\ker A$ basis vectors and the denominator corresponds to the number of choices for the basis of a given 2-dimensional subspace. So there are $(q^3-1)q^2$ factorizations for case 6.

Hence $Y_1 = 2|GL(3,\mathbb{F}_q)| + \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]_q (q^3-1)(q^3-q)^2 + 2(q^3-1)(q^2+q)(q^3-q) + q^2(q^3-1)$.

X_0: This is trivial: $X_0 = 1$.

Y_0: We divide the collection (A, B) such that $AB = \mathbf{0}$ into disjoint sets,

$$T_i = \{(A, B) : AB = \mathbf{0}, \operatorname{rank}(A) = i\}.$$

Clearly $|T_0| = q^9$ since $A = \mathbf{0}$, and B can then be any matrix in $\operatorname{Mat}(3, \mathbb{F}_q)$. For any $A \in R_1$, B is chosen such that $\operatorname{im}(B) \subset \ker A$. Each column of B can be an arbitrary vector in the $\ker A$ so $|T_1| = |R_1| \cdot (q^2)^3$. Similarily for $|T_2|$ for each $A \in R_2$, B is chosen such that each column is a vector in $\ker A$. So $|T_2| = |R_2|q^3$. Finally whenever $A \in GL(3,\mathbb{F}_q)$, B must be $\mathbf{0}$ so $|T_3| = |GL(3,\mathbb{F}_q)$. Conclude $Y_0 = |T_0 \sqcup T_1 \sqcup T_2 \sqcup T_3| = q^9 + \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]_q (q^3-1)q^6 + \left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]_q (q^3-1)(q^3-q)q^3 + (q^3-1)(q^3-q)(q^3-q^2)$.

Combining all of this and simplifying one obtains

$$|\overline{Rep}_2(m(5_2), \mathbb{F}_q^3)| = q^{18} - q^{17} - q^{16} + 2q^{14} + q^{13} - 2q^{12} - q^{11} + q^{10} + q^9 - q^7 + q^6.$$

□

Proposition 3.2.3. *The total 3-dimensional representation number for $m(5_2)$ is*

$$q^{-9/2}[(q^3-1)(q^3-q)(q^3-q^2)]^{-1}(q^{18} - q^{17} - q^{16} + 2q^{14} + q^{13} - 2q^{12} - q^{11} + q^{10} + q^9 - q^7 + q^6).$$

Proof. In this case $(\mathcal{B}, \delta) = (-End(\mathbb{F}_q^3), 0)$. Therefore, $|\mathcal{B}_k^2| = \begin{cases} q^9 & k \equiv 0 \mod 2 \\ 1 & \text{otherwise} \end{cases}$. In addition, the degree distribution for $m(5_2)$ is $r_{-2} = 1$, $r_0 = 3$, $r_1 = 4$ $r_2 = 1$ and $r_j = 0$ for all other values of j. Hence,

$$\lim_{N\to\infty} \prod_{\substack{k\in\mathbb{Z} \\ |k|\leq N}} |\mathcal{B}_0^2|^{-(\chi^k/2)} = (q^9)^{-(\chi^{-2}+\chi^0+\chi^2)/2} = (q^9)^{-(1-1+1)/2} = q^{-9/2} \tag{3.1}$$

Now since $(\mathbb{F}_q^3, 0)$ is endowed with $\delta = 0$, and the knot has exactly one basepoint,

$$|(\mathcal{B}_0^m)^* \cap \ker\delta|^{-l} = |(\mathcal{B}_0^2)^*| = |GL(3, \mathbb{F}_q)|^{-1} = [(q^3-1)(q^3-q)(q^3-q^2)]^{-1} \tag{3.2}$$

The result now follows by Definition 3.1.3, Proposition 3.2.2, and Equations 3.1 and 3.2. □

Remark. There is a convenient way to compute $Rep_2(K, \mathbb{F}_q^n)$ when $r(K) = 0$ by the specialization $P_{n,K}(a,q)|_{a^{-1}=0} = Rep_2(K, \mathbb{F}_q^n)$ where $P_{n,K}(a,q)$ is the n-colored HOMFLY-PT polynomial. (see [11]). Using the HOMFLY-PT implementation in *CoCalc* we see that the representation number in Proposition 3.2.3 coincides with this specialization. We will briefly discuss the 1-graded analog of this connection at the end of Chapter 4.

Chapter 4

Representations of a Satellite

The goal of this section is to review key results from [11] and provide a partial solution to an open problem: What is the "correct" definition of 1-graded n-colored ruling polynomial, i.e. the definition that will recover the 1-graded total n-dimensional representation number?

4.1 The Satellite Construction

Let K be a Legendrian knot equipped with a basepoint $*$ and let $L \subset J^1S^1$ be an oriented Legendrian *link*. For our purposes the 1-jet J^1S^1 can be viewed as $S^1 \times \mathbb{R}^2$. Projections are constructed by slicing the torus at $x = 0$. We count strands by the intersection of L with $x = 0$. The xz and xy diagrams differ by applying Ng's resolution procedure and by concatenation with a dip (as seen in Figure 4.1) on the right in the xy-projection. The xy-satellite of K with pattern L denoted by $S(K, L)$ is then constructed diagrammatically as follows:

1. Take the xy- diagram of L (with the dip). Assign basepoints $*_1, *_2, \ldots, *_n$ to each strand of L to the left of all crossings in L. Where the labeling of $*_i$ corresponds to the i^{th}-strand of L (numbered in decreasing y- coordinates).

2. We place $\pi_{xy}(L)$ in a neighborhood of $*$ and form the n-copy of K using the blackboard

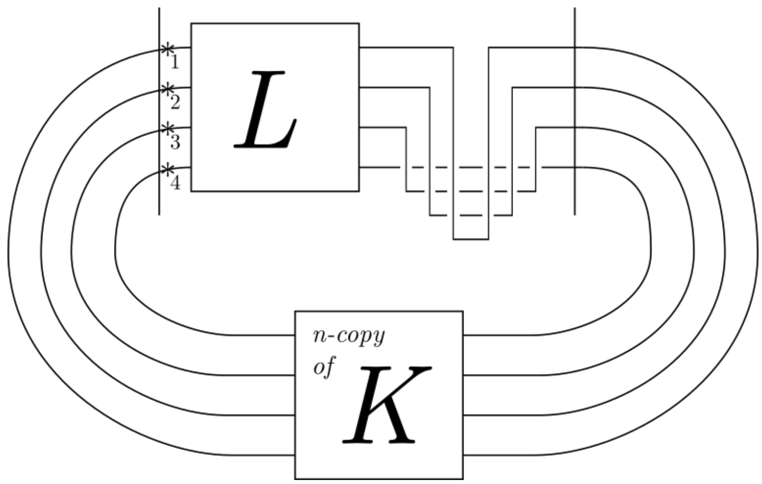


Figure 4.1: The satellite construction of K with pattern L

framing. The resulting link is then oriented using the orientation of L.

The n-copy of K, as the name suggests, is n copies of K that (outside of $\pi_{xy}(L)$) run parallel (because of the blackboard framing) to each other. Here n corresponds to the number of stands in L.

Convention. In the case of a positive permutation braid (a braid in which each pair of strands crosses at most once) β, the orientation of $S(K, \beta)$ is inherited by an orientation on K.

For each permutation in the symmetric group S_n fix a positive permutation braid (with reduced braid word) Then,

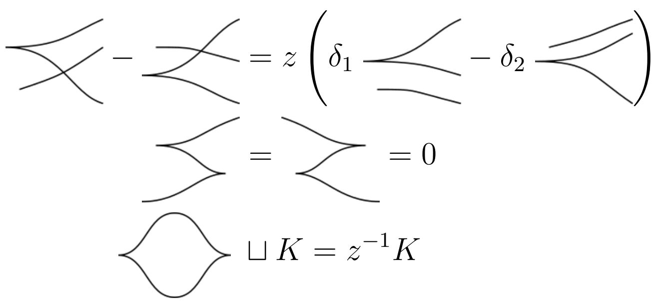
$$GL(n, \mathbb{F}) = \bigsqcup_{\beta \in S_n} B_\beta \tag{4.1}$$

where B_β is the path subset of β as presented in [11].

Remark. This decomposition was also shown [11, Proposition 4.14] to be the Bruhat decomposition of $GL(n, \mathbb{F}_q)$ into double cosets. We suggest reviewing [11, Definitions 4.2 and 4.7] for a precise and illustrative definition of how these path subsets are constructed.

4.2 Colored Ruling Polynomials

Definition 4.2.1. The m-**graded ruling polynomial of** K is the polynomial $R^m_K(z)$ determined by the skein relations:



With the normalization that $R^m_{\diamond}(z) = z^{-1}$, and δ_1 (resp. δ_2) is $+1$ if in the first (resp. second) diagram the strands have equal Maslov potential mod m, otherwise δ_1 (resp. δ_2) is 0.

Theorem 4.2.1. *(Leverson-Rutherford) Let $\beta \subset J^1S^1$ be a positive permutation braid, then*

$$R^m_{S(K,\beta)}(z)\big|_{z=q^{1/2}-q^{-1/2}} = q^{-\lambda_m(\beta)/2}(q^{1/2}-q^{-1/2})^{-n}\sum_d \widetilde{Rep}_m(K,(V_\beta,d),B^m_\beta)$$

where the sum is over all strictly upper triangular differentials d.

For an n-stranded positive permutation braid, V_β is the $\mathbb{Z}$-graded $\mathbb{F}$-vector space $V_\beta = \mathrm{Span}_{\mathbb{F}}\{e_i : 1 \leq i \leq n\}$ where the grading is determined by a choice of $\mathbb{Z}$-valued Maslov potential μ_β on β: $|e_i| = \mu_\beta(*_i)$.

Remark. The choice of ($\mathbb{Z}$-valued) Maslov potentials on K (possibly discontinous at $*$) and β, leads to a $\mathbb{Z}$-valued Maslov potential on $S(K,\beta)$. The precise definition of $\lambda_m(\beta)$ is not given here as we will only be working with $\lambda_1(\beta) = \ell(\beta)$ where $\ell(\beta)$ is the length of β [11]. The preceding theorem suggests the notation $z := q^{1/2} - q^{-1/2}$.

Definition 4.2.2. Let $m \geq 0$ with $m|2r(K)$ and $m \neq 1$. The n-**colored** m-**graded satellite**

ruling polynomial is

$$R^m_{n,K}(q) = \frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} R^m_{S(K,\beta)}(z)$$

where the sum is over all positive permutation braids with Maslov potential 0, $l(\beta)$ denotes the length of the braid β, and

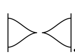
$$\alpha_n = (q^{1/2})^{n(n-1)/2} \prod_{j=1}^{n} \frac{q^{j/2} - q^{-j/2}}{q^{1/2} - q^{-1/2}}$$

Theorem 4.2.2. *(Leverson-Rutherford) Let K be Legendria n and $m \geq 0$ have $m|2r(K)$. Assume $m \neq 1$ and that $r(K) = 0$ if m is even. Then, $R^m_{n,K}(q) = Rep_m(K, \mathbb{F}_q)$.*

Remark. In the proof of [11] Theorem 7.4, it is shown that if the sum in Theorem 4.2.1 is over the zero differential, then by summing over all permutation braids we recover total representation numbers.

Definition 4.2.3. The 2-colored 1-graded ruling polynomial of K is

$$R^1_{2,K}(q) = \frac{1}{\alpha_2} \sum_{\beta \in S_2} q^{l(\beta)/2} R^1_{S(K,\beta)}(z) - \frac{z}{q+1} R^1_{S(K,h)}(z)$$

where the sum is over all positive permutation braids, and h is the standard 2-stranded hook with front projection .

Lemma 4.2.3. *Let $(\mathcal{B}, \delta)$ be a finite DGA with trivial homology and $(\mathcal{A}, \partial)$ the DGA of a Legendrian K. Then,*

$$|\overline{\mathrm{Rep}}_1(K, (\mathcal{B}, \delta), \mathcal{B}^* \cap \ker \delta)| = |\mathcal{B}^* \cap \ker \delta| \cdot |\ker \delta|^{\sigma_1}$$

where σ_1 is the number of Reeb chords.

Proof. We may choose notation on $\mathcal{A} = \mathbb{Z}\langle t^{\pm 1}, a_1, a_2, \ldots, a_n \rangle$ such that the height $h(a_i)$ is increasing in i [3, Lemma 6.1]. We wish to count DGA homomorphisms $f : (\mathcal{A}, \partial) \rightarrow$

$(\mathcal{B}, \delta)$. Since t is invertible, $f(t) \in \mathcal{B}^* \cap \ker \delta$. We now define $f(a_i)$ inductively. Assume $f(t), f(a_1), \ldots, f(a_{i-1})$ have been defined and the representation equation $f \circ \partial = \delta \circ f$ holds on $\mathbb{Z}\langle t^{\pm 1}, a_1, a_2, \ldots, a_{i-1}\rangle$. It suffices to define $f(a_i)$ so that the representation equation holds on $\mathbb{Z}\langle t^{\pm 1}, a_1, a_2, \ldots, a_i\rangle$. By the height filtration [3, Corollary 6.2], we have $\partial a_i \in \mathbb{Z}\langle t^{\pm 1}, a_1, a_2, \ldots, a_{i-1}\rangle$ and so $f \circ \partial a_i$ has already been defined. Next observe

$$\delta(f \circ \partial a_i) = (\delta \circ f)(\partial a_i) = (f \circ \partial)(\partial a_i) = f \circ \partial^2 a_i = 0$$

which implies $f \circ \partial a_i \in \ker \delta = \mathrm{im}\, \delta$. Hence $(f \circ \partial) a_i = \delta z$ for some $z \in \mathcal{B}$ defining $f(a_i) = z$ satisfies the representation equation on $\mathbb{Z}\langle t^{\pm 1}, a_1, a_2, \ldots, a_{i-1}\rangle$ completing the induction on $\mathbb{Z}\langle t^{\pm 1}, a_1, a_2, \ldots, a_i\rangle$. The preceding induction argument shows that there are $|\ker \delta|$ possible images for each of the Reeb chord generators, and so

$$|\overline{Rep}_1(K, (\mathcal{B}, \delta), \mathcal{B}^* \cap \ker \delta)| = |\mathcal{B}^* \cap \ker \delta| \cdot |\ker \delta|^{\sigma_1}$$

as desired. □

Theorem 4.2.4. *Let K be Legendrian, then $Rep_1(K, \mathbb{F}_q^2) = R^1_{2,K}(q)$.*

Proof. Observe that $\lambda_1(\beta) = l(\beta)$ and $\alpha_2 = q^{1/2} \frac{q - q^{-1}}{q^{1/2} - q^{-1/2}} = q + 1$. Then by definition,

$$\begin{aligned}
R^1_{2,K}(q) &= \frac{1}{\alpha_2} \sum_{\beta \in S_2} q^{l(\beta)/2} R^1_{S(K,\beta)}(z) - \frac{z}{q+1} R^1_{S(K,h)}(z) \\
&= \frac{1}{\alpha_2} \sum_{\beta \in S_2} q^{l(\beta)/2} q^{-\lambda_m(\beta)/2} \left(z^{-2} \sum_d \widetilde{Rep}_1(K, (V_\beta, d) B_\beta) \right) - \frac{z}{q+1} R^1_{S(K,h)}(z) \\
&= \frac{1}{\alpha_2} \sum_{\beta \in S_2} z^{-2} \sum_d \widetilde{Rep}_1(K, (V_\beta, d) B_\beta) - \frac{z}{q+1} R^1_{S(K,h)}(z) \\
&= Rep_1(K, \mathbb{F}_q^2) + \frac{z^{-2}}{\alpha_2} \sum_{\beta \in S_2} \sum_{d \neq 0} \widetilde{Rep}_1(K, (V_\beta, d) B_\beta) - \frac{z}{q+1} R^1_{S(K,h)}(z).
\end{aligned} \tag{4.2}$$

The equality in 4.2 holds since

$$\frac{z^{-2}}{\alpha_2}\sum_{\beta\in S_2}\widetilde{Rep}_1(K,(V_\beta,0),B_\beta) = \frac{z^{-2}}{\alpha_2}\widetilde{Rep}_1(K,(\mathbb{F}_q^2,0),GL(n,\mathbb{F}_q^2)) \qquad \text{(by (4.1))}$$

$$= \frac{(q-1)(q^2-1)}{(q+1)(q-1)^2}Rep_1(K,\mathbb{F}_q^2) = Rep_1(K,\mathbb{F}_q^2).$$

Next notice that since $m=1$, all generators have degree 0 so

$$\lim_{N\to\infty}\prod_{\substack{k\in\mathbb{Z}\\ |k|\leq N}} |-End(V_\beta)^1_k|^{-\chi^k/2} = |-End(V_\beta)^1_0|^{-\chi^0/2} = |-End(V_\beta)|^{-\sigma_1/2}.$$

Now we compute:

$$\widetilde{Rep}_1(K,(V_\beta,d),B_\beta) = |-End(V_\beta)|^{-1/2}|(-End(V_\beta))^*\cap\ker\delta|\cdot Rep_1(K,(V_\beta,d),B_\beta)$$

$$Rep_1(K,(V_\beta,d),B_\beta) = |-End(V_\beta)|^{-\sigma_1/2}|(-End(V_\beta))^*\cap\ker\delta|^{-1}|\overline{Rep}_m(K,(V_\beta,d),B_\beta)|$$

Thus,

$$\sum_{\beta\in S_2}\sum_{d\neq 0}\widetilde{Rep}_1(K,(V_\beta,d)B_\beta) = \sum_{\beta\in S_2}\sum_{d\neq 0}|-End(V_\beta)|^{(-\sigma_1-1)/2}|\overline{Rep}_m(K,(V_\beta,d),B_\beta)|. \quad (4.3)$$

Using the Bruhat decomposition: $\bigsqcup_{\beta\in S_n} B_\beta = GL(n,\mathbb{F}_q)$ equation (4.3) becomes

$$\sum_{d\neq 0}|\mathbb{F}_q^4|^{(-\sigma_1-1)/2}|\overline{Rep}_m(K,(\mathbb{F}_q^2,d),GL(2,\mathbb{F}_q)| = \sum_{d\neq 0}(q^2)^{-\sigma_1-1}|\overline{Rep}_m(K,(\mathbb{F}_q^2,d),GL(2,\mathbb{F}_q)|.$$

Now since d is strictly upper triangular and nonzero, $(-End(\mathbb{F}_q^2),\delta)$ has trivial homology so by Lemma 4.2.3,

$$\sum_{d\neq 0}(q^2)^{-\sigma_1-1}|\overline{Rep}_m(K,(\mathbb{F}_q^2,d),GL(2,\mathbb{F}_q)| = \sum_{d\neq 0}(q^2)^{-\sigma_1-1}\cdot|GL(2,\mathbb{F}_q)\cap\ker\delta|\cdot|\ker\delta|^{\sigma_1}. \quad (4.4)$$

Recall the differential d from the vector space (V,d) induces the differential on the algebra

$-End(V)$ by the (graded) commutator of d a with a transformation T. So in particular, any invertible $T \in \ker\delta$ must satisfy $d \circ T = T \circ d$. Checking this we conclude that this is equivalent to saying that T is of the form

$$T = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

and so we deduce there are q^2 such $T \in \ker\delta$ and $q(q-1)$ such $T \in GL(2,\mathbb{F}_q) \cap \ker\delta$, for a given $d \neq 0$. Thus (4.4) becomes

$$\sum_{d \neq 0} (q^2)^{-\sigma_1 - 1}(q^2)^{\sigma_1} q(q-1) = q^{-1}(q-1)^2.$$

Now since $\alpha_2 = q+1$, we have

$$\begin{aligned} R^1_{2,K}(q) &= Rep_1(K,\mathbb{F}_q^2) + \frac{z^{-2}}{q+1} q^{-1}(q-1)^2 - \frac{z}{q+1} R^1_{S(K,h)}(z) \\ &= Rep_1(K,\mathbb{F}_q^2) + \frac{z^{-2}(q^{1/2}-q^{-1/2})^2}{q+1} - \frac{z}{q+1} R^1_{S(K,h)}(z) \\ &= Rep_1(K,\mathbb{F}_q^2) + \frac{1}{q+1} - \frac{z}{q+1} R^1_{S(K,h)}(z). \end{aligned}$$

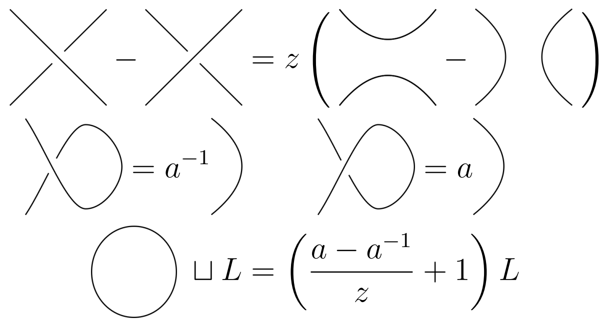
Since $S(K,h)$ is the standard unknot, $R^1_{S(K,h)}(z) = z^{-1}$. Therefore, $R^1_{2,K}(q) = Rep_1(K,\mathbb{F}_q^2)$.

□

4.3 The 2-colored Kauffman Polynomial

Throughout this section we assume that all links and diagrams have the blackboard framing. This section illustrates the interplay between the 2-colored Kauffman polynomial (an invariant of framed links) and 1-graded total representation numbers. A similar connection is given between the 2-graded representation numbers and the n-colored HOMFLY-PT polynomial in [11].

Definition 4.3.1. The **(framed) Kauffman polynomial (Dubrovnik version)** $D_K(a,z)$ is the Laurent polynomial $D_K \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ given by skein relations:



The normalization for the unknot is $D_{\bigcirc} = \frac{a-a^{-1}}{z} + 1$.

Definition 4.3.2. The n-colored Kauffman polynomial is defined to be

$$D_{n,K}(a,q) = D_{S(K,\mathcal{S}_n)}(a,z)|_{z=q^{1/2}-q^{-1/2}}$$

where $\mathcal{S}_n$ is the symmetrizer from the BMW-algebra, BMW_n (see [1], [9])

Theorem 4.3.1 ([7] [15])**.** *For any Legendrian K, $deg_a D_K \leq 0$ and $R^1_K(z) = D_K(a,z)|_{a^{-1}=0}$.*

Theorem 4.3.2. *Let K be Legendrian, then $Rep_1(K, \mathbb{F}_q^2) = D_{2,K}(a,q)|_{a^{-1}=0}$*

Proof. In [9] the authors compute (using our conventions) $\mathcal{S}_2 = \frac{1}{q^{1/2}+q^{-1/2}}(T_1 + q^{-1/2} + \frac{q^{1/2}-q^{-1/2}}{1-q^{1/2}a}h')$ where T_1 is the positive permutation braid with a single crossing and h' is a smooth hook. Equivalently,

$$\begin{aligned}\mathcal{S}_2 &= \frac{1}{q+1}\left(q^{1/2}T_1 + \mathbf{1}_{BMW_2} - \frac{q-1}{1-q^{1/2}a}h'\right)\\ &= \frac{1}{\alpha_2}\left(q^{\ell(T_1)/2}T_1 + q^{\ell(\mathbf{1}_{BMW_2})/2}\mathbf{1}_{BMW_2} - \frac{q-1}{1-q^{1/2}a}h'\right)\end{aligned}$$

Therefore,

$$\begin{aligned}
D_{n,K}(a,q) &= D_{S(K,\mathcal{S}_2)}(a,z)|_{z=q^{1/2}-q^{-1/2}} \\
&= \frac{1}{\alpha_2}\left(\sum_{\beta\in S_2} q^{\ell(\beta)/2} D_{S(K,\beta)}(a,z) + \frac{q-1}{1-q^{1/2}a} D_{S(K,h')}(a,z)\right)\Bigg|_{z=q^{1/2}-q^{-1/2}} \\
&= \frac{1}{\alpha_2}\left(\sum_{\beta\in S_2} q^{\ell(\beta)/2} D_{S(K,\beta)}(a,z) + \frac{q-1}{1-q^{1/2}a}\left(\frac{a-a^{-1}}{z}+1\right)\right)\Bigg|_{z=q^{1/2}-q^{-1/2}} \\
&= \frac{1}{\alpha_2}\left(\sum_{\beta\in S_2} q^{\ell(\beta)/2} D_{S(K,\beta)}(a,q^{1/2}-q^{-1/2}) - 1 - q^{1/2}a^{-1}\right). \qquad (*)
\end{aligned}$$

We now compute, using Theorem 4.2.4

$$\begin{aligned}
Rep_1(K,\mathbb{F}_q^2) &= R^1_{2,K}(q^{1/2}-q^{-1/2}) \\
&= \frac{1}{\alpha_2}\left(\sum_{\beta\in S_2} q^{\ell(\beta)/2} R^1_{S(K,\beta)}(z) - zR^1_{S(K,h)}(z)\right) \\
&= \frac{1}{\alpha_2}\left(\sum_{\beta\in S_2} q^{\ell(\beta)/2} D_{S(K,\beta)}(a,q^{1/2}-q^{-1/2})|_{a^{-1}=0} - zR^1_{S(K,h)}(z)\right) \\
&= \frac{1}{\alpha_2}\left(\sum_{\beta\in S_2} q^{\ell(\beta)/2} D_{S(K,\beta)}(a,q^{1/2}-q^{-1/2})|_{a^{-1}=0} - 1\right) \\
&= D_{n,K}(a,q)|_{a^{-1}=0} \qquad (*)
\end{aligned}$$

which completes the proof. □

The preceding exposition suggests the following conjecture:

Conjecture 4.3.1. Let K be Legendrian, then $Rep_1(K,\mathbb{F}_q^n) = D_{n,K}(a,q)|_{a^{-1}=0}$.

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