

Student-Faculty Colloquium

Some Proofs of the Existence of Irrational Numbers

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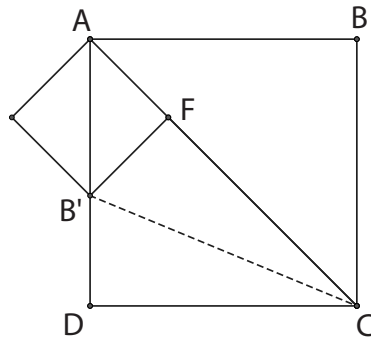


Neal Coleman is a freshman majoring in pure mathematics at Ball State. He served as the secretary of the Ball State Math Club during the 2006-07 school year; during the fall semester, he participated in the Student-Faculty Colloquium, where, advised by Dr. Rich Stankewitz, he prepared a presentation on proofs of the existence of irrational numbers.

Over the course of this article, we will discuss irrational numbers and several different ways to prove their existence. As is commonly known, the real numbers can be partitioned into rational numbers and irrational numbers. Rational numbers are those which can be represented as a ratio of two integers — i.e., the set $\{\frac{a}{b} : a, b \in \mathbf{Z}, b \neq 0\}$ — and the irrational numbers are those which cannot be written as the quotient of two integers. We will, in essence, show that the set of irrational numbers is not empty. In particular, we will show $\sqrt{2}$, e , π , and π^2 are all irrational.

Geometric Proof of the Irrationality of $\sqrt{2}$

The ancient Greeks were some of the first mathematicians. However, instead of approaching mathematical problems numerically or abstractly, the Greeks were very concrete: they envisioned problems geometrically. They considered numbers to be analogous to line segments. Two segments are considered *commensurate* if and only if there exists some other segment called a “common measure” which can be overlaid an integer number of times onto both. That is pretty convoluted; it is easier to think of two segments as commensurate when you can find some other “unit” segment to measure them both, where each has integer length with respect to the segment of this “unit.”



We begin by discussing why the diagonal and side of a square are incommensurate, i.e., why it is impossible to find some segment, however small, that can measure both the diagonal and square.

We start our proof from [2, p. 23], by constructing a square ABCD. Because we are assuming the side and diagonal are commensurate for contradiction, we shall denote the segment of common measure between the side AD and the diagonal AC by e . To begin the proof, on the diagonal AC , starting from C , measure out a segment congruent to $AD \cong CD$, and call the point where the subsegment terminates F . Next construct a line perpendicular to AC from F out to AD . Call the point where it meets the side B' . Note that because angle $B'AF$ is 45° and AFB' is 90° , $AF \cong B'F$. Moreover, $\triangle CFB' \cong \triangle CDB'$ because both are right triangles, $CD \cong CF$, and they share the hypotenuse CB' . We may therefore conclude that $B'D \cong B'F \cong AF$. Now, create a new square with $B'F$ and FA as sides. Note that the new diagonal is going to be $B'A$. Since AC and $FC \cong CD \cong AD$ are commensurate by e , then so are AF and $B'A$, one side of the new square and its diagonal.

Clearly, we can repeat this process of constructing a new square indefinitely, with each new square *strictly* smaller than the last. Note also that, inductively, the sides and diagonal of each new square are measured by our original e . This is a contradiction by infinite descent: for example, suppose that $|AC| = 1000e$; since each successive diagonal is measurable by e , and we can construct a new diagonal indefinitely, the 1001st diagonal we construct will no longer be commensurate by e .

We conclude that the side and diagonal of a square are incommensurable. This is equivalent to stating that $\sqrt{2}$ is irrational: two line segments are commensurable if and only if there exists another segment such that each of the two initial segments is an integer multiple of that third one, i.e., that the ratio of their lengths can be represented as a rational number. Hence, since the diagonal and side of a square are incommensurable, their ratio, $\sqrt{2}$, cannot be represented as a ratio of two integers, and is therefore irrational. Incidentally, it is to be noted that the discovery of irrational numbers shocked and startled the Greeks: according to legend, the Pythagorean who first completed this proof atoned by perishing in a shipwreck [2].

Algebraic Proof of the Irrationality of $\sqrt{2}$

We can take the essence of the geometric proof presented above and distill it into an algebraic argument as follows.

To prove the irrationality of $\sqrt{2}$ by contradiction, we suppose it is rational. Then consider the set $S = \{(m, n) : \left(\frac{m}{n}\right)^2 = 2, m, n \in \mathbf{Z}^+\}$. Invoking the Well-Ordering Principle, set $m_0 = \min\{m : (m, n) \in S\}$, the smallest of the set of numerators. Then let $n_0 \in \mathbf{Z}^+$ be the matching denominator for m_0 such that $2 = \left(\frac{m_0}{n_0}\right)^2$, and thus $2n_0^2 = m_0^2$. Now, we have, by subtracting m_0n_0 from both sides, $2n_0^2 - m_0n_0 = m_0^2 - m_0n_0$; that is,

$$n_0(2n_0 - m_0) = m_0(m_0 - n_0).$$

And so

$$\frac{m_0}{n_0} = \frac{2n_0 - m_0}{m_0 - n_0}.$$

Since $\sqrt{2} > 1$, we know $m_0 > n_0$; however, this implies that $2m_0 > 2n_0$, which indicates that $m_0 > 2n_0 - m_0$. This is a contradiction to the minimality of m_0 , establishing that $\sqrt{2}$ is irrational.

The basic idea behind these first two proofs is to find some violation of the Well-Ordering Principle: to show that if $\sqrt{2}$ were rational, it would be possible to construct a set of positive integers with no smallest element (in the first proof, this set with no lower bound is the set of diagonals: they are integer multiples of finite e , and yet can decrease indefinitely; in the second proof, and a bit more explicitly, the set with no lower bound is the set of all numerators of rational realizations of $\sqrt{2}$).

It should be noted that the commonplace proof of the irrationality of $\sqrt{2}$ invokes the Prime Factorization theorem in order to have the phrase “reduced-form fraction” be well-defined; the two proofs presented above, however, do not require this secondary result, and so are somewhat stronger than the most common proof.

Irrationality of e

The number $\sqrt{2}$ is not, of course, the only irrational number; it is possible to show that some important numbers which naturally occur in geometry and analysis are also irrational. For example, as seen in [1], let us consider the natural exponential e . Given the MacLaurin expansion of e^x , we see that

$$e = \sum_{j=0}^{\infty} \frac{1}{j!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots.$$

Suppose for contradiction that $e = \frac{a}{b}$ is rational, with $a, b \in \mathbf{Z}^+$. Now, fix any $k > 1$ such that $b|k!$, for example, $k = b$.

Define

$$\begin{aligned}\alpha &= k! \left[e - \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} \right) \right] \\ &= k! \left(\frac{a}{b} \right) - k! \left(\sum_{j=0}^k \frac{1}{j!} \right).\end{aligned}$$

Thus, because both terms in the final expression are integers, $\alpha \in \mathbf{Z}$. But consider now that

$$\begin{aligned}0 < \alpha &= \sum_{j=k+1}^{\infty} \frac{k!}{j!} \\ &= \frac{1}{(k+1)} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} + \cdots \\ &< \frac{1}{(k+1)} + \frac{1}{(k+1)(k+1)} + \frac{1}{(k+1)(k+1)(k+1)} + \cdots \\ &= \left(\frac{1}{k+1} \right) + \left(\frac{1}{k+1} \right)^2 + \left(\frac{1}{k+1} \right)^3 + \cdots \\ &= \frac{1}{k+1} \left(\frac{1}{1 - \frac{1}{k+1}} \right) \\ &= \frac{1}{k}.\end{aligned}$$

Since $k > 1$, we now have a startling and unsettling occurrence: the integer α satisfies $0 < \alpha < 1$. This is our contradiction: e is therefore irrational.

Irrationality of π and π^2

Nearly as ubiquitous as e , but arising initially from geometric instead of analytic considerations, is the number π , the ratio of any circle's circumference to its diameter. This last proof (from [1, pp. 46–47], with some detail added), of the irrationality of π , is rather more mysterious than the previously discussed proofs. Several very conveniently chosen functions conspire to show that it is impossible to write π or π^2 as quotients of integers.

We begin by defining, for any positive integer n ,

$$f(x) = \frac{x^n(1-x)^n}{n!} = \frac{1}{n!} \sum_{m=n}^{2n} c_m x^m,$$

where the $c_m \in \mathbf{Z}$. Note that for $0 < x < 1$, we have $0 < f(x) < \frac{1}{n!}$ and that $f(0) = 0$ and $f^{(m)}(0) = 0$ if $m < n$ or $m > 2n$. However, if $n \leq m \leq 2n$, $f^{(m)}(0) = c_m \frac{m!}{n!} \in \mathbf{Z}$.

Hence, $f(x)$ and all its derivatives take integral values at $x = 0$. Since $f(1-x) = f(x)$, the same is true at $x = 1$, because $f^{(k)}(1-x) = (-1)^k f^{(k)}(x)$.

Now, suppose that $\pi^2 = \frac{a}{b}$ is rational, where a and b are positive integers. Consider the function

$$G(x) = b^n \{ \pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) - \dots + (-1)^n f^{(2n)}(x) \}.$$

Note that both $G(0)$ and $G(1)$ are integers. With a little algebra, we have

$$\begin{aligned} G(x) &= b^n \left(\left(\frac{a}{b} \right)^n f(x) - \left(\frac{a}{b} \right)^{n-1} f^{(2)}(x) \right. \\ &\quad \left. + \left(\frac{a}{b} \right)^{n-2} f^{(4)}(x) - \dots + (-1)^n f^{(2n)}(x) \right) \\ &= \left(a^n f(x) - a^{n-1} b f^{(2)}(x) + a^{n-2} b^2 f^{(4)}(x) - \dots + (-1)^n b^n f^{(2n)}(x) \right). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dx} \{ G'(x) \sin \pi x - \pi G(x) \cos \pi x \} &= G''(x) \sin \pi x + \pi G'(x) \cos \pi x - \pi \{ G'(x) \cos \pi x - \pi G(x) \sin \pi x \} \\ &= G''(x) \sin \pi x + \pi^2 G(x) \sin \pi x \\ &= [G''(x) + \pi^2 G(x)] \sin \pi x \\ &= \left[b^n \{ \pi^{2n} f''(x) - \pi^{2n-2} f^{(4)}(x) + \dots + (-1)^{n-1} \pi^2 f^{(2n)}(x) + 0 \} \right. \\ &\quad \left. + \pi^2 b^n \{ \pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) - \dots + (-1)^n f^{(2n)}(x) \} \right] \sin \pi x \\ &= b^n \pi^{2n+2} f(x) \sin \pi x \\ &= \pi^2 (\pi^2)^n b^n f(x) \sin \pi x \\ &= \pi^2 \left(\frac{a^n}{b^n} \right) b^n f(x) \sin \pi x \\ &= \pi^2 a^n f(x) \sin \pi x. \end{aligned}$$

Therefore,

$$\begin{aligned} \pi \int_0^1 a^n f(x) \sin \pi x \, dx &= \left[\frac{G'(x) \sin \pi x}{\pi} - G(x) \cos \pi x \right]_0^1 \\ &= G(1) + G(0) \end{aligned}$$

which is an integer. But, because $0 < f(x) < \frac{1}{n!}$, for any x such that $0 < x < 1$,

$$0 < \pi \int_0^1 a^n f(x) \sin \pi x \, dx < \frac{\pi a^n}{n!},$$

which is strictly less than 1 for large n .

This is our contradiction. Consequently, π^2 is irrational, and so is π .

References

- [1] G.H. Hardy, and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press (1979).
- [2] H. Rademacher, and O. Toeplitz, *The Enjoyment of Mathematics*, Dover (1990).