Student-Faculty Seminar

Mathematical Modeling

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Seminar Overview, by Dr. Irene Livshits

Mathematical modeling is a scientific attempt to describe real life phenomena using mathematical tools. Each model consists of a set of variable parameters and rules of evolution for these parameters. By applying these rules to the model's parameters, one can learn a lot about the phenomenon in question, and, based on the results, make informed decisions, predict the future, or make the optimal choice between different options. Many mathematical models are implemented on computers; their simulations can run in a matter of minutes, or even seconds, often replacing real life experiments that consume vast amounts of time and resources.

Mathematical modeling is widely applied in a variety of fields, ranging from medicine to engineering, from physics to physiology, from economy to image processing. It is essential to almost every science. There is a great variety of mathematical models; the following are some of the main categories.

Linear or nonlinear: each mathematical model is defined by its parameters and the relations between these parameters. If the relations are defined by linear operators, the model is linear; otherwise it is considered nonlinear. Typical examples of mathematical models that can be either linear or nonlinear are those defined by linear or nonlinear differential equations.

The second important distinction between different models is whether the model is deterministic or stochastic. In deterministic models, the evolution of the system is completely determined by the initial (starting) value of the model; for example, a model of planetary motion is deterministic. In contrast, stochastic models allow for randomness—they are well-suited to describe processes that are not well defined, like the stock market—and the model variables' values are defined by their probability distribution rather than a unique value.

Finally, static models aim to describe real life processes at a given time, while dynamic models describe an evolution in time. Many dynamic models involve time-dependent differential equations.

In the fall of 2007, the student-faculty seminar was devoted to different applications of mathematical modeling. The topics presented by students and discussed with faculty included packaging costs, soda bubbles, nuclear arms races, and others. We used the book *An Introduction to Mathematical Modeling* by E.A. Bender [1].

An Example Presentation, by Neal Coleman

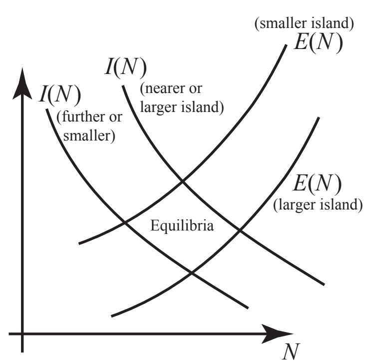
I gave a presentation to the seminar on qualitative graphing to learn about the behavior of different systems. In particular, I focused on two examples: stability in a nuclear arms race and biodiversity on islands. Here, I am going to outline the presentation I gave on biodiversity; the basic material for both examples can be found in [1].

The biosphere can be broken up into isolated habitats—perhaps a rainforest divided in two by a wide river, or perhaps cornfields separated by a four-lane interstate highway. The most extreme example, barring extraterrestrial life, is that of an island separated from a mainland by an ocean. We therefore will attempt to construct a crude model capable of accounting for general relationships between the island's biodiversity, its size, and its distance from the mainland.

Empirically, islands that are larger as well as islands that are closer to the mainland tend to be more diverse. Assuming that the model does not span an evolutionary or geologic time scale, so the island and mainland are fixed and species do not evolve to their environments, we can describe this phenomenon by the immigration of species to the island and the extinction of species on the island. (We neglect, also, seasonal variation in favor of a more general picture.)

To make this simple, let us consider only the number N of species on the island. This, of course, glosses over many of the finer details; as we have said, though, we are only pursuing a crude model. When this number is relatively constant, we can safely say that the number of species immigrating is roughly the same as the number of species dying out on the island. Thinking in terms of the broad picture, we can pretty safely deal with the rate of change in the species due to immigration and extinction. We shall write these rates as functions of N.

The extinction curve $E(N)$ is determined chiefly by the number of species on the island: the more species there are, the more competition for resources, and the greater the chance a species will become extinct in a given time period. Thus, E is an increasing function and has positive slope. The immigration curve $I(N)$, on the other hand, has negative slope because if there are more species on the island, there are fewer additional species off the island, so fewer species can immigrate and increase N.



The extinction rate is determined solely by factors on the island, so it is independent of the distance from the island to the mainland. The immigration rate, on the other hand, is determined by both distance and size. The closer the island, the more species will be able to make the trip; the larger the island, the larger it is as a target for immigrating species and the easier it is to establish a population.

By consulting the figure above, we can see that $I(N)$ and $E(N)$ intersect at an equilibrium species count. The equilibrium is stable because both the extinction and the immigration rates have opposite slopes: if the number of species is greater than the equilibrium, more will become extinct than immigrate, so N will decrease. Vice-versa, if the number of species is less than the equilibrium, more will immigrate than become extinct, so N will increase.

The model, crude though it is, does describe the empirical phenomenon laid out above: islands that are larger or closer to the mainland are more diverse. We can also tell something about the turnover rate: the equilibrium extinction rate is higher for closer islands. So, if we compare two islands of equal size, the species composition of the island nearer the mainland should change faster. Similarly, smaller islands should turnover more quickly than larger islands.

References

[1] E. Bender, An introduction to mathematical modeling, Dover (2000).