The O_n Problem

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Introduction

Interest in graph theory has greatly increased during the past decade. This increase in activity can be attributed both to the applications and intrinsic appeal of graph theory. Besides the strictly mathematical ramifications of graph theory in numerical analysis, group theory, topology, matrix theory, probability, and combinatorics, it also has widespread applications to computer science, physics, chemistry, communication science, psychology, economics, electrical and civil engineering, architecture, genetics, sociology, anthropology, linguistics, and operations research (6). In fact, as Harary (6) points out, "graph theory serves as a mathematical model for any system involving a binary relation."

The father of graph theory is considered to be Euler who, in 1736, solved the Königsberg bridge problem, an example of a physical problem in operations research. It was the custom in Euler's time for the city fathers of German cities to promenade around their town each Sunday. The city of Königsberg consisted of the opposite sides of a river and two islands in the river, interconnected by bridges (fig. 1). The problem was to find a means of traversing the bridges in such a way as to visit each land mass, crossing each bridge exactly once, and returning to the

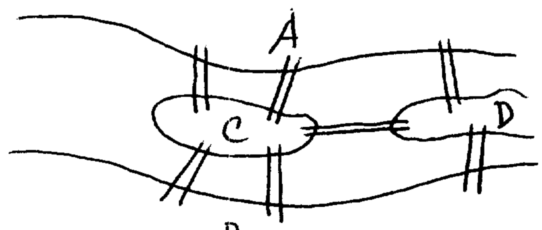


fig. 1

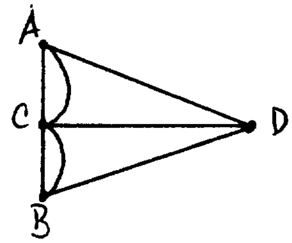


fig. 2

starting point. Euler recognized the problem as that of traversing the graph (fig. 2) in a continuous cyclic path. Rather than merely solve this specific problem, Euler chose to settle the question of which graphs in general can be so traversed. He found that a necessary and sufficient condition is that an even number of edges adjoin every vertex (14).

For the next century little was done in graph theory other than to perhaps use it in relation to puzzles. Then, in 1847, Kirchoff developed the theory of the class of graphs known as trees to solve the system of simultaneous linear equations which represents the current in each branch of an electrical network. In 1857 Cayley used trees in enumerating the isomers of alkanes. Both men used trees in physical applications before Jordan independently developed trees in a purely mathematical sense (6). Meanwhile, in 1852, in a letter to Hamilton, de Morgan mentioned a problem brought to him by one of his students. The problem has since become known as the Four Color Conjecture. In 1890 Heawood found the error in the first "proof" of this simple problem by Kempe, and then proved the Five Color Theorem. Many great mathematicians have since attempted to solve the 4 C C, and many erroneous proofs have been proposed, but it remains today as one of mathematics' greatest unsolved problems.

Hamilton's "Around the World" game (1859) introduced the concept of a circuit (called a Hamiltonian circuit in his honor) which goes through every vertex of a graph exactly once. The finding of reasonable conditions for the existence of Hamiltonian circuits is another of graph theory's unsettled questions.

The first books which can be said to deal with graph theory were König's _Theorie der endlichen und unendlichen Graphen_ in 1936 (9), followed by Berge's _Theorie des Graphes et ses Applications_ in 1958 (1a), which became the first book on graph theory published in English when it was translated in 1962 (1b). In the twentieth century applications of graph theory began to be realized and efforts in graph theory multiplied as a multitude of books and papers were published. Since 1969 about one third of the research problems in _The American Mathematical Monthly_ have dealt with graph theory.

Definitions

Nomenclature is by no means uniform in graph theory. Each mathematician writing a paper in graph theory seems to have his own personal terminology. Consequently, it is necessary to define the terms to be used throughout the rest of this paper.

A graph G is an ordered pair (V,E) consisting of a set V of vertices and a set E of edges. The cardinality of V and E is also denoted by V and E, respectively. However, the context will make the meaning clear. Although a graph may be infinite, we shall confine our study to finite graphs only, that is, V and E are finite sets. An edge $e \in E$ is an ordered pair (v_i, v_k) of vertices from V, where e joins v_i to v_k. If $e = (v_i, v_k)$, then v_i is said to be adjacent to v_k and e is said to be incident with both v_i and v_k. If $i = k$, then e is said to be a loop. If e_i and e_j have one vertex v_k in common, then e_i and e_j are said to be adjacent edges. If there exist several edges each having v_i and v_k as endpoints, then the edges are called multiple edges (for the vertices v_i and v_k). For the most part, research in graph theory today is concerned with graphs having no loops or multiple edges. This is because theorems proven for graphs with no loops or multiple edges are more general (and usually more difficult) than theorems for graphs with loops or multiple edges. For the remainder of this paper, all graphs are assumed to be without loops or multiple edges unless stated otherwise.

The degree of a vertex v in a graph G is defined as the number n of edges which have v as an endpoint, and is denoted by deg(v). A loop contributes two to the degree of a vertex. We also define $\delta(G) = \min_{v \in G} \deg(v)$ and $\Delta(G) = \max_{v \in G} \deg(v)$.

A subgraph G' of G is an ordered pair of sets (V',E') where $V' \subset V$ and $E' \subset E$. A subgraph is said to be a spanning subgraph of a graph if their vertex sets are equal. Two graphs G and H are said to be isomorphic (denoted $G \cong H$) if there exists a one-to-one correspondence between their vertex sets which preserves adjacency of vertices. An induced subgraph $\langle S \rangle$ for any subset S of V is a maximal subgraph of G with vertex set S (13).

The line graph of G = (V,E) is the graph L(G) = (E, V_1), in which the vertices of L(G) are the edges of G, and with two vertices in L(G) adjacent whenever the corresponding edges in G are adjacent (6,7). A theorem due to Whitney states that for every graph G_1 there exists a unique line graph of another graph G_2 such that $G_1 \not\cong G_2$ and $G_1 = L(G_2)$, provided that $G_1 \neq K_3$ (6,7).

Let $F = \{S_1, S_2, \ldots, S_p\}$ be a family of distinct nonempty subsets of a set S such that $\bigcup_{i=1}^{p} S_i = S$. Then the intersection graph of F, denoted $\Omega(F)$, is the ordered pair (F,E) where e = $(S_i, S_j) \in E$ if $i \neq j$ and $S_i \cap S_j \neq \emptyset$. A graph G is an intersection graph on S if there exists a family F of subsets of S such that $G = \Omega(F)$. If E is the edge set of a graph G, then the line graph of G may alternately be defined as $L(G) = \Omega(E)$ (6).

A path is a finite sequence $v_1, v_2, \ldots, v_n$ of vertices such that v_i and v_{i+1} are adjacent for all i, $1 \leq i \leq n - 1$. If $v_1 = v_n$ then the path is called a cycle or circuit. The length of a path (cycle) is the number of included edges in the path, where each edge is counted according to its number of occurrences in the path. If a cycle contains all of the vertices in the graph exactly once, the cycle is called a Hamiltonian cycle, H-cycle, or H-circuit. Thus an H-circuit is itself a spanning subgraph of the given graph.

A subgraph S of G is called a 1-factor of G if $\delta(S) = \Delta(S) = 1$. Thus if V is even, an H-circuit in G decomposes into two edge-disjoint 1-factors.

A graph may be identified with its geometric realization in the plane. Every intersection of edges in a graph which is not a vertex in the graph is called a crossing. The edges of any graph may be bent within the plane in such a way as to minimize the number of crossings in the geometric realization of the graph. The minimum number of crossings in a graph is called the crossing number of G and is denoted by c(G). If c(G) = 0, then G is said to be planar. When the edges of a planar graph G have been bent until they form a geometric realization G" of G in which there are no crossings, the subsets of the plane partitioned by the edges of G" are called the faces of G". The face not enclosed by the edges of G" is called the infinite face. A face may also be denoted by the set of edges which constitutes its boundary. Two faces are said to be contiguous if they have an edge in common as part of their respective boundaries (10). Since the edges of any planar graph G can be bent until they are in a geometric realization of G with no crossings, all references to planar graphs will henceforth assume that the graphs are in a planar geometric realization. Many theorems in graph theory deal only with planar graphs. The Five Color Theorem and the Four Color Conjecture deal with the coloring of the faces of any planar graph. An important result for planar graphs due to Euler states that V - E + F = 2, where F denotes the cardinality of the set of faces.

If G is a planar graph, then the dual graph G* of G is the ordered pair (V*,E*), where V* = the set of faces of G, and E* = the set of all unordered pairs of contiguous faces of G.

When distinct colors are assigned to the various faces of a graph, the graph is said to be face-colored. Likewise, if the colors are assigned to the vertices or edges of the graph, then the graph is said to be vertex-colored or edge-colored, respectively. If exactly k distinct colors are used to color the faces of a graph G such that contiguous faces have different colors, then G is said to be face-k-colored. G is vertex-k-colored (edge-k-colored) if k distinct colors are used to color the vertices (edges) of G such that adjacent vertices (edges) have different colors. The face-chromatic number χ_f is the minimum k such that G is face-k-colorable. Similarly, χ_v and χ_e are the vertex- and edge-chromatic numbers of G, respectively. The chromatic numbers of a graph G are related in the following way: $\chi_e(G) = \chi_v(L(G))$, and if G is planar, $\chi_f(G) = \chi_v(G^*) = \chi_e(L(G^*))$.

Colorings of graphs are very important in graph theory. The Five Color Theorem says that any planar graph can be face-5-colored. The Four Color Conjecture asserts that every planar graph is face-4-colorable. The O_n problem is concerned with edge-coloring. An important theorem due to Szekeres and Wilf states that for any graph G, $\chi_v(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraphs G' of G (18).

If $\delta(G) = \Delta(G) = n$, then G is said to be a regular graph of degree n, and we write deg(G) = n. If G has exactly n vertices and deg(G) = n - 1, then G is said to be a complete graph and is denoted K_n. Thus an H-circuit is a regular subgraph of degree 2 and a 1-factor is a regular subgraph of degree 1.

Let A be a set having 2n - 1 elements ($n > 1$). Let V be the set of all subsets of A having exactly n - 1 elements. Let E be the set of all unordered pairs of disjoint subsets from V. We denote the graph (V,E) by O_n. Thus 2E = nV. (Figure 3 shows O_n for n = 2,3,4.)

Statement of the Problem

In the November 1972 issue of The American Mathematical Monthly, Norman Biggs presented the following research problem. Because of the amusing nature of its presentation and its relative shortness, the problem will be quoted in its entirety (3).

> In the little English hamlet of Croam the consuming passion of the inhabitants is Association Football. In fact, the members of the village football team have become so ruthless in their will to win that no other team will play against them.
>
> Thus the eleven footballers of Croam (who are, incidentally, the only able-bodied men in the village) are forced to arrange their own matches between two teams of five, with the eleventh man as referee.

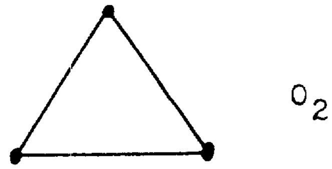
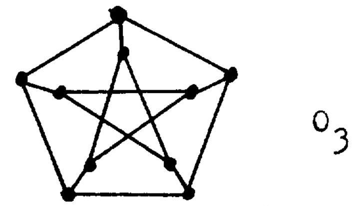


fig. 3a



(also known as Petersen's graph)

fig. 3b

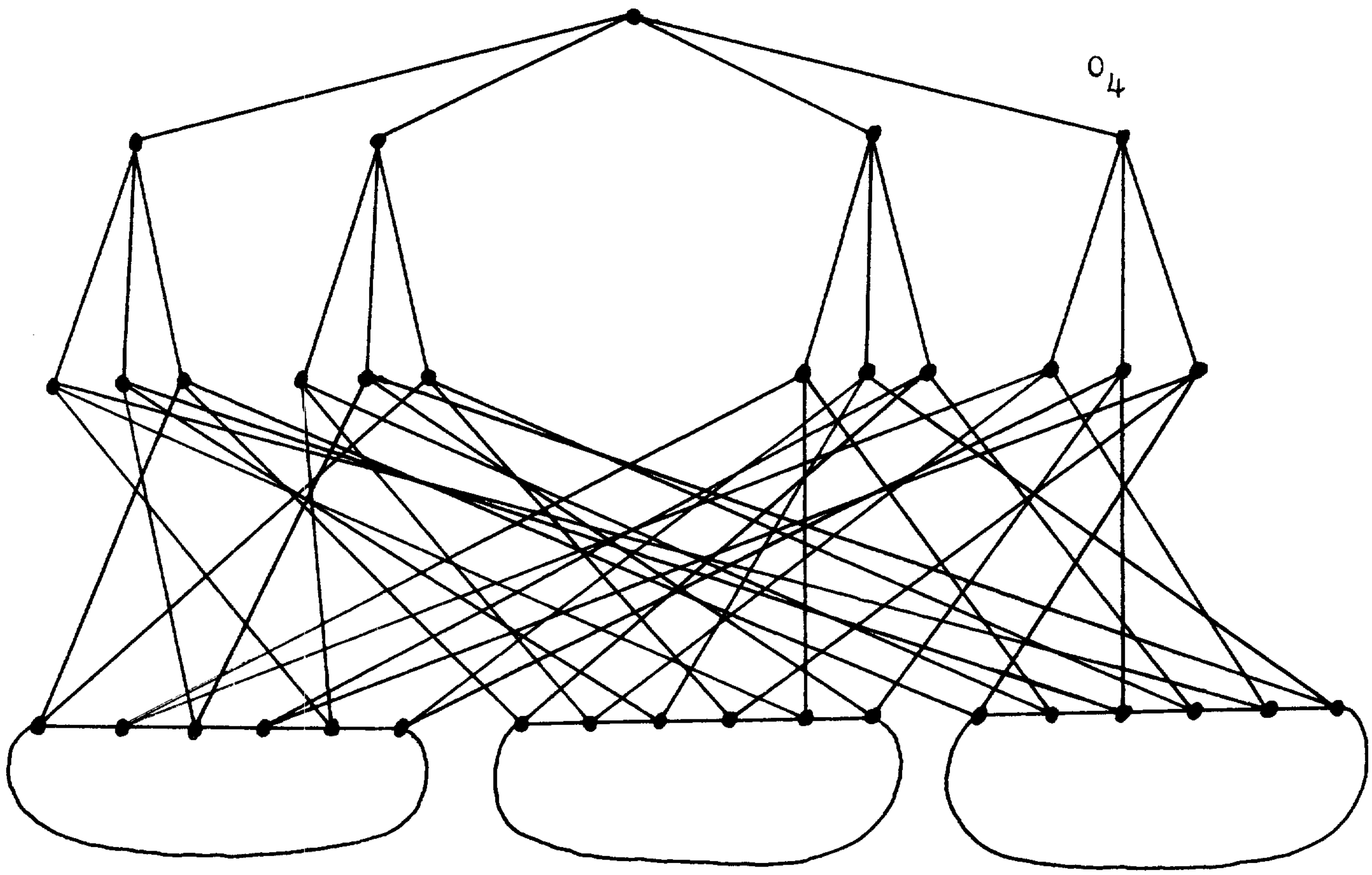


fig. 3c

Further, such is the bitterness of recrimination which follows even these matches, that it has proved necessary to rule that only one match can be played with the same teams and the same referee. This rule was originally regarded with some misgiving, as it was felt that it might seriously limit the number of matches which could be played. However, a villager who had a head for figures worked out that there are 1386 different ways of splitting the eleven men into two teams of five and a referee. This number is thought to be adequate but not generous, for the footballers of Croam are dedicated men.

But there is a second rule which these men, united by their love of football, but embittered by isolation, have been forced to make in order to keep the peace. No five men will play together as a team more than once on any given day of the week. Therein lies the problem. Can all the possible matches be played under this restriction? Can all the matches be played if Sunday games are not allowed?

The graph theoretic equivalent of this problem may be stated as follows. Let each vertex represent a team, with an edge joining two vertices representing a game played between the two teams. In other words, consider the graph O_6. A simple calculation will show that for O_6, $V = \binom{11}{5} = 462$ and $E = \frac{n}{2}V = 1386$. The questions in the problem can alternately be stated as: 1) Is O_6 edge-7-colorable? (Can all the possible matches be played on the seven days of the week under the restrictions of the problem?) and 2) Is O_6 edge-6-colorable? ("Can all the possible matches be played if Sunday games are not allowed?"). Biggs' originally conjectured that O_n for $n \geq 2$ is not edge-n-colorable.

Research Techniques

Earlier we represented O_n as the ordered pair (V,E), where V = the set of all (n - 1)-element subsets of a (2n - 1)-element set A, and E = the set of all unordered pairs of disjoint subsets from V. This abstract definition can be put into a numerical form which can be more easily manipulated.

Denote the (2n - 1)-element set A ($n > 1$) by the class of (2n - 1)-digit binary numbers. Let V be the set of subsets of A having exactly n - 1 ones and n zeros. Then, in this representation, two vertices in V represent disjoint subsets of A and hence are joined by an edge in E if and only if their sum (base two) is a (2n - 1)-digit binary number having exactly 2n - 2 ones and one zero.

Binary representation has its advantages and its disadvantages. It is clearly an advantage in that it allows us to work with sequences of numbers to represent sequences of vertices, as in an H-circuit, rather than sequences of sets. Binary addition is very simple, so that it takes only a moment to determine whether or not two vertices represent disjoint subsets of A. This is an obviously easier task than determining if two abstract sets are disjoint. Binary representation is also a link between the graph theorist and the computer. As graph theory continues to expand, greater emphasis will be placed upon computer and algorithmic techniques in graph theory research and problem-solving. A primary disadvantage of binary numbers is their bulky size. To represent a vertex of O_6 as a binary number requires the use of an eleven-digit number. As the degree increases, so does the amount of work in writing long sequences of vertices. An H-circuit in O_6 would involve 462 eleven-digit binary numbers! Because of this inconvenience, it is often easier when working by hand to convert the binary numbers to base ten.

Some attempts have been made to use binary arrays (adjacency matrices) representing subgraphs of O_n in this research, where a one represents an edge and a zero represents the absence of an edge. This technique has yet to give much information about O_n. However, it might be of future help when considered along with factorization of O_n, and perhaps when related to variations of Menger's Theorem (12), such as Kőnig's variation (8), or Hall's Theorem (5) mentioned below.

As of this date, no one has been successful in determining those n for which O_n has H-circuits. Both H-circuits and 1-factors are of importance in the O_n problem and have been avenues of research in order to develop them as tools. The 1-factor (indeed, n-factorization in general) is closely connected to Hall's Theorem, sometimes called the Marriage Theorem. This theorem states the conditions necessary and sufficient to guarantee the existence of "a system of distinct representatives for a family of distinct sets (6)." The application of Hall's Theorem to the O_n problem in an endeavor to factor O_n has been a major thrust of this research.

Results

The answer to Biggs' first question is immediate: all the possible matches can be played under the restrictions of the problem, using all seven days of the week. This follows from a powerful theorem due to Vizing(18) which implies that $\chi_e(O_n) \leq n + 1$. The result also follows from a theorem by Szekeres and Wilf (16) which states that $\chi_v(O_n) \leq 1 + \max \delta(O_n')$ (where O_n' represents an induced subgraph of O_n) $= 1 + n$, since the maximum degree of all the induced subgraphs of O_n is n. Thus O_6 is edge-7-colorable.

An H-circuit has two edges at each of its vertices, and hence the number of edges in an H-circuit must equal the number of vertices in the graph. For O_n, $V = \frac{2}{n}E$ = the number of edges in an H-circuit in O_n. Thus O_n contains at most $[\![\frac{n}{2}]\!]$ edge-disjoint H-circuits, where $[\![\ ]\!]$ denotes the greatest integer function. If O_n can be so decomposed into edge-disjoint H-circuits, and if V is even, then $\chi_e(O_n) = n$ (3). However, if V is odd, then, since $2E = nV$, $\frac{E}{n} = \frac{1}{2}V$ = the average number of edges per color in an edge-n-coloration of O_n, and it follows that at least $\frac{1}{2}V + \frac{1}{2}$ edges have the same color. These edges have $V + 1$ endpoints, so that at least two called the Marriage Theorem. This theorem states the conditions necessary and sufficient to guarantee the existence of "a system of distinct representatives for a family of distinct sets (6)." The application of Hall's Theorem to the O_n problem in an endeavor to factor O_n has been a major thrust of this research.

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Meredith and Lloyd (4) have independently shown that O_4 is the union of two edge-disjoint H-circuits, and have settled O_5 and O_6 by demonstrating that O_5 is the union of two edge-disjoint H-circuits and a 1-factor, and that O_6 is the union of three edge-disjoint H-circuits. Szekeres has also settled O_5, independently. Thus Biggs' second question is also answered in the affirmative, and his conjecture is proven false. The current conjecture for O_n in general is that O_n consists of the union of exactly $\frac{n}{2}$ edge-disjoint H-circuits (and exactly one 1-factor, if n is odd) and is thus edge-n-colorable for all $n \neq 2^k$, n 3. In addition, Owens (15) has proven that O_n is 1-factorable for all $n \neq 2^k$. Thus the research on the O_n problem has progressed considerably, though it is far from a general solution.

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Table 1
V for O_4

7
11
13
14
19
21
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25
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56
67
69
70
73
74
76
81
82
84
88
97
98
100
104
112

Table 2
H-circuits for O_4

Red	Black
7	7
56	88
67	35
28	28
98	97
21	22
42	41
81	82
38	37
88	26
37	100
74	19
52	76
11	50
100	13
25	98
70	25
41	38
84	73
35	52
76	67
49	44
14	81
97	14
26	112
69	11
50	84
73	42
22	69
104	56
19	70
44	49
82	74
13	21
112	104
7	7

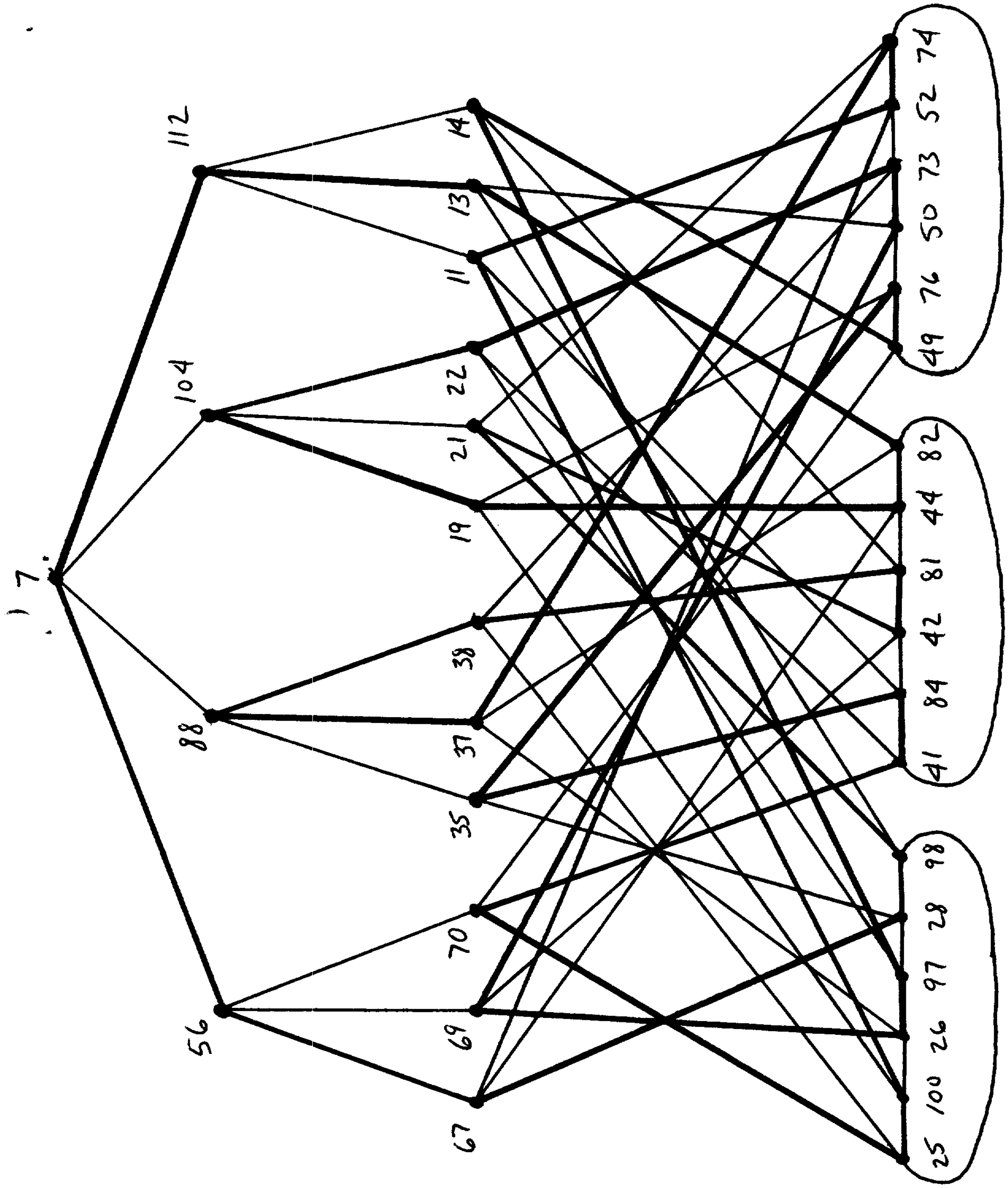

fig. 4 Decomposition of O_4

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