A CONVERGING TELESCOPIC PRODUCT

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ABSTRACT. The key result of this paper is a sequence of products which converges to a number of the form m^r, where m is an integer such that $m \geq 2$ and r is rational. Before the key theorem is presented, the reader is given a brief summary of product sequences and the techniques used to treat them. Then, after the statement and proof of the key theorem, the result is generalized in a corollary to encompass sequences converging to numbers of the form q^r, where both q and r are rational.

Before proceeding to the statement of the result, let us review some of the basic conventions and procedures used to evaluate infinite products. One finds that the development of ideas concerning infinite products is intentionally analogous to the development of infinite series. To begin, here is the definition of a sequence of partial products.

Definition. Given a sequence of factors $\{f_j\}$, we define its sequence of partial products, $\{p_k\}$, by

$$p_k = \prod_{j=1}^{k} f_j = f_1 \cdot f_2 \cdot \cdots \cdot f_k.$$

If the sequence $\{p_k\}$ tends towards some finite number $P \neq 0$, we say the product converges to the limit P. We can then denote the limit of the partial products as the infinite product

$$\prod_{j=1}^{\infty} f_j = \lim_{k\to\infty} \prod_{j=1}^{k} f_j = \lim_{k\to\infty} p_k = P.$$

Otherwise, the product is said to diverge.

Immediately we see that if a product converges, all the factors $\{f_j\}$ must be nonzero since its limit cannot be zero. Without a loss of generality, it follows that

The ideas for this paper are the culmination of an investigation which curiously began by naïvely attempting to "divide all of the even numbers by all of the odd numbers." Needless to say, this thesis has matured substantially. At this time, I would like to thank Ball State's Honors College for giving me the opportunity to freely express my aspiring mathematical ideas, and I would like to extend my deep appreciation to Dr. John W. Emert, who has supported, encouraged, and guided this project long before either of us realized it would mature so nicely. Thank You .

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$f_j > 0$ $(j = 1, 2, \dots)$, since this could be false for only finitely many factors in the case that the limit $P \neq 0$.

The reasons for these conditions may at first seem obscure, but they are easily motivated by noting that when they are followed, one can take complete advantage of the existing theorems about infinite series by relating products and series through the logarithmic transformation

$$\ln \prod_{j \in J} f_j = \sum_{j \in J} \ln f_j.$$

Continuing the analogy, it is interesting to note that a product is said to converge *absolutely* if all but finitely many of its factors are either less than 1 or greater than 1, for then the series of logarithms will converge absolutely. Pulling all of these details together is the following lemma, which will be helpful in proving the results of this paper. For a proof of this lemma and a more thorough discussion of infinite products, see [1] or [5].

Lemma. *Given a sequence $\{a_j\}$, the following conditions ensure that the product $\prod(1 + a_j)$ converges:*

(1) *$a_j \neq -1$ $(j = 1, 2, \dots)$, so there are no factors of zero,*
(2) *either $a_j \geq 0$ or $a_j \leq 0$ for all j sufficiently large, and*
(3) *the series $\sum a_j$ converges (absolutely, by the conditions imposed).*

Finally we are prepared to state and prove the key result.

Theorem. *If m is an integer such that $m \geq 2$, and r is a rational number, then for any constant c,*

$$\lim_{k \to \infty} \prod_{j=k+1}^{mk} \frac{j + r + c}{j + c} = m^r.$$

Proof.

Comment. The proof will be broken into three main steps: proving the existence of a limit, proving that the limit is independent of the constant c, and calculating the limit to be m^r.

Definition. Let $\{T_k\}$ denote the telescopic sequence of products above. That is, let $\{T_k\}$ be defined by

$$T_k = \prod_{j=k+1}^{mk} \frac{j + r + c}{j + c} = \prod_{j=k+1}^{mk} (1 + \frac{r}{j + c}).$$

(I use the word telescopic to describe $\{T_k\}$ since another sequence could easily be constructed so that every time one of its partial products is evaluated, all but the final $(k + 1)$-st to (mk)-th terms cancel each other.)

Step 1. *Proof of convergence to some limit T.*

Applying the above lemma to the product sequence $\{T_k\}$, we easily see that the first two conditions hold for sufficiently large k. To see that the third condition holds (that $\sum \frac{r}{j+c}$ converges), it suffices to show that the following telescopic "tail end" of the harmonic series converges. That is, it suffices to show that the sequence $\{h_k\}$ converges, where

$$h_k = \sum_{j=k+1}^{mk} \frac{1}{j}.$$

First, $\{h_k\}$ is bounded. This can be shown by noting that the '$(m-1)k$' terms of each particular h_k are bounded below by $1/mk$ and above by $1/k$. Thus, $\{h_k\}$ is bounded above and below, since for all k,

$$\frac{m-1}{m} = [(m-1)k](\frac{1}{mk}) \leq h_k \leq [(m-1)k](\frac{1}{k}) = m-1.$$

Also,$\{h_k\}$ is increasing since

$$\begin{aligned} h_{k+1} - h_k &= \sum_{j=k+2}^{mk+m} \frac{1}{j} - \sum_{j=k+1}^{mk} \frac{1}{j} \\ &= \sum_{j=mk+1}^{mk+m} \frac{1}{j} - \frac{1}{k+1} \\ &\geq [m](\frac{1}{mk+m}) - \frac{1}{k+1} = 0. \end{aligned}$$

Because the final condition of the lemma holds ($\{h_k\}$ converges since it is bounded and increasing), $\{T_k\}$ must converge to some T, as proposed.

Step 2. *Proof that the constant c in $\{T_k\}$ is arbitrary.*

Comment. The proof of this step will be broken into two cases by first supposing $r \geq 0$, and then supposing $r < 0$.

Case 1. $r \geq 0$.

Definition. Let the sequence $\{S_k\}$ be defined in the following way:

$$S_k = \prod_{j=k+1}^{mk} \frac{j+r+d}{j+d}.$$

Note that the only difference between $\{T_k\}$ and $\{S_k\}$ are their respective constants c and d.

Now, since the respective limits S and T of $\{S_k\}$ and $\{T_k\}$ exist (Step 1), to show that the constants c and d are arbitrary, it suffices to show either of the following:

(1) $\lim_{k\to\infty} T_k = \lim_{k\to\infty} S_k$, or

(2) $\lim_{k\to\infty} \ln \frac{T_k}{S_k} = 0.$

Expanding and simplifying T_k/S_k, we have

$$\frac{T_k}{S_k} = [\prod_{j=k+1}^{mk} \frac{j+r+c}{j+c}]/[\prod_{j=k+1}^{mk} \frac{j+r+d}{j+d}]$$
$$= \prod_{j=k+1}^{mk} [\frac{(j+r+c)(j+d)}{(j+c)(j+r+d)}]$$
$$= \prod_{j=k+1}^{mk} \frac{j^2+j(r+c+d)+cr+cd+(dr-cr)}{j^2+j(r+c+d)+cr+cd}$$
$$= \prod_{j=k+1}^{mk} [1+\frac{r(d-c)}{j^2+j(r+c+d)+cr+cd}].$$

Now since $r(d-c) \geq 0$, in the case that $r \geq 0$ and $d > c$, from the work above, we have that a lower bound for T_k/Sk is 1. An upper bound for the limit of T_k/S_k is found by taking the natural logarithm of T_k/S_k, and appealing to the fact that for $x > -1$, $\ln(1+x) \leq x$. Thus,

$$0 = \ln 1 \leq \lim_{k\to\infty} \ln \frac{T_k}{S_k} = \lim_{k\to\infty} \sum_{j=k+1}^{mk} \ln [1+\frac{r(d-c)}{j^2+j(r+c+d)+cr+cd}]$$
$$\leq \lim_{k\to\infty} \sum_{j=k+1}^{mk} [\frac{r(d-c)}{j^2+j(r+c+d)+cr+cd}]$$
$$\leq \lim_{k\to\infty} [(m-1)k][\frac{r(d-c)}{k^2+k(r+c+d)+cr+cd}]$$
$$= 0,$$

so that for any constants c and d, we have that

$$T = \lim_{k\to\infty} T_k = \lim_{k\to\infty} S_k = S.$$

Hence, for the case that $r \geq 0$, it has been shown that the constants are arbitrary since they do not affect limits of $\{T_k\}$ and $\{S_k\}$.

Case 2. $r < 0$.

It is interesting to see how this case quickly follows from the previous case. To see this, choose some $r < 0$ and let T be the value of the following limit:

$$T = \lim_{k\to\infty} \prod_{j=k+1}^{mk} \frac{j+|r|}{j}.$$

Then let c be given, and choose c' so that $c = |r| + c'$. Then we see that

$$\lim_{k\to\infty} \prod_{j=k+1}^{mk} \frac{j+r+c}{j+c} = \lim_{k\to\infty} \prod_{j=k+1}^{mk} \frac{j+c'}{j+|r|+c'} = \lim_{k\to\infty} \frac{1}{[\prod_{j=k+1}^{mk} \frac{j+|r|+c'}{j+c'}]} = \frac{1}{T}.$$

Thus, the limit of $\{T_k\}$ is independent of c when $r < 0$ since T is independent of c' (Case 1).

Step 3. *Calculating the limit.*

The crux of the following argument is this: we could either try to directly calculate the limit of $\{T_k\}$ to be m^r, or we could let $r = a/b$ and then show that

$$\lim_{k\to\infty} (T_k)^b = m^a.$$

Since we know that the limit of $\{T_k\}$ exists and is independent of the constant c, we could achieve this by multiplying together 'b' sequences that are defined similarly, except for the fact that they have different respective constants. Let us denote these 'b' sequences by the family $\{T_{k,i}\}_{i=1}^{b}$. We will find that if we let the respective constants of the $T_{k,i}$'s be $\frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b}$, and 1, all but '$a$' terms of the product of the sequences $\{T_{k,i}\}_{i=1}^{b}$ will telescopically cancel each other. Each of these 'a' terms will approach m in the limit, so that we will get the desired result.

That is, let each element of the set of sequences $\{T_{k,i}\}_{i=1}^{b}$ be defined by

$$T_{k,i} = \prod_{j=k+1}^{mk} \frac{j + r + \frac{i}{b}}{j + \frac{i}{b}} = \prod_{j=k+1}^{mk} \frac{j + \frac{a+i}{b}}{j + \frac{i}{b}}.$$

Now, since each of these sequences have the same limit (Step 2), it must be that

$$\lim_{k\to\infty} \prod_{i=1}^{b} T_{k,i} = \lim_{k\to\infty} \prod_{i=1}^{b} [\prod_{j=k+1}^{mk} \frac{j + \frac{a+i}{b}}{j + \frac{i}{b}}] = T^b,$$

where T is the common limit of all the sequences.

But since for each particular, finite k, we can interchange the order of multiplication in the above products, we have that

$$\begin{aligned}\prod_{i=1}^{b} T_{k,i} &= [\prod_{j=k+1}^{mk} \frac{j + \frac{a+1}{b}}{j + \frac{1}{b}}][\prod_{j=k+1}^{mk} \frac{j + \frac{a+2}{b}}{j + \frac{2}{b}}] \cdots [\prod_{j=k+1}^{mk} \frac{j + 1 + \frac{a}{b}}{j + 1}] \\ &= \prod_{j=k+1}^{mk} [(\frac{j + \frac{a+1}{b}}{j + \frac{1}{b}})(\frac{j + \frac{a+2}{b}}{j + \frac{2}{b}}) \cdots (\frac{j + 1 + \frac{a}{b}}{j + 1})] \\ &= (\frac{mk + 1 + \frac{1}{b}}{k + 1 + \frac{1}{b}}) \cdot (\frac{mk + 1 + \frac{2}{b}}{k + 1 + \frac{2}{b}}) \cdots (\frac{mk + 1 + \frac{a}{b}}{k + 1 + \frac{a}{b}}),\end{aligned}$$

in which only the first 'a' terms in the denominator and the last 'a' terms in the numerator remain. From here it is easily seen that each of these fractions approach m as $k \to \infty$, so it follows that

$$T^b = \lim_{k\to\infty} \prod_{i=1}^{b} T_{k,i} = m^a,$$

which proves

$$\lim_{k\to\infty} \prod_{j=k+1}^{mk} \frac{j + r + c}{j + c} = m^r. \quad \square$$

Corollary. *If q and r are rational numbers such that $q = m/n > 1$,then for any constant c,*

$$\lim_{k\to\infty} \prod_{j=nk+1}^{mk} \frac{j+r+c}{j+c} = q^r.$$

Proof.

First note that the case of $n = 1$ is the above theorem. Interestingly enough, also note that Wallis' Product for π can be used to easily prove the special case that $r = 1/2$. (Hint: Let $c = -\frac{1}{2}$ and square the factors in the above sequence.) For more discussion about Wallis' Product and π, an interesting source is [2].

Now, if we let $\{R_k\}$ represent the above sequence in the case that $n \geq 2$, we can interpret $\{R_k\}$ as the *ratio* of two of the telescopic sequences in the theorem above. That is,

$$R_k = \prod_{j=nk+1}^{mk} \frac{j+r+c}{j+c} = [\prod_{j=k+1}^{mk} \frac{j+r+c}{j+c}]/[\prod_{j=k+1}^{nk} \frac{j+r+c}{j+c}].$$

It immediately follows from the theorem that

$$\lim_{k\to\infty} \prod_{j=nk+1}^{mk} \frac{j+r+c}{j+c} = \frac{m^r}{n^r} = q^r. \quad \square$$

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