Resources for Advanced Logic Students

Honors Thesis
(HONRS 499)

By

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Abstract

This paper is designed as a resource for students in introductory logic classes, such as PHIL 200, which readily grasp the normal material and desire an expanded study of related material. The paper starts with elementary logical notation, treats two connectives not usually covered in introductory classes, and discusses other possible connectives. It then gives a method for generating and examining tables based on connectives which serve as a bridge into lattice theory, a form of abstract algebra. Lattice theory is discussed and a lattice is generated for a two-variable model. Next, set theory is introduced and its relation to both logic and lattice theory is discussed. Finally, some shorter problems are included for those not wishing to put forth the effort or expend the time necessary to cover the lattice theory material. An appendix is included with suggestions on where to find more of these shorter problems.

Introduction

This work is intended for use by instructors in logic classes or advanced students in logic who desire more stimulation than provided in the classroom setting. This work can be used as a resource for instructors in supplying challenging and diverse problems for advanced students. It may also be used by advanced students to broaden their "logical horizons" beyond the scope of typical classroom exercises. A background in propositional logic, (such as PHIL 200) is presupposed in this work.

The work begins with some basic propositional logic, it then treats the stroke and dagger operators. Possible connectives are discussed and illustrative tables constructed. From the tables lattice theory, a branch of abstract algebra, is introduced. Next, set theory and its relation to both logic and lattice theory is explored. For those not wishing to pursue such an in-depth exploration, some smaller puzzles and paradoxes have been included and appendix B contains an annotated bibliography of sources for more problems of similar type.

Propositional Calculus

For the present work, the following calculus will be employed. {p,q,r,s} for propositional variables, {(,)} as grouping indicators, and {&,v,⊃,≡,↓,|,~} as truth-functional connectives.

Well-Formed Formula (wff)

1. Any propositional variable is a wff.
2. Any propositional variable preceded by a negation is a wff.
3. Any two propositional variables joined by a connective and enclosed in parenthesis is a wff.[1]
4. Nothing else is a wff.[2]

Rules of Inference

The following rules of inference will be employed:[3]

Conjunction (Con)
From any two lines-
p
and
q
one may infer-
p & q

Addition (Add)
From any line-
p
one may infer-
p v q

Simplification (Simp)
From any line-
p & q
one may infer-
p

Disjunctive Syllogism (DS)
From any two lines-
p v q
and
~p
one may infer-
q

Excluded Middle Introduction (EMI)
At any point one may introduce-
$p \vee \sim p$

Modus Ponens (MP)
From any two lines-
$p \supset q$
and
p
one may infer-
q

Modus Tollens (MT)
From any two lines-
$p \supset q$
and
$\sim q$
one may infer-
$\sim p$

Hypothetical Syllogism (HS)
From any two lines-
$p \supset q$
and
$q \supset r$
one may infer-
$p \supset r$

Constructive Dilemma (CD)
From any three lines-
$p \vee q$
$p \supset r$
and
$q \supset s$
one may infer-
$r \vee s$

Idempotence (Idem)
From any line-
$p \vee p$
one may infer-
p

Rules of Replacement

The following Rules of Replacement will be employed.[4]

Commutation (Com)
$p \& q :: q \& p$
$p \vee q :: q \vee p$

Distribution (Dist)
$p \&(q \vee r) :: (p \& q) \vee (p \& r)$
$p \vee(q \& r) :: (p \vee q) \& (p \vee r)$

Association (Assoc)
$p \&(q \& r) :: (p \& q) \& r$
$p \vee(q \vee r) :: (p \vee q) \vee r$

DeMorgan (Dem)
$\sim p \& \sim q :: \sim(p \vee q)$
$\sim p \vee \sim q :: \sim(p \& q)$

Double Negation (DN)
$p :: \sim\sim p$

Transposition (Trans)
$p \supset q :: \sim q \supset \sim p$

Exportation (Exp)
$(p \& q) \supset r :: p \supset(q \supset r)$

Material Implication (MI)
$p \supset q :: \sim p \vee q$

Equivalence (Equiv)
$p \equiv q :: (p \supset q) \& (q \supset p)$
$p \equiv q :: (p \& q) \vee (\sim p \& \sim q)$

Truth Tables

The connectives will be defined in terms of truth tables as follows-

p	q	(p & q)	(p v q)	(p ⊃ q)	(p ≡ q)	(p | q)	(p ↓ q)
T	T	T	T	T	T	F	F
T	F	F	T	F	F	T	F
F	T	F	T	T	F	T	F
F	F	F	F	T	T	T	T

Sheffer Stroke and Dagger

These are additional connectives that advanced students may be encouraged to experiment with.

Rules for Sheffer Stroke "|"[5]

Several deductive rules and rules of replacement may be derived from the knowledge presented above. Students should be encouraged to derive them themselves if possible.

Commutation- p | q :: q | p
Proof: truth table.

Equivalence- p | q :: ~(p & q)
Proof: truth table.

Addition- From any line ~p, one may derive p | q.
Proof:

1.	~p	Premise
2.	~p v ~q	1, Add
3.	~(p & q)	2, Dem
∴4.	p | q	3, Equiv.

Stroke Elimination- From any two lines p | q, and p, one may infer ~q.
Proof:

1.	p | q	Premise
2.	p	Premise
3.	~(p & q)	1, Equiv
4.	~p v ~q	3, Dem
5.	~~p	2, DN
∴6.	~q	4,5, DS

Conversion- (p | q) | r :: r ⊃(p ⊃ ~q)
Proof:

1.	(p | q) | r	Premise
2.	~((p | q) & r)	1, Equiv
3.	~(p | q) v ~r	2, Dem
4.	~(p & q) v ~r	3, Equiv
5.	(~p v ~q) v ~r	4, Dem
6.	~r v(~p v ~q)	5, Com
7.	r ⊃(~p v ~q)	6, MI
8.	r ⊃(p ⊃ ~q)	7, MI

Rules for the Dagger "↓"

As with the Sheffer Stroke, several rules for this connective can be derived.

Commutation- p ↓ q :: q ↓p
Proof: truth table.

Equivalence- p ↓ q :: ~(p v q)
Proof: truth table.

Simplification- From any line p ↓ q, one may infer ~p,
Proof:

1.	p ↓ q	Premise
2.	~(p v q)	1, Equiv
3	~p & ~q	2, Dem
∴4.	~p	3, Simp

Conversion- (p ↓ q)↓ r :: (p v q)& ~r
Proof:

1.	(p ↓ q)↓ r	Premise
2.	~((p ↓ q)v r)	1, Equiv
3.	~(p ↓ q) & ~r	2, Dem
∴4.	(p v q) & ~r	3, DN,Equiv

Sufficiency of Stroke and Dagger.

Both the Stroke and Dagger are capable of representing all 16 truth value combinations which a two-variable model may generate. That is, for any combination of p and q, the resulting proposition may be equivalently symbolized using only the stroke or dagger. The sixteen possible truth value combinations of a two variable model are given below. This is an important result, for a propositional calculus without the Stroke or Dagger requires a minimum of two connectives: the tilda and another two place connective, such s conjunction.

p	q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

The representing of all of these using only the Stroke or Dagger can be a challenging exercise for students. Here is one fairly compact set of solutions. (TVC# refers to the truth value combination as given in the above table. These numbers will be used throughout the remainder of the text.)

<table>
<tr><th>TVC#</th><th>Notation (For Stroke)</th></tr>
<tr><td>1</td><td>p|(p|p)</td></tr>
<tr><td>2</td><td>(p|p)|(q|q)</td></tr>
<tr><td>3</td><td>q|(p|p)</td></tr>
<tr><td>4</td><td>p</td></tr>
<tr><td>5</td><td>p|(q|q)</td></tr>
<tr><td>6</td><td>q</td></tr>
<tr><td>7</td><td>(p|q)|((p|p)|(q|q))</td></tr>
<tr><td>8</td><td>(p|q)|(p|q)</td></tr>
<tr><td>9</td><td>p|q</td></tr>
<tr><td>10</td><td>((p|q)|((p|p)|(q|q)))|((p|q)|((p|p)|(q|q)))</td></tr>
<tr><td>11</td><td>q|q</td></tr>
<tr><td>12</td><td>(p|(q|q))|(p|(q|q))</td></tr>
<tr><td>13</td><td>p|p</td></tr>
<tr><td>14</td><td>(q|(p|p))|(q|(p|p))</td></tr>
<tr><td>15</td><td>((p|p)|(q|q))|((p|p)|(q|q))</td></tr>
<tr><td>16</td><td>(p|(p|p))|(p|(p|p))</td></tr>
</table>

<table>
<tr><th>TVC#</th><th>Notation (For Dagger)</th></tr>
<tr><td>1</td><td>(p↓(p↓p)↓(p↓(p↓p))</td></tr>
<tr><td>2</td><td>(p↓q)↓(p↓q)</td></tr>
<tr><td>3</td><td>(p↓(q↓q))↓(p↓(q↓q))</td></tr>
<tr><td>4</td><td>p</td></tr>
<tr><td>5</td><td>((p↓p)↓q)↓((p↓p)↓q)</td></tr>
<tr><td>6</td><td>q</td></tr>
<tr><td>7</td><td>((p↓p)↓q)↓(p↓(q↓q))</td></tr>
<tr><td>8</td><td>(p↓p)↓(q↓q)</td></tr>
<tr><td>9</td><td>((p↓p)↓(q↓q))↓((p↓p)↓(q↓q))</td></tr>
<tr><td>10</td><td>(p↓q)↓((p↓p)↓(q↓q))</td></tr>
<tr><td>11</td><td>q↓q</td></tr>
<tr><td>12</td><td>(p↓p)↓q</td></tr>
<tr><td>14</td><td>p↓p</td></tr>
<tr><td>15</td><td>p↓q</td></tr>
<tr><td>16</td><td>(p↓p)↓p</td></tr>
</table>

Tables

Function tables can be generated for all the connectives, even those which are not treated above, for each truth value combination is the result of a function. To construct the tables, draw a 16 by 16 grid and number the outside as shown

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1																
2																
3																
4																
5																
6																
7																
8																
9																
10																
11																
12																
13																
14																
15																
16																

Then calculate which truth value combination would be produced by using the two truth value combinations indicated on the grid, for the function being analyzed. To do so, establish which function is to be represented. Next, assume that there is a function which produces that truth value combination when "p β q" is stated, and "β" is the connective in question. (Consider the top row, or x axis,as p). Then, insert the truth value combination denoted by the number on the x axis for p and the truth value combination denoted by the number on the y axis for q. Finally, determine the truth value combination for this arrangement and write its number in the appropriate box.

Note, the sixteen truth value combinations listed on page four are critical to this procedure. Each truth value combination's number serves as shorthand for that combination in the tables. For example, truth value combination five, is shorthand for a proposition (let's call it x) with the truth conditions as follows:

p	q	x
T	T	T
T	F	F
F	T	T
F	F	F

Let us now return to our project and illustrate the procedure with an example. To construct a grid for the first 4 truth value combination's using function 2 (exclusive disjunction), the table would be as follows.

	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

In this example, the function "v" is being plotted. For coordinate 1,1, the following was used.

p(1)	q(1)	p v q
T	T	T
T	T	T
T	T	T
T	T	T

Since the resulting truth value combination is identical to truth value combination one, the numeral one is placed in 1,1.

These tables, while there construction is an exercise in itself, can serve several functions once constructed. The can be used as a bridge into lattice theory, a branch of abstract algebra with similarities to logic and set theory (see below), and to many tasks in illustrating and examining the properties of the connectives themselves.

Derived Properties of tables-

There are several properties that apply to all sixteen tables (See appendix A for a complete listing). They can be ascertained by examining them once they are constructed. Some of them are given here.

I. For any function containing two T's and two F's, it's f-table will be an inverted mirror image of itself when divided in half vertically.

II. For each function there is, there is a distinct and unique table that can be generated by that function.

III. Tables 1,2,7,8,9,10,15,& 16 may be mirrored when halved along a diagonal from the upper left hand corner to the lower right hand corner.

IV. Tables 1,3,5,7,10,12,14,& 16 may be mirrored when halved along a diagonal from the upper right hand corner to the lower left hand corner.

V. Any table exchanged according to the following table will also have all of it's values exchanged in the same fashion. (Moving either top to bottom or bottom to top.)

1	2	3	4	5	6	7	8
16	15	14	13	12	11	10	9

VI. For any f-table, the proportion of values 1 through 8 to the values 9 through 16 will be the same as the proposition of T's to F's in the matrix for the truth value combination of that same number.

Within the tables, there are also other interesting characteristics. Let us look at the table for function 2 as an example.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
3	1	1	3	3	1	1	3	3	1	1	3	3	1	1	3	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	1	1	1	1	5	5	5	5	1	1	1	1	5	5	5	5
6	1	2	1	2	5	6	5	6	1	2	1	2	5	6	5	6
7	1	1	3	3	5	5	7	7	1	1	3	3	5	5	7	7
8	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
9	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	9
10	1	2	1	2	1	2	1	2	9	10	9	10	9	10	9	10
11	1	1	3	3	1	1	3	3	9	9	11	11	9	9	11	11
12	1	2	3	4	1	2	3	4	9	10	11	12	9	10	11	12
13	1	1	1	1	5	5	5	5	9	9	9	9	13	13	13	13
14	1	2	1	2	5	6	5	6	9	10	9	10	13	14	13	14
15	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 2 (Function 2)

Using the figure one as a guide, which applies to all 16 f-tables, we can analyze and distinguish the characteristics of the different functions.

Using figure one, we can note the following characteristics of the table of function 2: a={1→1}, b={1↓16}, c={1↓1}, d={1→16}, e={1↘16} and f={1↙1}. All arrows merely indicated direction of reading. For example, {1↘16} denotes that in the upper left hand corner (position 1,1) is the numeral one and advancing on a diagonal to the lower right hand corner (position 16,16) are the numbers ascending to 16. {1,2,3,4...14,15,16}. We can show this for all 16 function tables, and each table has a unique combination for {a,b,c,d,e,f}. Here is a complete listing:

	1		2		3		4
a	1...1	a	1...1	a	1...16	a	1...16
b	1...1	b	1...16	b	16...1	b	1...1
c	1...1	c	1...1	c	1...1	c	1...16
d	1...1	d	1...16	d	1...1	d	16...1
e	1...1	e	1...16	e	1...1	e	1...1
f	1...1	f	1...1	f	16...1	f	1...16

	5		6		7		8
a	1...1	a	1...1	a	1...16	a	1...16
b	1...1	b	1...16	b	16...1	b	16...16
c	1...16	c	1...16	c	1...16	c	1...16
d	16...1	d	16...16	d	16...1	d	16...16
e	1...1	e	1...16	e	1...1	e	1...16
f	1...16	f	1...16	f	16...16	f	16...16

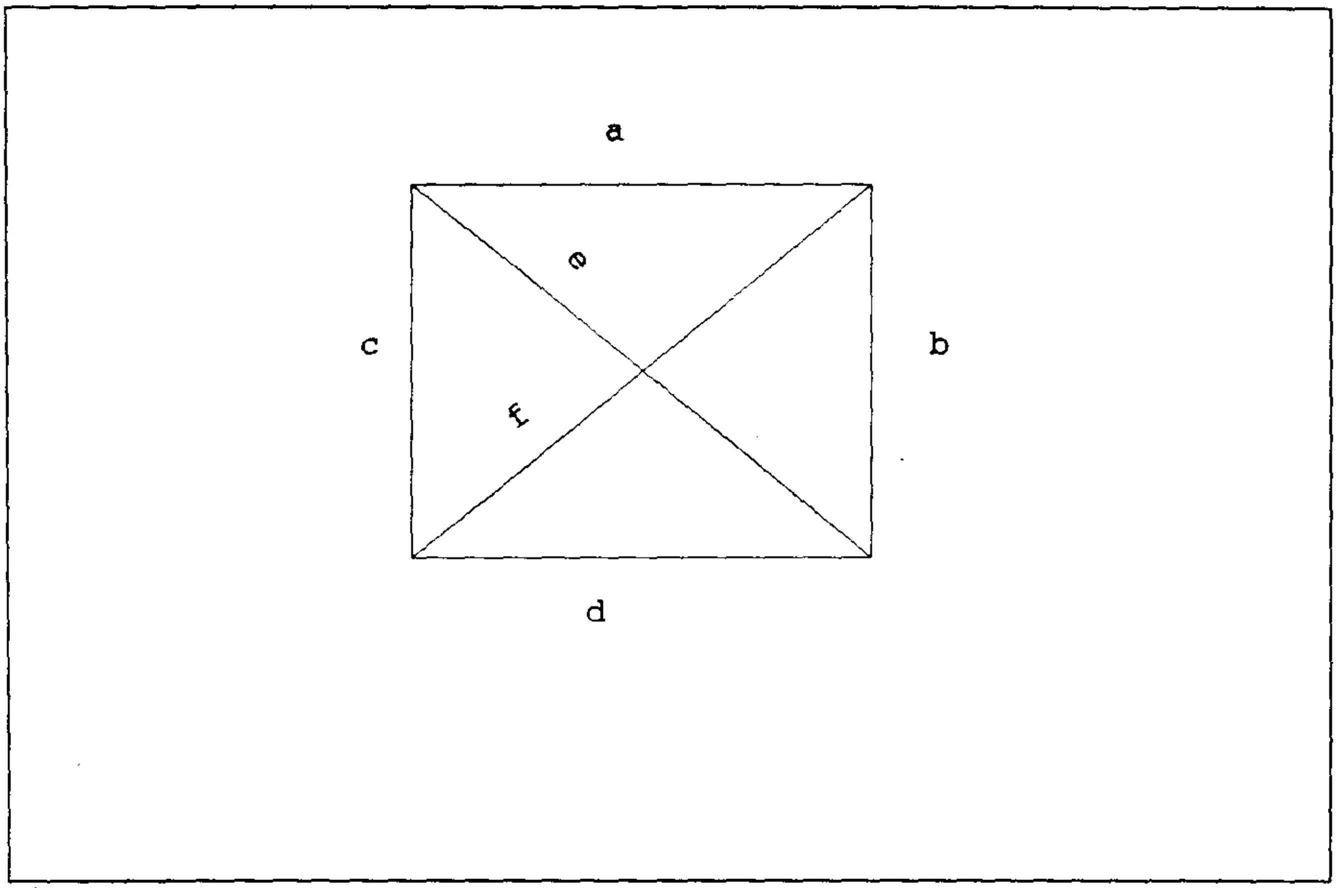

Figure One

	9		10		11		12
a	16...1	a	16...1	a	16...16	a	16...16
b	1...1	b	1...16	b	16...1	b	16...16
c	16...1	c	16...1	c	16...1	c	16...1
d	1...1	d	1...16	d	1...1	d	1...16
e	16...1	e	16...16	e	16...1	e	16...16
f	1...1	f	1...1	f	16...1	f	16...1

	13		14		15		16
a	16...1	a	16...1	a	16...16	a	16...16
b	1...1	b	1...16	b	16...1	b	16...16
c	16...16	c	16...16	c	16...16	c	16...16
d	16...1	d	16...16	d	16...1	d	16...16
e	16...1	e	16...16	e	16...1	e	16...16
f	1...16	f	1...16	f	16...16	f	16...16

All of the above information should be derivable by the advanced student in logic.

The tables can also be used to examine "loops of sufficiency", that is, to show how many different truth value combinations each connective, by itself, can represent. This is done by allowing the truth value combinations 3 and 6 (p and q) and n (where n = the truth value combination produce by the function with only p and q.) to be used. Resulting truth value combinations are then calculated by finding the numeral at the positions {3,3 3,6 3,n 6,6 6,n n,n 6,3 n,3 n,6}. The results as recorded and used in generating as many different truth value combinations as possible by the same method. The resulting list will represent the truth value combinations that can be represented using only the connective for that table. For example, function five can represent the following truth value combinations: {1,2,3,5,6}.

If done properly, tables 9 and 15 should be the only ones which give a list of one through sixteen inclusive, for they represent the stroke and dagger functions respectively.

<u>Lattice Theory</u>

What is a lattice? It is a graphical representation of the possible combinations of things in a set. It is best to start with an example.[6]

Let us consider the set of numbers {1,2,3,5,6,10,15,30}. All of these numbers are divisors of thirty. Now we know that not all of these numbers are divisors of each other. We can represent these relationships graphically in a lattice (figure 2).

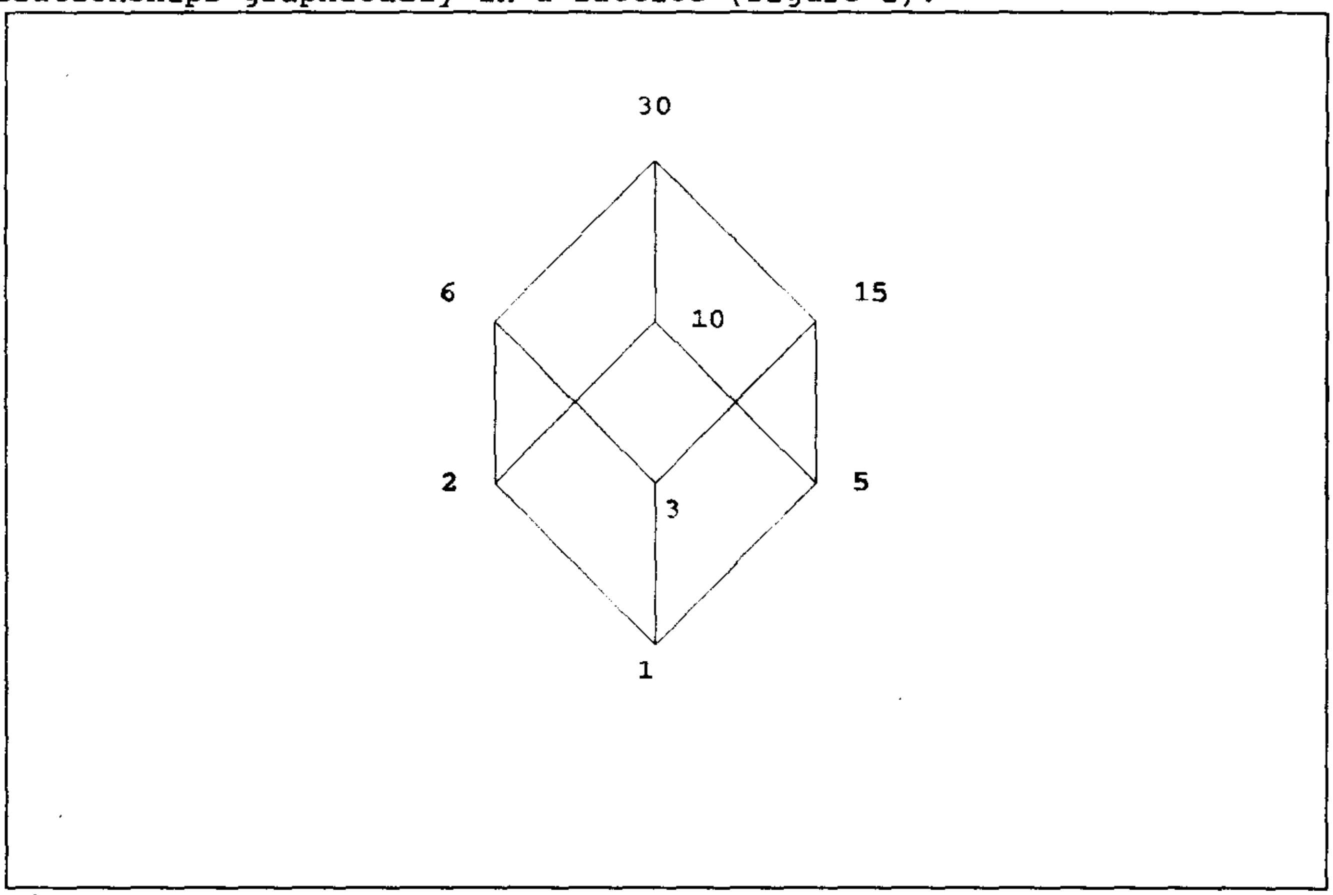

Figure Two

For each line between two points on this lattice, if we work from top to bottom, we notice that all the numbers "contained" (we will formally define this concept later) by the upper number are divisors of the numbers below it. For example, 6 is divisible by 1,2 and 3. These are the only numbers that six contains on this lattice. (It is useful to note here that where two lines cross without a point, they are not connected.) Similarly, 3 is not divisible by 2, 5 or 10.

It will be useful to define formally some concepts at this point. The first is the notion of containment mentioned above, for it is vital to our understanding of lattices.

Let us adopt the notation $x \supseteq y$ for "x contains y." So for figure one $10 \supseteq 2$ could be stated for "10 contains 2."

Let us now put forth three properties of containment.[7]
1. The transitive property. If $x \supseteq y$ and $y \supseteq z$ then $x \supseteq z$. (30 contains 6 and six contains 2, therefore 30 contains 2.)
2. The reflexive property. $x \supseteq x$. (Any number contains itself.)
3. The antisymmetric property. If $x \supseteq y$ then $y \not\supseteq x$ unless x=y. (Its not the case that 6 contains three and 3 contains 6.)

Now let us take any two points on a lattice, for example 2 and 3 on figure one. These two points are both contained by 6 and 30. We will therefore say that 6 and 30 are upper bounds of 2 and 3.[8] Conversely, we can see that 6 and 15 both contain 3 and 1.[9] We will then call 3 and 1 lower bounds of 6 and 15.

With these notions in hand, we can now define two more concepts, least upper bound and greatest lower bound. First, least upper bound. We will say that some point (β) is a least upper bound of two points, x and y, if $\beta \supseteq x$ & $\beta \supseteq y$ and for any point z that is an upper bound of x & y, $z \supseteq \beta$.[10] So, in figure one, we can see that 6 is the least upper bound for 2 and 3, while 30 is not.[11] It is also important to note that due to our reflexive rule, 30 is the least upper bound of 15 and 30.

A point (g) is a greatest lower bound of two points, x and y, when $x \supseteq g$ and $y \supseteq g$ and for any other point z that is a lower bound of x & y, $g \supseteq z$.[12] This is very similar to least upper bound, except it "moves" in the other direction.

We now have all the concepts necessary in order to give a formal definition of a lattice. A lattice satisfies the following conditions.[13]
1. For any point x, $x \supseteq x$.
2. For any two points x and y, if $x \supseteq y$ & $y \supseteq x$ then x=y.
3. For any three points x, y and z, if $x \supseteq y$ & $y \supseteq z$ then $x \supseteq z$.
4. For any two points, they will have exactly one least upper bound.
5. For any two points, they will have exactly one greatest lower bound.

All of this is necessary background to the next step, which ties the above material to propositional logic.

Tables can be constructed showing the least upper bounds and greatest lower bounds for individual lattices. For the example above (figure 2) the table for least upper bound would be as follows.

∪	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	2	2	6	10	6	10	30	30
3	3	6	3	15	6	30	15	30
5	5	10	15	5	30	10	15	30
6	6	6	6	30	6	30	30	30
10	10	10	30	10	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

To the perceptive, this table will look like an abbreviated table for function 2 above. Indeed, that is exactly what it is. The least upper bound function is equivalent to function two. With this knowledge, we can then use table 2 to construct a lattice for our sixteen truth value combinations. This is done by considering numbers one through four and drawing a lattice that satisfies table two as least upper bound for those four truth value combination's. Additional ones are then added in like manner until a lattice is completed. Then, it can be cross-checked by using table 8, as greatest lower bound, which is its equivalent. The resulting lattice (with cosmetic re-arrangement in order to make it symmetrical) is shown in figure three. A more common representation of this lattice is shown in figure four.[14] This is a more demanding exercise, as it requires a fair amount of time, patience, accuracy and paper.

By using the lattice constructed in figure three, there are several properties we can derive. First, the farther up the lattice a number is, the more t's that truth value combination contains. (10 and 7 are in the middle row, there were placed slightly above and below in order to achieve symmetry within the lattice.)

Therefore, those truth value combinations containing four t's, and subsequently being in the top row are {1}, containing 3 are {2,3,5,9}, containing 2 are {4,6,7,10,11,13}, containing one are {8,12,14,15} and containing none are {16}.

It can also be noted that all points on the lattice have four rays. The number of rays projecting upward corresponds to the number of F's in the truth value combination and the number of rays projecting downward correspond to the number of t's within the truth value combination.

The Lattice in figure three is a distributive lattice, that is it satisfies the following definition: (Where ∩ represents the intersection and ∪ represents the union)

$x\cap(y\cup z) = (x\cap y)\cup(x\cap z)$[15]

The lattice in figure three is also a complemented lattice as it satisfies the following definition:

Given a lattice L with universal bounds Z and I. (A universal bound is an extreme end point of a lattice, here Z and I correspond to 1 and 16, respectively.)

If for every a belonging to L there exists and a' such that

$a\cup a'=I$ and $a\cap a'=Z$

Then L is called a complemented lattice.[16]

This serve to demonstrate that the lattice in figure three represents a boolean algebra (two-valued logic).[17]

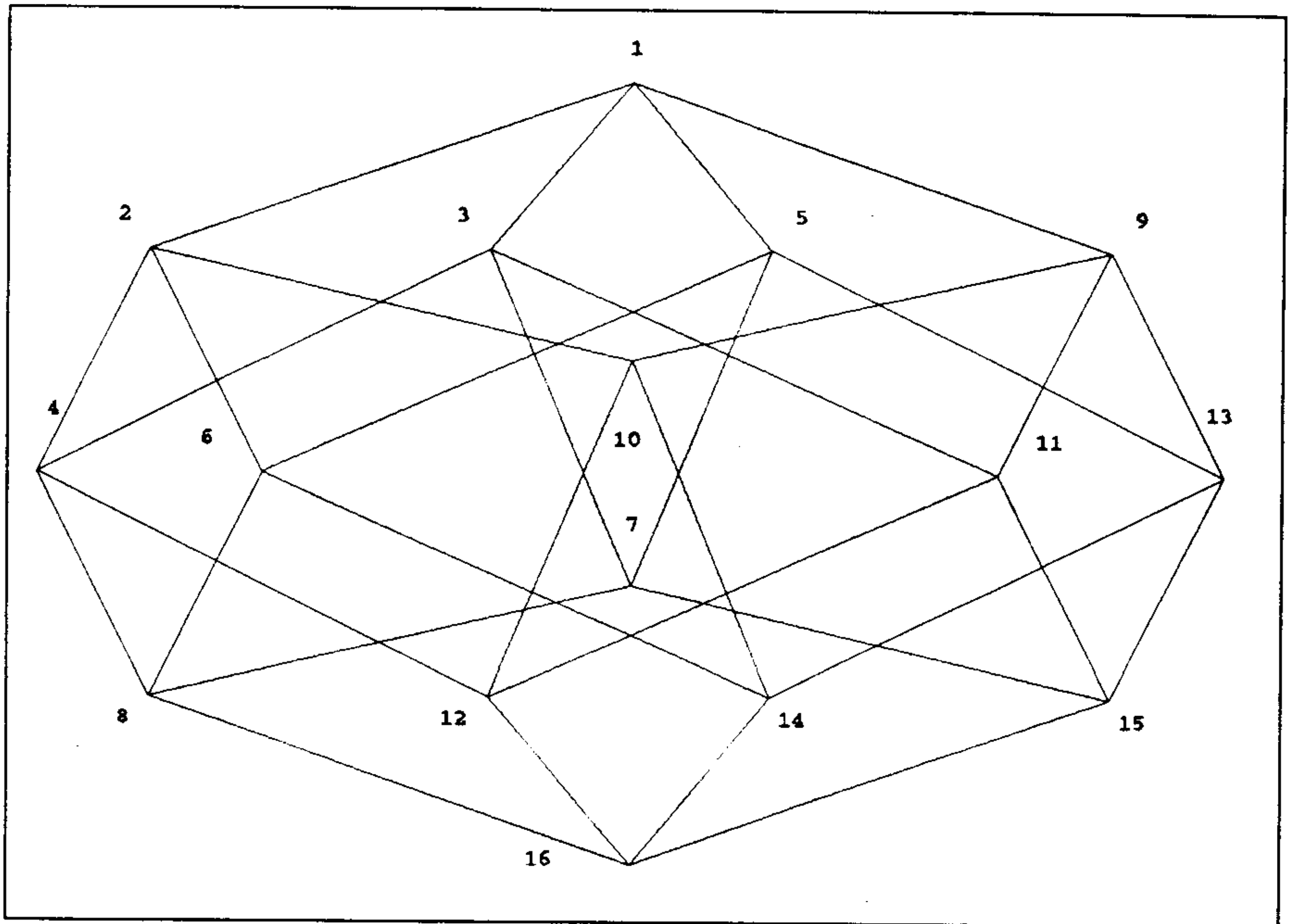

Figure Three

Here are some theorems which can be tested by students to determine whether they apply to the lattice in figure three or not. The following definitions are necessary for their proof.

A lattice in which $p \supset q$ exist for every pair of elements is called an implicative lattice.[18]
A lattice in which $\sim(p \supset q)$ exist for every pair of elements is called a subtractive lattice.[19]

Theorems:
A lattice is called a Skolem lattice (or algebra) if union and intersection hold for it as well as either implication or subtraction.
If both implication and subtraction hold, then it is a double-Skolem lattice.[20]
Every Skolem lattice is distributive.[21]
A finite Distributive Lattice is both Implicative and subtractive, therefore it is a double Skolem lattice.[22]
Every Boolean Algebra is a Skolem algebra.[23]

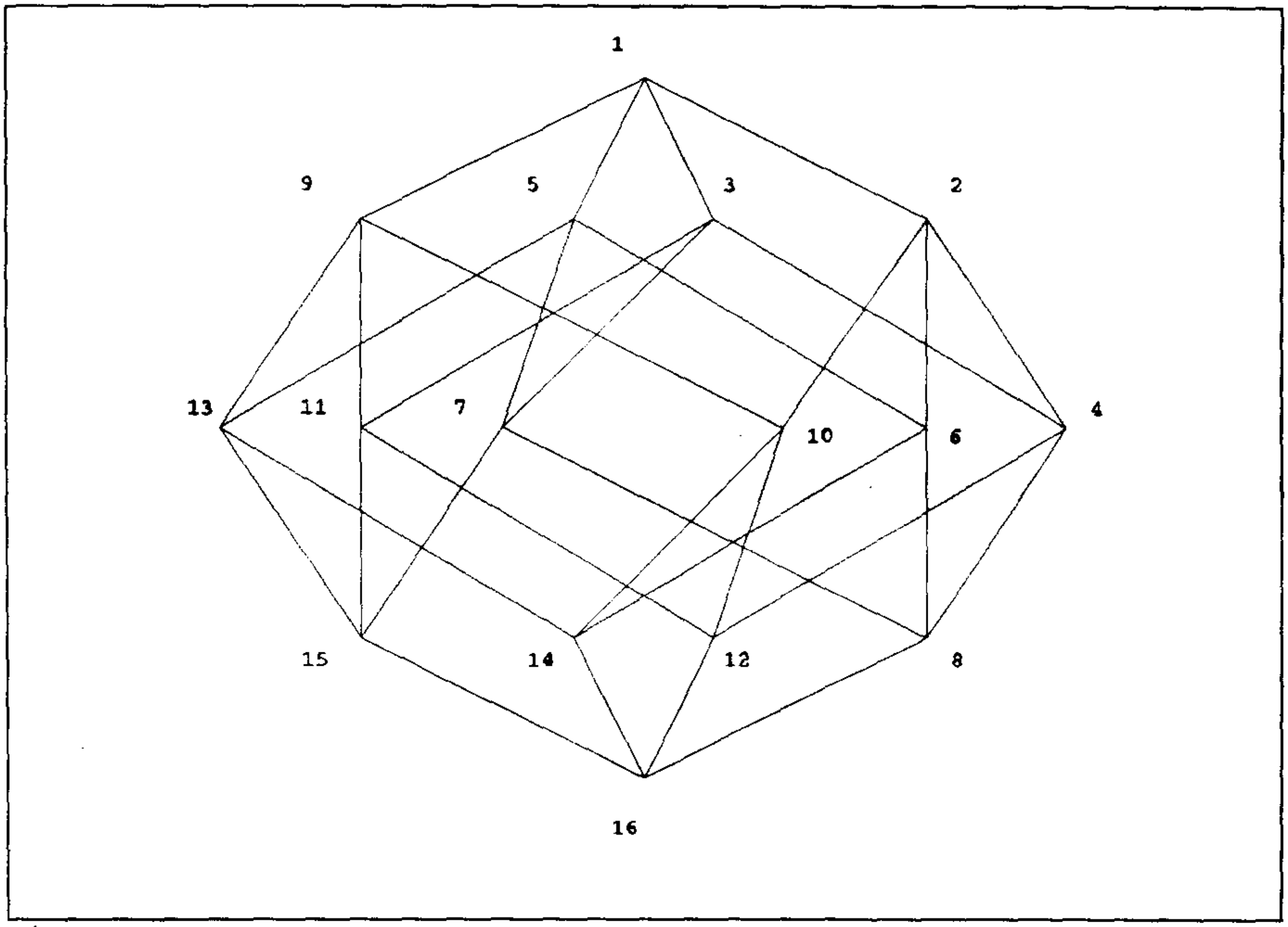

Figure Four

Set Theory

Of relation to lattice theory is set theory. The two share much in the way of notion and concepts. In the literature, lattice theory is usually explained in terms of set theory.

A set can be simply defined as any group of things.[24] We speak of a sports team as having members, it is a group of things, namely players. The same with sets. We distinguish between the set and the member. We agree that the quarterback is a member of the team, but the team is not a member of the quarterback. Membership is expressed as $q \in T$[25] (Where q = quarterback and T = team.)

The notion of membership should not be confuse with the notion of set inclusion, however. A set can be said to be included in another set when it is a subset of it.[26] Take for example the set of offensive linemen. This is a subset of the team, that is, the set of linemen is included in the set of the team. Inclusion is symbolized thus; $L \subseteq T$[27] (Where L is the set of linemen.)

Let us consider some other important notions related to set theory. An intersection is where two (or more) sets "overlap".[28] Let us use the follow two sets for examples-

R = {1,3,5,7} and S = {1,2,3,4}

The intersection of R and S ($R \cap S$) = {1,3}. The intersection is the set that contains all the members that are common to both R and S.

The union, on the other hand, of two (or more) sets contains all of the members of both sets.[29] Therefore $R \cup S$ = {1,2,3,4,5,7}.

A set is said to be empty when it has no members. This is also known as the null set and is symbolized as $\emptyset$.[30] For example, let E = the set of even integers and O = the set of odd integers. $E \cap O = \emptyset$.

As with the lattice theory section, the above information is necessary for work within set theory that will help tie it to propositional logic.

All of the 16 connectives can be expressed in terms of Venn diagrams. they are shown as such in figure 4.[31] Read the left circle as p, the right as q. Shaded areas represent what can be true in order to satisfy the appropriate truth value combination.

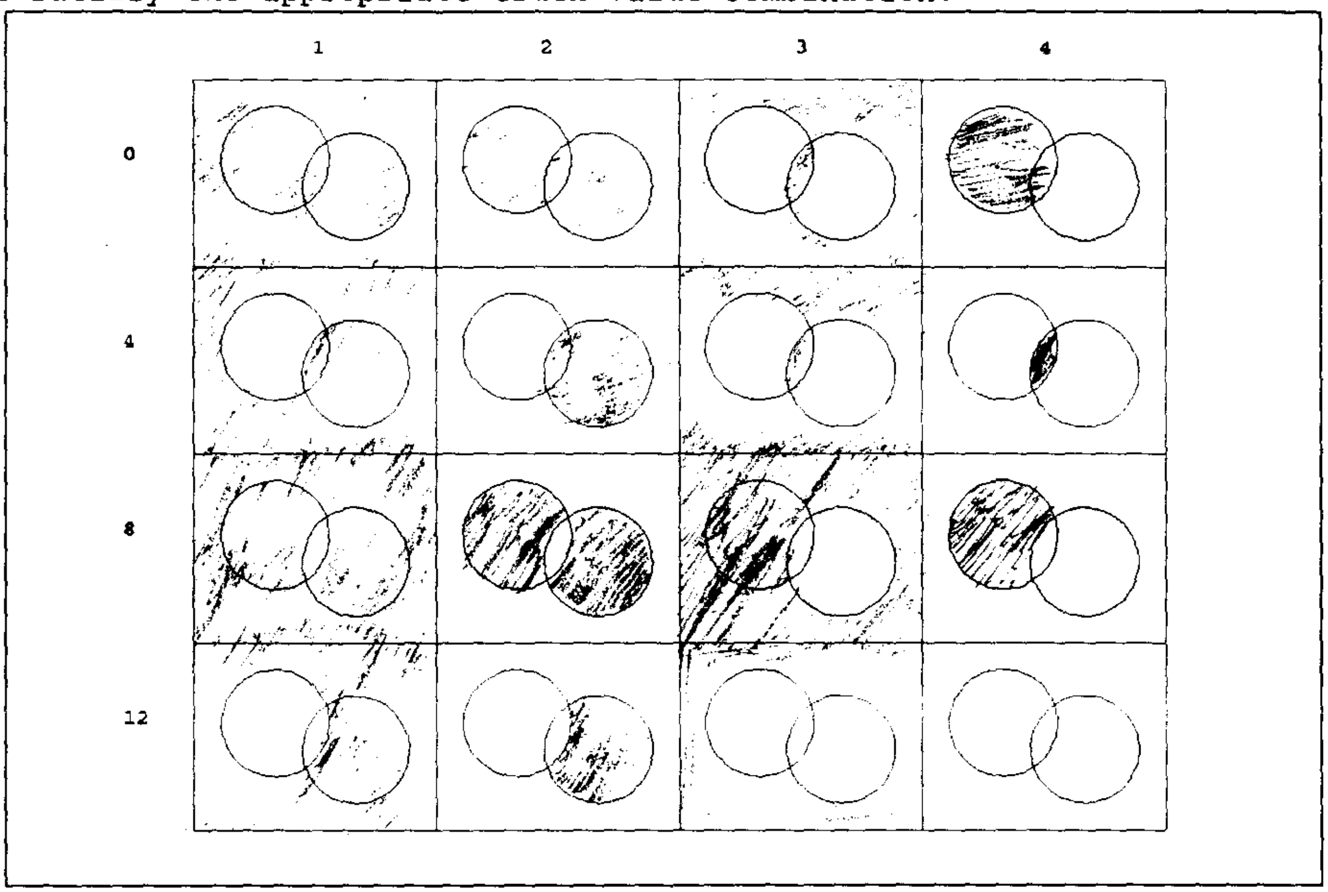

Figure Five

This diagram can be generated by students as well.

Relation of Lattice Theory to Set Theory.

The material on lattice theory was included because of its relation to many other fields.[32] Lattice theory, logic and set theory all share two common functions, as evidenced above. These are Conjunction and disjunction, in logical terms. This allows logic and set theory and lattice theory to enjoy some degree of interdefinability. This could lead to relating quantificational logic to lattice theory, as venn diagrams are commonly used to represent quantificational statements.[33]

Summary

The above material allows a student to begin with propositional logic, and using it as a springboard, to explore the stroke and dagger functions, possible truth value combinations, and function tables. Lattice theory was then introduced and explored in terms of propositional logic and was also related to set theory, allowing an advanced student to gain access to both of these fields.

The material below is provided for those who may not wish to engage in the above, somewhat involved, exercise. The following contains a few examples of other items that can be used to challenge logic students and perhaps wet their appetites for the above material.

Paradoxes and Puzzles

Most instructors will be familiar with the various logical paradoxes and several logical puzzles. Here is a collection of a few particularly intriguing ones and suggestions on where to find more.

A classic genre of puzzle is the "knights and knaves" puzzles, any of which are contained in Richard Smullyan's books. Below is a similar puzzle from William Poundstone's Labyrinths of Reason.

> On the island of Liars and Truth Tellers, where liars always lie, and truth tellers are always honest, you meet three people, A, B, and C. You ask A whether she is a liar or a truth teller. She answers in a local dialect which you cannot understand. Then you ask B what A said. B replies, "She said she's a liar." B then adds that C is also a liar. C interjects, "A is a truth teller." Are A, B, and C liars or truth tellers?[34]

Answer: A & C are truth tellers and B is a liar.

Reasoning: A could not have said what B reported her to. If A was a liar she would have lied about it and said she was a truth teller. Therefore, B is a liar, and by implication A and C are not.

"Coin problems" are another interesting diversion. In these problems, you are given a number of coins, a balance and a piece of information regarding any counterfeit coin(s) in the number given to you. Your job is to find the counterfeit in the least number of balances possible. For example, say you are given eight coins and are told that the counterfeit is slightly heaver than the genuine ones. What is the least number of balances needed to single out the counterfeit in? Answer, two.

The reasoning: with the setup in figure 6, you will be able to either eliminate six of the coins or limit your search to three of them, for if the balance balances, the weight of all six coins on it can be assumed to be normal. If you eliminate six of the coins, placing the remaining two on the balance will find the counterfeit. If you limit yourself to three coins, place one on each side of the balance and leave one off. If it balances, the one left off is counterfeit, if not, the side that tips is the bad one. While not purely "logical" in a propositional sense, these kind of problems encourage analytic thinking and problem solving.

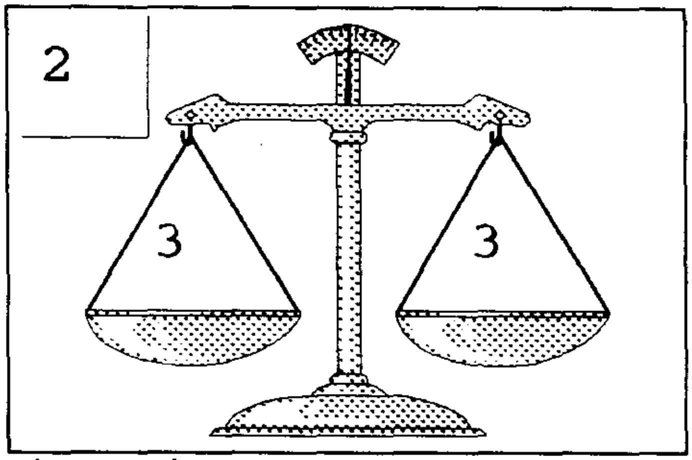

Figure Six

I have not found a resource for these problems, however, they are easy to construct, although the solutions may not be as easy to find. One suggestion is to give the problem to the students an the person who can defend the lowest number of balances "wins".

Another, perhaps more "logical" exercise is to construct a truth value combination of some length n. (Where n is a number to the fourth power). A student would be given the truth value combination and asked to create a proposition which satisfies the truth value combination. For example:

p	q	r	
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

Answer: (((p & q)& ~r)v((p & r)& ~q))v((q & ~p)& ~r).

The method for such problems is really rather simple, once it is known. Trying to tackle such a problem, especially with five or six variables, by hit and miss is nearly impossible. First, locate all of the lines in which the proposition is true. Second, by reference to the truth tables left columns, write conjunctions which satisfy each of the combinations shown. Third, connect the resulting conjunctions with disjunctions which creates the final desired proposition. This makes a great bonus problem for students who have just mastered truth tables.

Paradoxes

For those willing to explore problems which have no widely accepted solutions, paradoxes can be fascinating. Here is given one example of this genre of problem.

The Gallows

The law of a certain land is that all those who wish to enter the city are asked to state their business there. Those who reply truly are allowed to enter and depart in peace. Those who reply falsely are hanged. What should happen to the traveller who, when asked his business, replies,"I have come to be hanged"?[35]

Historical Developments and Systems

Also of considerable interest is the history and development of logic. A student aspiring to learn more could investigate alternate notational systems such as Jan Lukasiewicz's or Gottleb Frege's. I.M. Bochenski's Ancient Formal Logic and History of Formal Logic are both excellent resources for the student aspiring to know more about the history and development of logic. Also worthy of consideration is William & Martha Kneale's work The Development of Logic.

Appendix A

Possible Truth Value Combinations (Connectives)

p	q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

(Top row is "p")

Table 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1	2	1	2	1	1	1	1
3	1	1	3	3	1	1	3	3	1	1	3	3	1	2	1	2
4	1	2	3	4	1	2	3	4	1	2	3	4	1	1	3	3
5	1	1	1	1	5	5	5	5	1	1	1	1	1	2	3	4
6	1	2	1	2	5	6	5	6	1	2	1	2	5	5	5	5
7	1	1	3	3	5	5	7	7	1	1	3	3	5	6	5	6
8	1	2	3	4	5	6	7	8	1	2	3	4	5	5	7	7
9	1	1	1	1	1	1	1	1	9	9	9	9	5	6	7	8
10	1	2	1	2	1	2	1	2	9	10	9	10	9	9	9	9
11	1	1	3	3	1	1	3	3	9	9	11	11	9	10	9	10
12	1	2	3	4	1	2	3	4	9	10	11	12	9	9	11	11
13	1	1	1	1	5	5	5	5	9	9	9	9	9	10	11	12
14	1	2	1	2	5	6	5	6	9	10	9	10	13	13	13	13
15	1	1	3	3	5	5	7	7	9	9	11	11	13	14	13	14
16	1	2	3	4	5	6	7	8	9	10	11	12	13	13	15	15

Table 3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15
3	1	2	1	2	5	6	5	6	9	10	9	10	13	14	13	14
4	1	1	1	1	5	5	5	5	9	9	9	9	13	13	13	13
5	1	2	3	4	1	2	3	4	9	10	11	12	9	10	11	12
6	1	1	3	3	1	1	3	3	9	9	11	11	9	9	11	11
7	1	2	1	2	1	2	1	2	9	10	9	10	9	10	9	10
8	1	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9
9	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
10	1	1	3	3	5	5	7	7	1	1	3	3	5	5	7	7
11	1	2	1	2	5	6	5	6	1	2	1	2	5	6	5	6
12	1	1	1	1	5	5	5	5	1	1	1	1	5	5	5	5
13	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
14	1	1	3	3	1	1	3	3	1	1	3	3	1	1	3	3
15	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 4

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
5	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
10	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
11	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
12	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
13	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
15	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 5

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
3	3	3	1	1	3	3	1	1	3	3	1	1	3	3	1	1
4	4	3	2	1	4	3	2	1	4	3	2	1	4	3	2	1
5	5	5	5	5	1	1	1	1	5	5	5	5	1	1	1	1
6	6	5	6	5	2	1	2	1	6	5	6	5	2	1	2	1
7	7	7	5	5	3	3	1	1	7	7	5	5	3	3	1	1
8	8	7	6	5	4	3	2	1	8	7	6	5	4	3	2	1
9	9	9	9	9	9	9	9	9	1	1	1	1	1	1	1	1
10	10	9	10	9	10	9	10	9	2	1	2	1	2	1	2	1
11	11	11	9	9	11	11	9	9	3	3	1	1	3	3	1	1
12	12	11	10	9	12	11	10	9	4	3	2	1	4	3	2	1
13	13	13	13	13	9	9	9	9	5	5	5	5	1	1	1	1
14	14	13	14	13	10	9	10	9	6	5	6	5	2	1	2	1
15	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1
16	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Table 6

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13
14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

Table 7

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
6	6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11
7	7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
8	8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
9	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
11	11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
12	12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
13	13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
14	14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3
15	15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2
16	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Table 8

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	2	4	4	6	6	8	8	10	10	12	12	14	14	16	16
3	3	4	3	4	7	8	7	8	11	12	11	12	15	16	15	16
4	4	4	4	4	8	8	8	8	12	12	12	12	16	16	16	16
5	5	6	7	8	5	6	7	8	13	14	15	16	13	14	15	16
6	6	6	8	8	6	6	8	8	14	14	16	16	14	14	16	16
7	7	8	7	8	7	8	7	8	15	16	15	16	15	16	15	16
8	8	8	8	8	8	8	8	8	16	16	16	16	16	16	16	16
9	9	10	11	12	13	14	15	16	9	10	11	12	13	14	15	16
10	10	10	12	12	14	14	16	16	10	10	12	12	14	14	16	16
11	11	12	11	12	15	16	15	16	11	12	11	12	15	16	15	16
12	12	12	12	12	16	16	16	16	12	12	12	12	16	16	16	16
13	13	14	15	16	13	14	15	16	13	14	15	16	13	14	15	16
14	14	14	16	16	14	14	16	16	14	14	16	16	14	14	16	16
15	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

Table 9

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
2	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1
3	14	13	14	13	10	9	10	9	6	5	6	5	2	1	2	1
4	13	13	13	13	9	9	9	9	5	5	5	5	1	1	1	1
5	12	11	10	9	12	11	10	9	4	3	2	1	4	3	2	1
6	11	11	9	9	11	11	9	9	3	3	1	1	3	3	1	1
7	10	9	10	9	10	9	10	9	2	1	2	1	2	1	2	1
8	9	9	9	9	9	9	9	9	1	1	1	1	1	1	1	1
9	8	7	6	5	4	3	2	1	8	7	6	5	4	3	2	1
10	7	7	5	5	3	3	1	1	7	7	5	5	3	3	1	1
11	6	5	6	5	2	1	2	1	6	5	6	5	2	1	2	1
12	5	5	5	5	1	1	1	1	5	5	5	5	1	1	1	1
13	4	3	2	1	4	3	2	1	4	3	2	1	4	3	2	1
14	3	3	1	1	3	3	1	1	3	3	1	1	3	3	1	1
15	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 10

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
2	15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2
3	14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3
4	13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
5	12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
6	11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
7	10	99	12	11	14	13	16	15	2	1	4	3	6	5	8	7
8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
9	8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
10	7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
11	6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11
12	5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
13	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
14	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
15	2	1	4	3	6	5	8	7	10	11	12	11	14	13	16	15
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 11

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
2	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
3	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14
4	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13
5	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
6	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
7	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
8	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
9	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
10	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
11	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
12	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
13	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
14	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 12

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
2	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16
3	14	14	16	16	14	14	16	16	14	14	16	16	14	14	16	16
4	13	14	15	16	13	14	15	16	13	14	15	16	13	14	15	16
5	12	12	12	12	16	16	16	16	12	12	12	12	16	16	16	16
6	11	12	11	12	15	16	15	16	11	12	11	12	15	16	15	16
7	10	10	12	12	14	14	16	16	10	10	12	12	14	14	16	16
8	9	10	11	12	13	14	15	16	9	10	11	12	13	14	15	16
9	8	8	8	8	8	8	8	8	16	16	16	16	16	16	16	16
10	7	8	7	8	7	8	7	8	15	16	15	16	15	16	15	16
11	6	6	8	8	6	6	8	8	14	14	16	16	14	14	16	16
12	5	6	7	8	5	6	7	8	13	14	15	16	13	14	15	16
13	4	4	4	4	8	8	8	8	12	12	12	12	16	16	16	16
14	3	4	3	4	7	8	7	8	11	12	11	12	15	16	15	16
15	2	2	4	4	6	6	8	8	10	10	12	12	14	14	16	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 13

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
2	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
3	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
4	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
5	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
6	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
7	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
8	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
9	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
10	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
11	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
12	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
13	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
14	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
15	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Table 14

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
2	16	16	14	14	12	12	10	10	8	8	6	6	4	4	2	2
3	16	15	16	15	12	11	12	11	8	7	8	7	4	3	4	3
4	16	16	16	16	12	12	12	12	8	8	8	8	4	4	4	4
5	16	15	14	13	16	15	14	13	8	7	6	5	8	7	6	5
6	16	16	14	14	16	16	14	14	8	8	6	6	8	8	6	6
7	16	15	16	15	16	15	16	15	8	7	8	7	8	7	8	7
8	16	16	16	16	16	16	16	16	8	8	8	8	8	8	8	8
9	16	15	14	13	12	11	10	9	16	15	14	13	12	11	10	9
10	16	16	14	14	12	12	10	10	16	16	14	14	12	12	10	10
11	16	15	16	15	12	11	12	11	16	15	16	15	12	11	12	11
12	16	16	16	16	12	12	12	12	16	16	16	16	12	12	12	12
13	16	15	14	13	16	15	14	13	16	15	14	13	16	15	14	13
14	16	16	14	14	16	16	14	14	16	16	14	14	16	16	14	14
15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

Table 15

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
2	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15
3	16	16	14	14	16	16	14	14	16	16	14	14	16	16	14	14
4	16	15	14	13	16	15	14	13	16	15	14	13	16	15	14	13
5	16	16	16	16	12	12	12	12	16	16	16	16	12	12	12	12
6	16	15	16	15	12	11	12	11	16	15	16	15	12	11	12	11
7	16	16	14	14	12	12	10	10	16	16	14	14	12	12	10	10
8	16	15	14	13	12	11	10	9	16	15	14	13	12	11	10	9
9	16	16	16	16	16	16	16	16	8	8	8	8	8	8	8	8
10	16	15	16	15	16	15	16	15	8	7	8	7	8	7	8	7
11	16	16	14	14	16	16	14	14	8	8	6	6	8	8	6	6
12	16	15	14	13	16	15	14	13	8	7	6	5	8	7	6	5
13	16	16	16	16	12	12	12	12	8	8	8	8	4	4	4	4
14	16	15	16	15	12	11	12	11	8	7	8	7	4	3	4	3
15	16	16	14	14	12	12	10	10	8	8	6	6	4	4	2	2
16	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Table 16

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
2	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
3	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
4	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
5	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
6	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
7	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
8	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
9	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
10	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
11	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
12	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
13	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
14	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16

Appendix B

Several collections of logical puzzles and paradoxes exist. Some of the best are by Martin Gardner and especially Raymond Smullyan. Titles include The Scientific American Book of Mathematical Puzzles by Gardner and What is the Name of This Book? by Smullyan. Also of interest are Paradoxes, by R. M. Sainsbury and Labyrinths of Reason by William Poundstone. All of these are worth recommending to the student desiring something more thought provoking than elementary deductions.

For those interested in the lattice and set theory aspects of this work, Garrett Birkhoff's Lattice Theory is one of the first major works written on this subject. Stephen Anthony Kiss' Structures of Logic also provides an in depth examination of some of the concepts presented here.

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1. For ease of notation, extreme outside parenthesis may be omitted.

2. Adapted from Bonevac, 119.

3. All except Idem. adapted from Gustason & Ulrich, 84-85, 100-101. Idem from Bonevac, 210-211.

4. Adapted from Gustason & Ulrich, 88, 102.

5. Bochenski, 319

6. Adapted from Dubisch, 3ff.

7. Dubisch, 5.

8. Dubisch, 5.

9. Dubisch, 7.

10. Dubisch, 6.

11. Dubisch, 6.

12. Dubisch, 7.

13. Dubisch, 8.

14. As shown in several of the volumes listed in the works cited.

15. Stabler, 226-227.

16. Stabler, 227.

17. Stabler, 227.

18. Donnellan, 251.

19. Donnellan, 251.

20. Donnellan, 251.

21. Donnellan, 252.

22. Donnellan, 253.

23. Donnellan, 254.

24. Thomason, 282.

25. Thomason, 282.

26. Thomason, 282.

27. Thomason, 283.

28. Thomason, 283.

29. Thomason, 284.

30. Abbott, 12.

31. Abbott 49, with adaptations to the system used in this work.

32. Dubisch, v.

33. Bonevac 236ff. A foundational article for the relation of sets and lattices is Birkhoff, Garret & Orrin Frink. "Representations of Lattices by Sets." Transactions of the American Mathematical Society. 64 (1948) 299-316.

34. Poundstone, 98.

35. Sainsbury, 145.

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Index