An Undergraduate Fellow Chases Chaos

An Honors Thesis (ID 499)

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May 4, 1990

Expected Date of Graduation:

May 5, 1990

The attached paper, "Chasing Orbits in Polar Coordinates" presents the results of an undergraduate fellowship in which I worked with Dr. Kathryn Porter, Assistant Professor of Mathematics at Ball State University. Originally, when Dr. Porter asked if I would be interested in applying for the fellowship, she suggested chaos as a topic for research. I knew very little about chaos but I was interested in experiencing research so we applied for and were granted an undergraduate fellowship for the two semesters of the upcoming academic year.

During the summer of 1989, I became familiar with chaos through James Gleick's book, Chaos: Making a New Science which chronicles the development of chaos theory. According to Gleick, chaos is the name given to a scientific movement that is changing the way researchers in many fields understand the world. Previously, irregularity and randomness in nature were attributed to small errors or "noise" that disrupted the physical laws under which nature was believed to operate. Physics could explain the laws of nature theoretically and even apply these laws and make accurate predictions - in the laboratory. Outside of laboratory experiments, however, prediction in even the simplest of nature's systems was impossible. For example, physicists who claimed to understand fluid systems still could not predict how close two bits of foam floating next to each other at the bottom of a waterfall had been at the top. (Gleick, 8)

Chaos was discovered gradually by scientists who were intrigued by the complicated behavior and unpredictability of ordinary, everyday systems such as waterfalls, cigarette smoke, clouds, and weather. They found in these simple systems a strange order beneath the disorder. Moreover, as varied studies were performed, the hidden order appeared to be universal, cutting across all scientific fields.

One of the concepts at the foundation of chaos theory was accidentally discovered by Edward Lorenz during the winter of 1961. Lorenz was a research meteorologist simulating a weather system on his computer. His simple graphics revealed that certain weather patterns repeated themselves but the repetitions were never exact. He decided to follow one particular pattern for a longer time period. Instead of starting from the beginning, he used calculations from the middle of the previous printout as the initial conditions for a new printout, expecting both runs to be identical. He left the office for a cup of coffee and, when he returned, found a completely different weather system.

The calculations Lorenz used for initial conditions had been rounded off by the computer. The error was so slight that Lorenz naturally assumed the effect would be negligible. Instead, the difference completely changed the pattern. Experiments in other systems revealed the same phenomenon. Small input differences yielded drastically different results. Two patterns following the same path would, at some point, begin to diverge. This behavior is called "sensitive dependence on initial conditions".

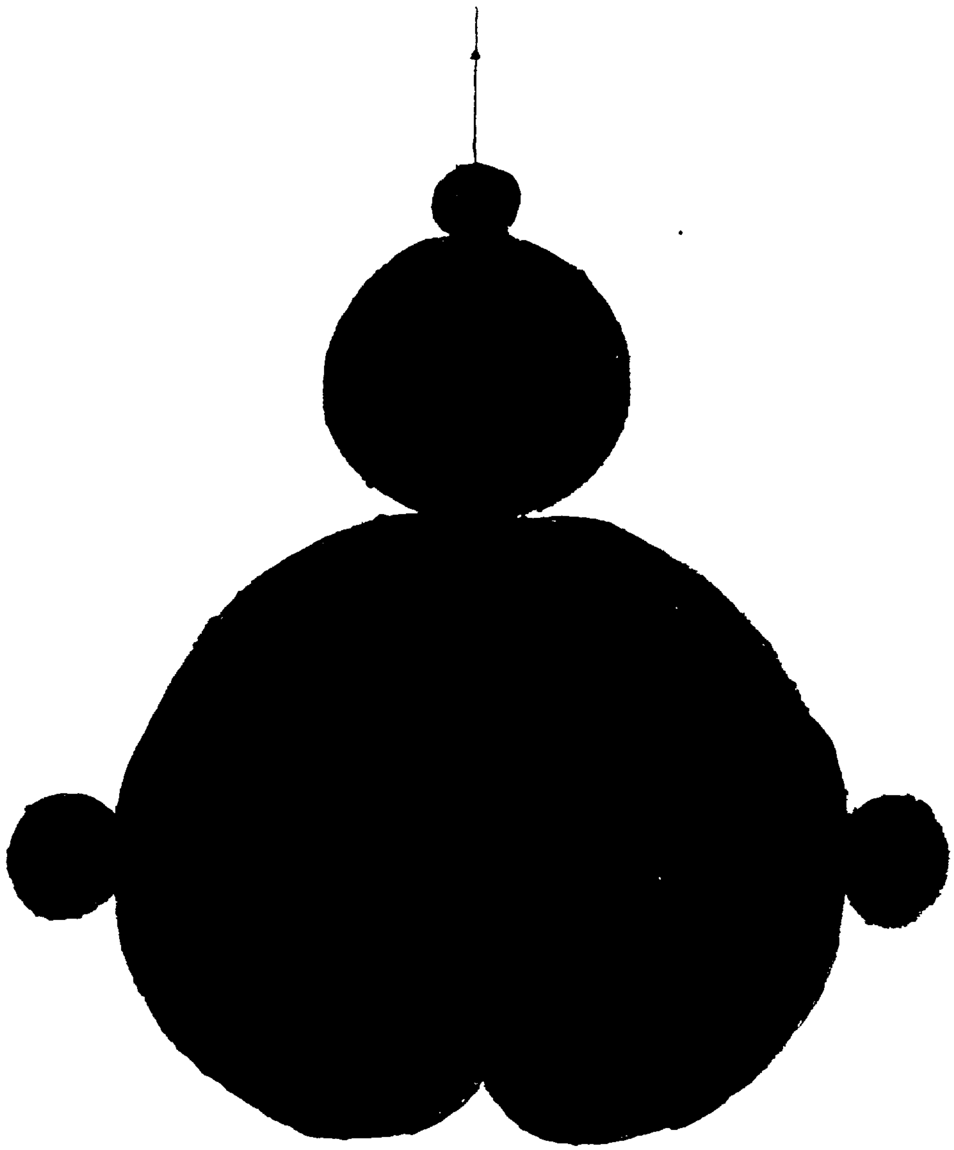
At the time they were published, Lorenz's discoveries and his implication that the same complex behavior occurred in other scientific fields were not well received. Lorenz was suggesting that many of the properties science had accepted long ago might be wrong. Predicting the behavior of a system with sensitive dependence on initial conditions might be impossible. Lorenz had also stumbled upon a strange type of order. Though exact predictions were impossible, systems always stayed within certain boundaries. For example, a desert would not experience a flood. Weather patterns and behavior in other systems appeared to be attracted by a periodic cycle, never exactly repeating but never completely leaving the cycle. This strange attractor would become the basis of chaos.

James Yorke received, from a friend, a copy of Lorenz's "Deterministic Nonperiodic Flow" almost ten years after it was printed in the Journal of the Atmospheric Sciences. Yorke was a mathematician who saw great value in Lorenz's work that he felt physicists in particular should know about. In 1976, he wrote a paper about periodicity entitled "Period Three Implies Chaos" to be published in the American Mathematical Monthly. In the same year, Mitchell Feigenbaum, at the expense of his health, developed a rule about systems at the moment when regular behavior becomes irregular. Feigenbaum believed his theory applied universally. His work contributed the mathematical evidence necessary to attract the attention of skeptical scientists. Gradually, the concepts of sensitive dependence on initial conditions and order beneath apparent disorder spread throughout the scientific community. Attached to these concepts was Yorke's term: Chaos.

At approximately the same time chaos was emerging, fractal geometry was making a name for itself. Fractal structures display increasing detail under magnification. However, fractals also exhibit self-similarity. This concept is best illustrated by the Mandelbrot Set, named after the IBM mathematician who developed fractal geometry, Benoit Mandelbrot.

The Mandelbrot Set is formed by iterating in the complex plane on a computer. Iterating a function involves calculating the function at a particular value, then calculating the same function at the new value and so on. Some points stay within certain bounds under iteration. Others go to infinity at different rates. In the Mandelbrot Set, the points that remain are colored black. Different shades of gray surrounding the black area indicate how quickly the other points leave the set with white designating the fastest. The illustration on the following page is the basic shape of the Mandelbrot Set in an extremely simplified form. For obvious reasons, Stewart calls this form "the gingerbread man". (Stewart, 236) Magnifying the set reveals various sizes of gingerbread man "heads" spaced regularly along the border, surrounded by beautifully complex and intricate designs. (In the simple illustration here, the "arms" are smaller copies of the "head".) Further magnification shows that each "head" has small replicas of itself along its own border engulfed in more complicated patterns. At some levels of

The Mandelbrot Set



magnification, the whole gingerbread man surprisingly reappears, complete in every detail as the original. Mandelbrot found within the complexity of a fractal, a repetitive structure, that is, a type of order within disorder. Other scientists have found that the strangely attracting patterns in chaotic systems are sometimes fractal. Chaos and fractals developed independently yet they are interrelated.

The abundance of chaotic behavior and fractal structure in nature is startling. Chaos can be seen in everyday occurrences that switch from orderly to turbulent behavior: a spinning top, the smoke from a cigarette, water flowing in a stream, a dripping faucet. Leaves, branches, frost crystals, snowflakes, whirlpools, lightning bolts, blood vessels, and coastlines all exhibit fractal structure. The pervasiveness of chaos in nature suggests that the growing knowledge about chaotic dynamical systems may hold information for every field of discipline about unexplained occurrences. In the medical field, chaos is being considered in connection with ventricular fibrillation, epilepsy, manic-depression, schizophrenia, some forms of leukemia, and measles outbreaks. Engineers hope chaos has real world applications in developing more stable and efficient designs. Chaos also may play a role in animal populations, computer reliability, stellar mechanics, weather prediction, the stock market, locating oil, and even predicting war.

Despite the connotations attached to the word "chaos" in a nonscientific sense, chaos as a scientific property can be necessary and beneficial. Evolution due to irregularity in the gene pool provides animals and plants a better chance of survival in a fluctuating world. Perhaps more importantly, randomness and unpredictability generate the beauty and richness in the universe. With good reason, scientists are excited about chaos. For many years, science has concentrated on isolated particles and cells. Chaos focuses on the universal behavior of nature, connecting all disciplines and providing a refreshing holistic approach to scientific research.

The excitement is contagious. After reading James Gleick's book and an assortment of articles about chaos in a variety of magazines and journals, I was anxious to become a part of this grand and mysterious scientific revolution. However, I was afraid that my limited knowledge would prevent me from exploring chaos in mathematics. When the fellowship officially began in August, Dr. Porter provided a series of books by Ralph H. Abraham and Christopher D. Shaw. Part 1 of Dynamics - The Geometry of Behavior discusses periodic behavior and Part 2 explains chaotic behavior. Abraham and Shaw present information via illustrations and diagrams. Being able to visualize fixed points, periodic points, attractors, repellors, phase portraits, and other concepts of dynamics helped a great deal when our research became more mathematical. We began studying Robert L. Devaney's An Introduction to Chaotic Dynamical Systems. Simple dynamics in itself was a new area for me. For each consecutive section of the textbook, I needed more assistance from Dr. Porter to understand the material. By the time we covered the chapter on chaos, the first semester of the two allotted for the fellowship was almost over.

With such little time remaining, we needed to decide quickly on a manageable topic for research. Chaos itself was too complex. Since we had spent so much time studying dynamics, Dr. Porter suggested trying to apply Devaney's definitions and methods regarding dynamics. We wanted to do original work and polar coordinates provided the opportunity. Devaney approached dynamics in the rectangular coordinates of the Cartesian plane only. Although polar coordinates are related directly to rectangular coordinates, Devaney's methods would have to be altered for the polar coordinate system. Since I already understood this system and was now comfortable with the basics of dynamics, I would be able to contribute to the project by doing some research on my own. Dr. Porter and I decided to "chase" orbits (the iterates of a function) in polar coordinates.

To produce the graphs for our study, I used MathCAD, Version 2.0, a software program developed by Allen Raxdow, David Blohm, Joshua D. Bernoff, and Diane Gioseffi for MathSoft, Inc., Cambridge, Massachusetts. From the instruction manual and with much experimentation, I learned to plot and accurately print graphs of functions in both coordinate systems.

As Lorenz discovered, small round-off errors can have drastic results, therefore, I began with polar curves for which some iterations could be calculated exactly. Most of Devaney's elementary definitions for dynamics in rectangular coordinates held in polar coordinates as well. The method for graphical analysis had to be redefined because points in polar coordinates are expressed in terms of distance and angle measurements, not in distance alone, as are points in rectangular coordinates. After Dr. Porter defined an analogous method, I studied many curves to test the new procedure. The method worked perfectly for some graphs but, due to the nature of the polar coordinate system, was difficult and sometimes ambiguous for others. For graphical analysis in the rectangular coordinates, one need only know the graph of the function and the method for analysis. With polar curves, more knowledge about the function is necessary. For example, points may falsely appear fixed because each point can be expressed in more than one way. The points $(\pi/2, \pi/2)$ and $(3\pi/2, -\pi/2)$ occupy the same position on a graph but the coordinate values obviously are different.

I encountered another significant distinction between the coordinate systems. Polar curves often have periodic attractors other than fixed points. Devaney's examples do not have this characteristic so he only briefly mentions nonfixed periodic attractors. For nonfixed attractors and other characteristics, I used the mathematics of dynamics to support our findings from graphical analysis.

Most of our work with polar coordinates supported Devaney's definitions for rectangular coordinates. The most interesting part of our research was discovering the differences between the two coordinate systems and the difficulties involved in studying the orbits of points in the polar coordinate system.

"Chasing Orbits in Polar Coordinates" may not hold revelations for the scientific world, but the work from which the paper is produced is personally significant. The paper represents my first attempt at research and technical writing. Studying the dynamics of polar curves was often frustrating. At times the graphs were ambiguous and misleading. Some curves yielded little or no concrete information. When analyzing a graph produced important results, however, research was enjoyable and exciting. Summarizing our conclusions in the style of a standard mathematics article was challenging as well.

The undergraduate fellowship program provided the opportunity to experience a career option in mathematics. During the time I was involved in the fellowship project, I was considering career plans. Research as an occupation had been a mystery. I not only learned more about mathematical research and technical writing but about my own abilities and interests regarding mathematics. The knowledge and experience I acquired with the help of Dr. Kathryn Porter through the Undergraduate Fellowship Program at Ball State University will be a tremendous asset as I make decisions for my future.

Sources

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Chasing Orbits in Polar Coordinates

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Introduction: The study of dynamical systems has its origins with Galileo (1564-1642) and Kepler (1571-1620) as a branch of natural philosophy or physics. Dynamics involves change, rate of change, and how, when, and why change occurs in natural phenomena. Modern dynamical systems begins with Poincaré (1854-1912) and his methods for understanding global behavior of solutions of systems of equations.

The area of dynamics in which we are interested is the study of iterated maps. This area arose from the desire of mathematicians to understand the behavior of solutions of differential equations. Solving differential equations frequently involves some method of numerical integration. Numerical integration is usually an iterative process or procedure.

In this paper we shall discuss the similarities and differences of the study of orbits in the rectangular and polar coordinate systems. R. Devaney [2] has studied the behavior of orbits in the rectangular coordinate system, whereas we are interested in how to adapt the analysis for use in the polar coordinate system. In the last section, we shall indicate some shortcomings of the adaptation.

Preliminaries: The following definitions can be found in Devaney's book on dynamical systems [2].

For a given function, f, the *forward orbit* of x, $O^+(x)$, is defined to be the set $O^+(x) = \{x, f(x), f^2(x), f^3(x), \ldots\}$ where $f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x)$. If f is a homeomorphism (i.e., one-to-one, onto, continuous, with a continuous inverse) then we define the *backward orbit* of x, $O^-(x)$, by $O^-(x) = \{x, f^{-1}(x), f^{-2}(x), f^{-3}(x), \ldots\}$ where $f^{-n}(x) = \underbrace{f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}}_{n \text{ times}}(x)$. The *full orbit* of x is

$$O(x) = O^+(x) \cup O^-(x) = \{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\}$$

Example: Let $f(x) = 3x$. Then $f^n(x) = 3^n x$ and $f^{-n}(x) = \frac{1}{3^n}x$ since $f^{-1}(x) = \frac{1}{3}x$. So $O^+(1) = \{1, 3, 9, 27, \ldots\}$, $O^-(1) = \{\ldots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1\}$, and the full orbit is $O(1) = \{\ldots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, \ldots\}$.

A point x is a *fixed point* for f if $f(x) = x$. The set of all fixed points for f will be denoted by $Fix(f)$. For a given function, f, x is called a *periodic point of period* n if $f^n(x) = x$. The *prime period* of x is the least positive integer, n, for which $f^n(x) = x$. Let $Per_n(f)$ denote the set of all periodic points of period n (not necessarily prime).

Example: Let $f(x) = -x$. Then $Fix(f) = \{0\}$ and $Per_2(f) = (-\infty, \infty)$, since $f(0) = 0$ and $f^2(x) = -(-x) = x$, for all $x \in (-\infty, \infty)$.

For a function, f, a point x is *eventually periodic of period* n if there exists $m > 0$ such that $f^{n+i}(x) = f^i(x)$ for all $i \geq m$.

Example: Let $f(x) = -x^2$. $Fix(f) = \{-1, 0\}$. The point $x = 1$ is eventually fixed (i.e., eventually periodic of period 1) since $f^{1+i}(1) = -1, \quad \text{for all } i \geq 1$.

If p is a periodic point of period n, then x is defined to be *forward asymptotic* to p provided $\lim_{i \to \infty} f^{in}(x) = p$. The set of all points forward asymptotic to p is denoted by $W^s(p)$, and is called the *stable set* of p.

Example: Let $f(x) = x^2$. Then $Fix(f) = \{0, 1\}$; their stable sets are: $W^s(1) = \{1\}$ and $W^s(0) = \{x : |x| < 1\}$ since $f^i(x) = x^{2i}$ and $\lim_{i \to \infty} x^{2i} = \begin{cases} 0, & x \in (-1, 1) \\ \infty, & \text{otherwise.} \end{cases}$

Let p be a periodic point of prime period n. The point p is called a *hyperbolic point* provided $|(f^n)'(p)| \neq 1$.

Example: For $f(x) = -x^3$, $Fix(f) = \{0\}$ and $Per_2(f) = \{-1, 0, 1\}$. All three values are hyperbolic since $|f'(0)| = 0 \neq 1$, $f^2(x) = x^9$, and $|(f^2)'(\pm 1)| = 9 \neq 1$.

Suppose p is a hyperbolic periodic point of period n. If $|(f^n)'(p)| < 1$, then the point p is called an *attracting periodic point* or *attractor* and all the points in some open interval about p approach p under iteration of f. If $|(f^n)'(p)| > 1$, the point p is called a *repelling periodic point*, or *repellor* and there exists an open interval about p such that all points in the interval move away from p under iteration of f.

Example: Let $f(x) = x^2$. The point $x = 0$ is an attracting fixed point since $f'(x) = 2x$ and $|f'(0)| = 0 < 1$. The point $x = 1$ is a repelling fixed point since $|f'(1)| = 2 > 1$.

Dynamics in the Rectangular Coordinate System: As Devaney [2] has observed, it is usually impossible to explicitly find the periodic points of a given function. Even with the help of a computer, the equation $f^n(x) = x$ often cannot be solved without round off error occuring. Small errors will add up, causing the computer to miss some periodic points.

In his book [2] Devaney introduced a graphical technique, called *graphical analysis*, which can be used to trace the iterations of a function. We now describe this technique. Let $g(x) = x$ be called the *diagonal*. Note that $Fix(g) = (-\infty, \infty)$. The coordinate curves of any plane system are defined to be the curves which result when one sets each variable of the system equal to a constant. In the rectangular coordinate system, for $k \in (-\infty, \infty)$, the coordinate curves, $x = k$ and $y = k$, are vertical lines and horizontal lines, respectively. To trace the iterations of f at a point p, begin on the diagonal, $g(x) = x$, at (p, p). Draw a vertical segment from (p, p) to the graph of f. The point you have reached is $(p, f(p))$. Next trace a horizontal line back to the diagonal, meeting the diagonal at $(f(p), f(p))$. Note that both movements have been made along the coordinate curves, first a vertical line and then a horizontal one. By repeating this procedure, we trace the iterates of f at p on the diagonal. The set of values on the diagonal will be $\{p, f(p), f^2(p), \ldots\}$ which is the forward orbit of p.

Example: Let $f(x) = x^2$. We have illustrated several orbits of f in Figure [1]. The fixed point, $x = 0$, attracts all $x \in (-1, 1)$. All other points, except $x = -1$, which is eventually fixed, are repelled by the fixed point $x = 1$.

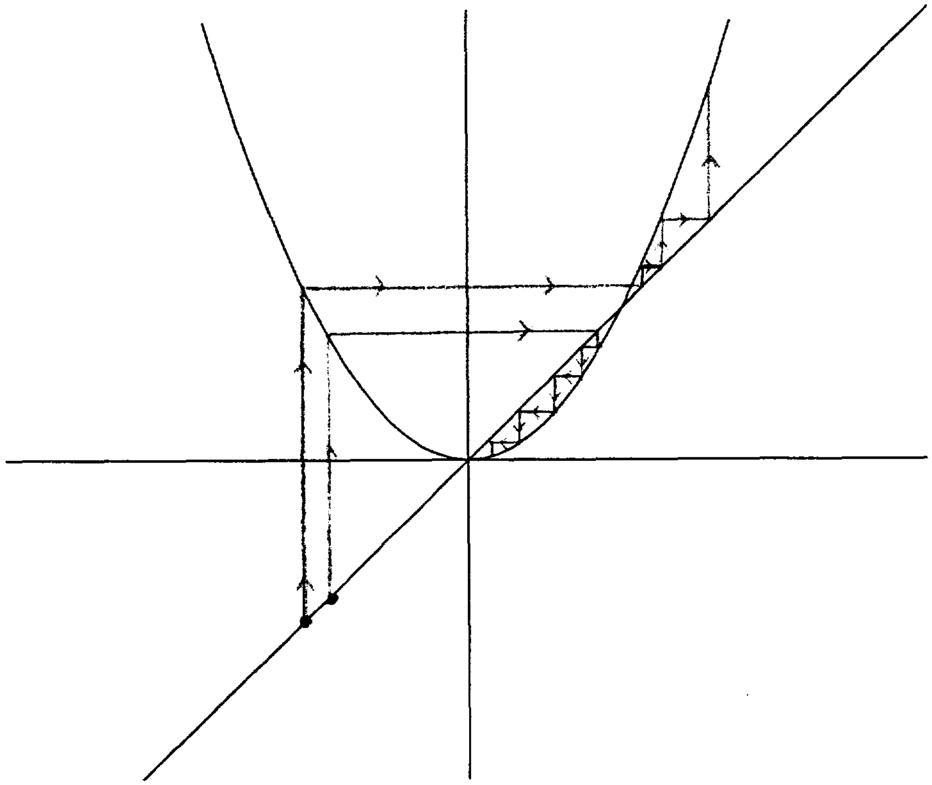

Figure [1]: Graphical analysis of $f(x) = x^2$.

Dynamics in the Polar Coordinate System: In the polar coordinate system, points are defined by a pair (r, θ) where r is the distance of the point from the origin and θ is the angle measured in radians from the positive x-axis to the segment joining the point to the origin. Polar coordinates, (r, θ), are related to rectangular coordinates, (x, y) by the following equations:

$$x = r\cos(\theta) \quad \textit{and} \quad y = r\sin(\theta).$$

We define periodic points in polar coordinates as in rectangular coordinates. A point θ is *periodic of period n* if $r^n(\theta) = \theta$ and θ is a fixed point if $r(\theta) = \theta$.

Example: Let $r(\theta) = k\theta$ for $k > 0$. The graph is a spiral. If $k = 1$, all points are fixed. If $0 < k < 1$, $Fix(r) = \{0\}$ and all points are attracted to zero, so that $W^s(0) = (-\infty, \infty)$. If $k > 1$, all points are repelled from zero and hence, $W^s(0) = \{0\}$.

It seems logical that one should approach graphical analysis in the polar coordinate system in a manner similar to that of the rectangular coordinate system; i.e., moving along the coordinate curves. In the polar coordinate system, the coordinate curves are $r = k$ and $\theta = k$, k constant, which are a circle with center at the origin and radius k and a radial line that passes through the origin and forms an angle of k with the positive x-axis, respectively. The analog, for the polar system, of the diagonal, $g(x) = x$, from the rectangular coordinate system, is the pair of spirals $\hat{r}(\theta) = \theta$. Note that $\hat{r}$ fixes all values. See Figure [2].

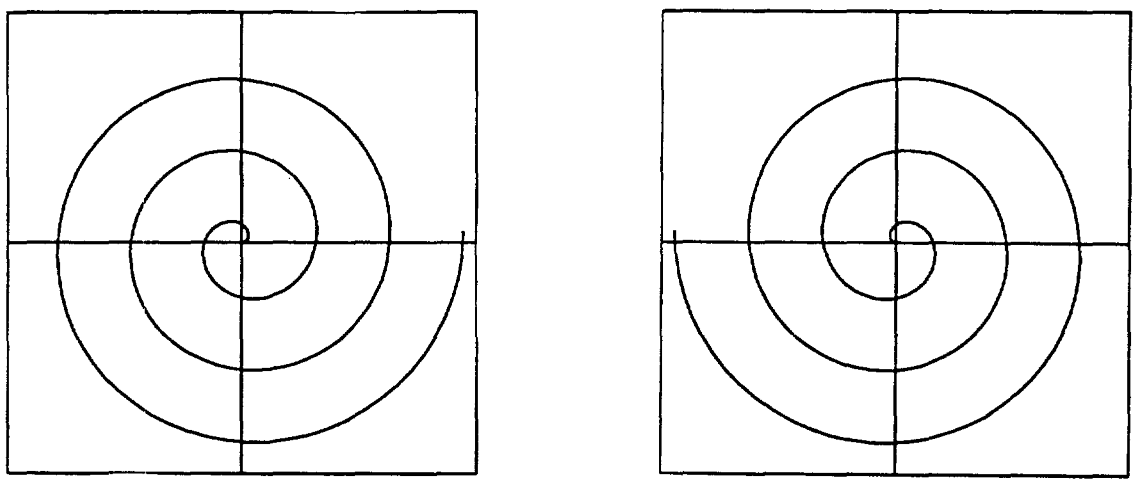

Figure [2]: $\hat{r}(\theta) = \theta$, $\theta \geq 0$ (left) and $\theta \leq 0$ (right).

If the range of the polar curve contains all positive or all negative values, only one spiral is necessary for graphical analysis. To analyze an orbit with a positive range, graph a polar curve, $r(\theta)$ and the spiral, $\hat{r}(\theta) = \theta, \theta \geq 0$ on the same axis system. Choose a value, say θ_0, and begin at (θ_0, θ_0) on the spiral. Draw the segment on the radial line $\theta = \theta_0$, from (θ_0, θ_0) to $(r(\theta_0), \theta_0)$. Then trace along the circle $r = r(\theta_0)$ until you meet the spiral. This point is $(r(\theta_0), r(\theta_0))$. As in rectangular coordinates, the forward orbit of θ_0 is traced on the spiral with this process.

Our first three examples are functions which are periodic with period 2π, hence, we shall only consider $\theta \in [0, 2\pi]$ for our initial values.

Example: $$r_1(\theta) = \frac{\pi}{2} - \frac{\pi}{2}\cos(\theta)$$

$Fix(r_1) = \{0, \frac{\pi}{2}, \pi\}$. As Figure [3] indicates, some points are attracted to $\theta = 0$ or $\theta = \pi$ and others are repelled by $\theta = \frac{\pi}{2}$. Note that $|r_1'(0)| = |r_1'(\pi)| = 0 < 1$ and $|r_1'(\frac{\pi}{2})| = \frac{\pi}{2} > 1$, so the previous definitions for attractors and repellors seem to apply in polar coordinates.

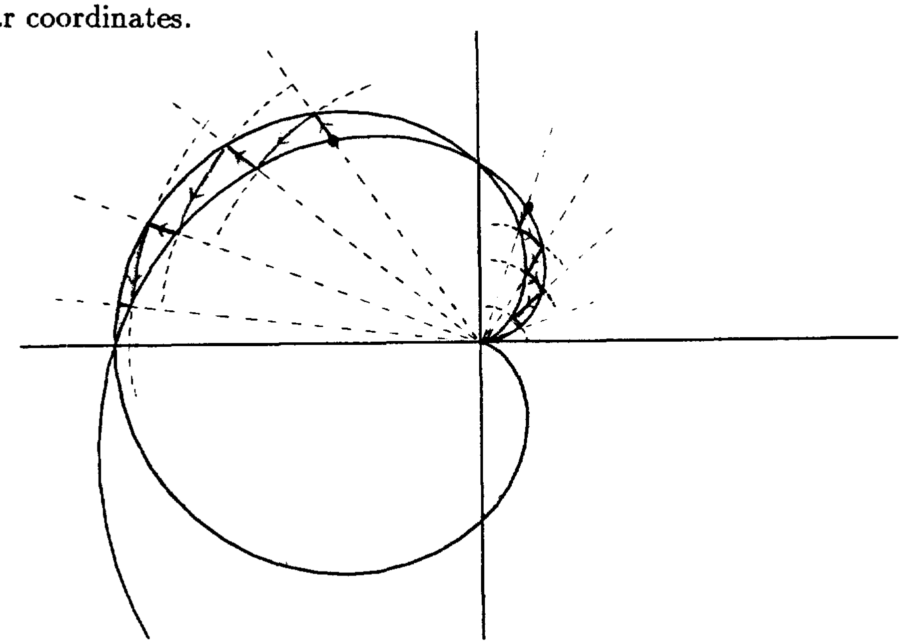

Figure [3]: Graphical analysis of $r_1(\theta) = \frac{\pi}{2} - \frac{\pi}{2}\cos(\theta)$.

Most polar functions are nonhomeomorphic and therefore may have eventually periodic points of period n. Consider the following example.

Example: $$r_2(\theta) = \frac{\pi}{2} + \frac{\pi}{2}\cos(\theta)$$

Note that $r_2(\theta)$ is nonnegative for all θ, which implies that $r_2^n(\theta)$ is nonnegative for all n and all θ. r_2 fixes $\theta = \frac{\pi}{2}$ and eventually fixes $\theta = \frac{3\pi}{2}$. $|r_2'(\frac{\pi}{2})| = \frac{\pi}{2} > 1$ so r_2 repels all points "close "to $\frac{\pi}{2}$. Also, since $\lim_{i\to\infty} r_2^i(\frac{3}{2}\pi) = \frac{\pi}{2}$, one can see that $W^s(\frac{\pi}{2}) = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. $Per_2(r_2) = \{0, \frac{\pi}{2}, \pi\}$. In Figure [4], graphical analysis shows that all other points are forward asymptotic to one of the periodic points.

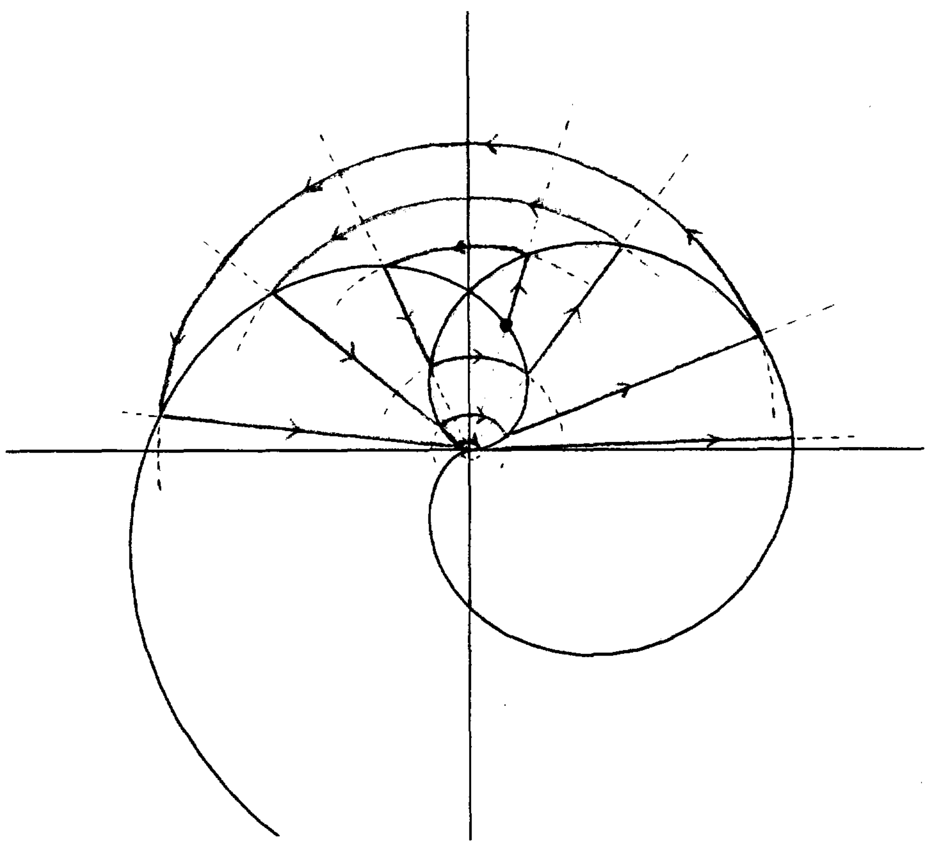

Figure [4]: Graphical analysis of $r_2(\theta) = \frac{\pi}{2} + \frac{\pi}{2}\cos(\theta)$.

To find $W^s(0)$ and $W^s(\pi)$, we evaluate $r_2^n(\theta)$:

$$r_2^n(\theta) = \frac{\pi}{2} + (-1)^{n-1} g^{n-1}(\frac{\pi}{2}\cos(\theta))$$

where

$$g(\beta) = \frac{\pi}{2}\sin(\beta).$$

For $0 < \theta < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$, $\beta = \frac{\pi}{2} cos(\theta) \in (0, \frac{\pi}{2})$ so $0 < g(\beta) < \frac{\pi}{2}$.

Note that $g(\beta) > \beta$ for all $\beta \in (0, \frac{\pi}{2})$, thus, $g^n(\frac{\pi}{2}\cos(\theta)) \to \frac{\pi}{2}$ as $n \to \infty$. Using the definition of forward asymptotic points, we see that

$$\lim_{i\to\infty} r_2^{2i}(\theta) = \lim_{i\to\infty} \frac{\pi}{2} + (-1)^{2i-1} g^{2i-1}(\frac{\pi}{2}\cos(\theta)) = 0$$

so that $W^s(0) = \{\theta : 0 \le \theta < \frac{\pi}{2} \text{ and } \frac{3\pi}{2} < \theta \le 2\pi\}$

For $\frac{\pi}{2} < \theta < \pi$ and $\pi < \theta < \frac{3\pi}{2}$, we have

$$\frac{-\pi}{2} < g(\frac{\pi}{2}\cos(\theta)) < 0.$$

In this interval, $g(\beta) < \beta$ and $g^n(\frac{\pi}{2}\cos(\theta)) \to -\frac{\pi}{2}$ as $n \to \infty$.

Therefore, $\lim_{i\to\infty} r_2^{2i}(\theta) = \pi$ and $W^s(\pi) = \{\theta : \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$.

Graphical analysis of a polar curve with nonpositive range values needs to be approached slightly differently. Orbits must be traced on the spiral that fixes all nonpositive values, i.e., $\hat{r}(\theta) = \theta$, $\theta \le 0$.

Example: Consider $r_3(\theta) = -\frac{\pi}{2} - \frac{\pi}{2}\cos(\theta)$

The range of r_3 is $[-\pi, 0]$. Therefore, orbits are traced on the spiral $\hat{r}(\theta) = \theta$, $\theta \leq 0$. $Fix(r_3) = \{-\frac{\pi}{2}\}$ and $Per_2(r_3) = \{0, -\frac{\pi}{2}, -\pi\}$. By graphical analysis in Figure [5], we find that points are forward asymptotic to $\theta = -\pi$ or $\theta = 0$ and are repelled by $-\frac{\pi}{2}$. By an argument similar to that in the preceding example, we find that $W^s(0) = \{\theta : 0 \leq \theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta \leq 2\pi\}$ and $W^s(-\pi) = \{\theta : \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$.

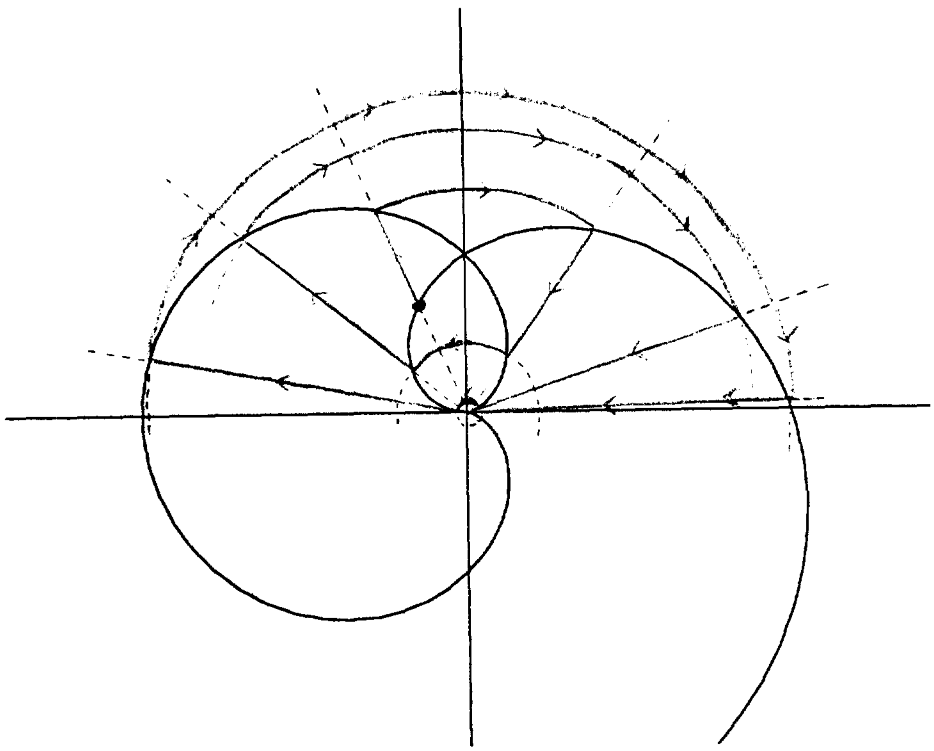

Figure [5]: Graphical analysis of $r_3(\theta) = -\frac{\pi}{2} - \frac{\pi}{2}\cos(\theta)$.

Curves with ranges containing both positive and negative values are more complex since both spirals are necessary for graphical analysis, since it must be determined when $r^n(\theta)$ is positive and when it is negative in order to use the correct spiral.

Example: Iterate $r_4(\theta) = -\theta$ for $\theta = \frac{3\pi}{4}$. $O^+(\frac{3\pi}{4}) = \{\frac{3\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, \ldots\}$. In graphical analysis (Figure [6]), begin at $(\frac{3\pi}{4}, \frac{3\pi}{4})$ on the spiral, $\hat{r}(\theta) = \theta$, $\theta \geq 0$. Trace the radial line from this point to $(r_4(\frac{3\pi}{4}), \frac{3\pi}{4})$ on the curve. Since $r_4(\frac{3\pi}{4}) = -\frac{3\pi}{4}$, the circular movement to a spiral should meet the spiral at $(-\frac{3\pi}{4}, -\frac{3\pi}{4})$. The appropriate spiral then is $\hat{r}(\theta) = \theta$, $\theta \leq 0$. The forward orbit is traced on both spirals. The curve $r_4(\theta) = -\theta$ is elementary enough that we know when the iterates are positive or negative. This is not the case for most polar curves.

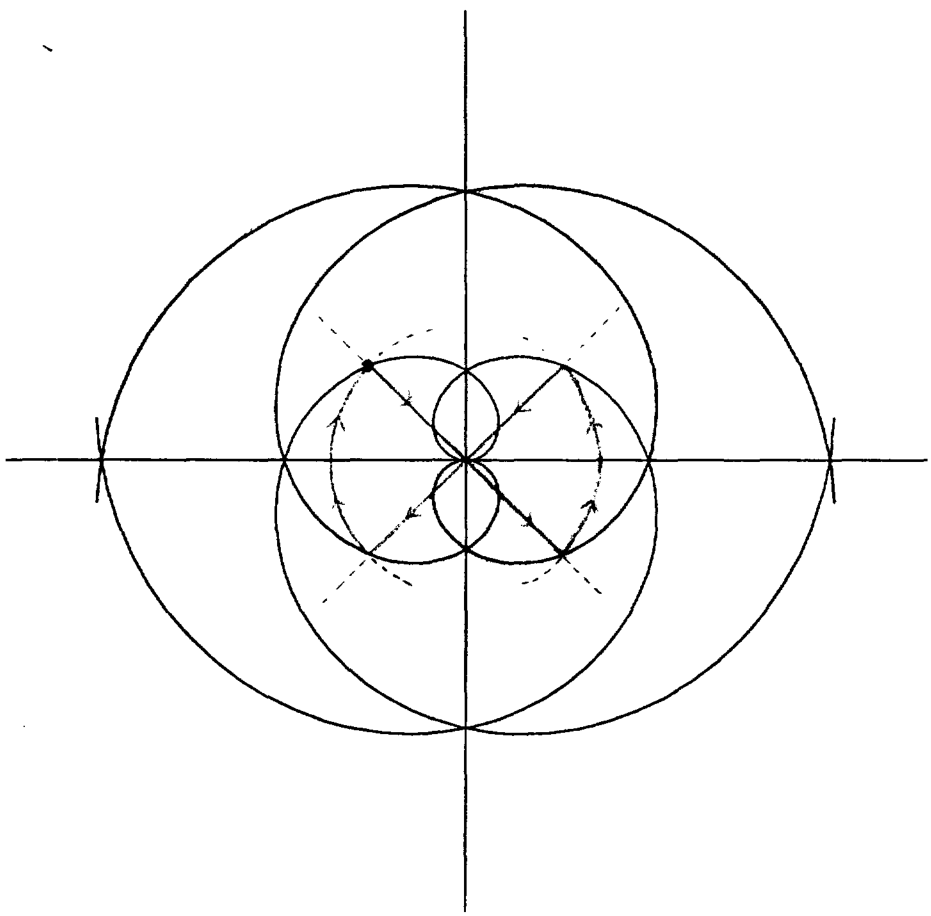

Figure [6]: Graphical analysis of $r_4(\theta) = -\theta$, $\theta = \frac{3\pi}{4}$.

Some Observations: As indicated in the previous section, most of the basic definitions from the rectangular coordinate system hold in the polar coordinate system. Also, graphical analysis appears to work in simple cases. However, there are some problems which arise when using the polar system.

One problem with graphical analysis in polar coordinates involves the origin. In the previous example, the radial line from $(\frac{3\pi}{4}, \frac{3\pi}{4})$ intersects the graph of r_4 first at the origin and then at $(r_4(\frac{3\pi}{4}), \frac{3\pi}{4})$. If we rely on the rules for graphical analysis alone, we should stop at the origin during the tracing of the orbit of $\frac{3\pi}{4}$, and, conclude that $\theta = \frac{3\pi}{4}$ is eventually fixed. However, since we know that $(r_4(\frac{3\pi}{4}), \frac{3\pi}{4})$ is not the origin, we continue on the radial line to the point $(r_4(\frac{3\pi}{4}), \frac{3\pi}{4})$.

In rectangular coordinates, fixed points occur at the intersections of the diagonal and the function. In polar coordinates, intersections of the spiral and the curve may not designate fixed points. Every point in polar coordinates has infinitely many representations, but if two polar curves intersect at a point, that point may not have the same coordinates on both curves. For example, $\dot{r} = 2\cos\theta$ and $\dot{r} = 2\sin\theta$ intersect at the origin; however, the origin is $(0,0)$ for $\dot{r}$ and $(0, \frac{\pi}{2})$ for $\dot{r}$. See Figure [7].

Example: $$r_5(\theta) = \frac{\pi}{2}\cos(2\theta)$$

By examining the graph of r_5 in Figure [8], one might guess that $\theta = \frac{\pi}{2}$ is a fixed point. Also, since $|r_5'(\frac{\pi}{2})| = 0 < 1$, $\frac{\pi}{2}$ also appears to be an attractor. However, $\theta = \frac{\pi}{2}$ is only eventually fixed. The actual attracting fixed point is $\theta = -\frac{\pi}{2}$.

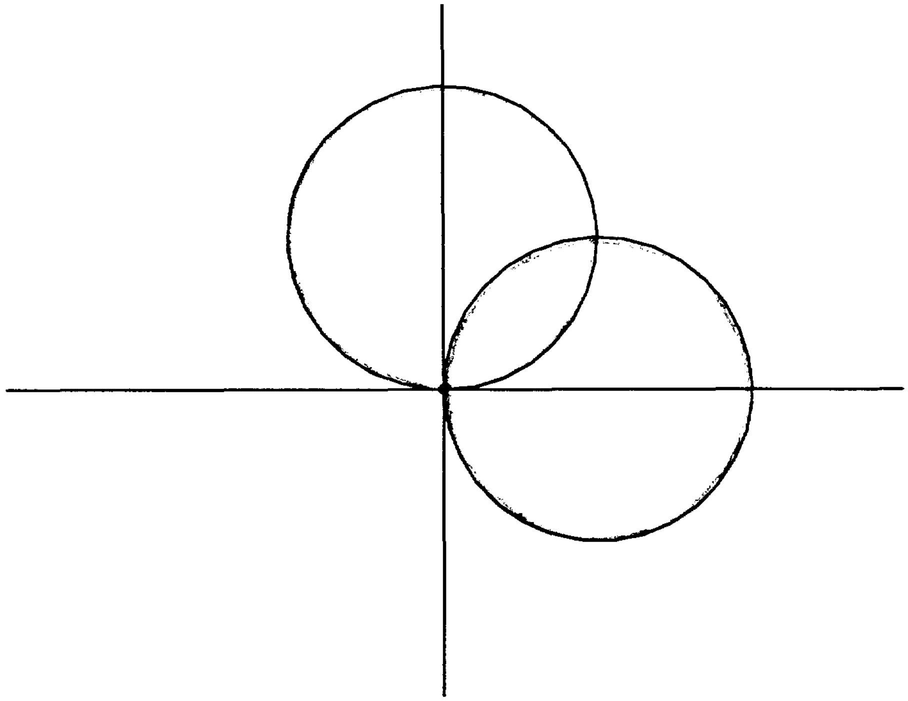

Figure [7]: $\dot{r} = 2\cos(\theta)$ and $\dot{r} = 2\sin(\theta)$.

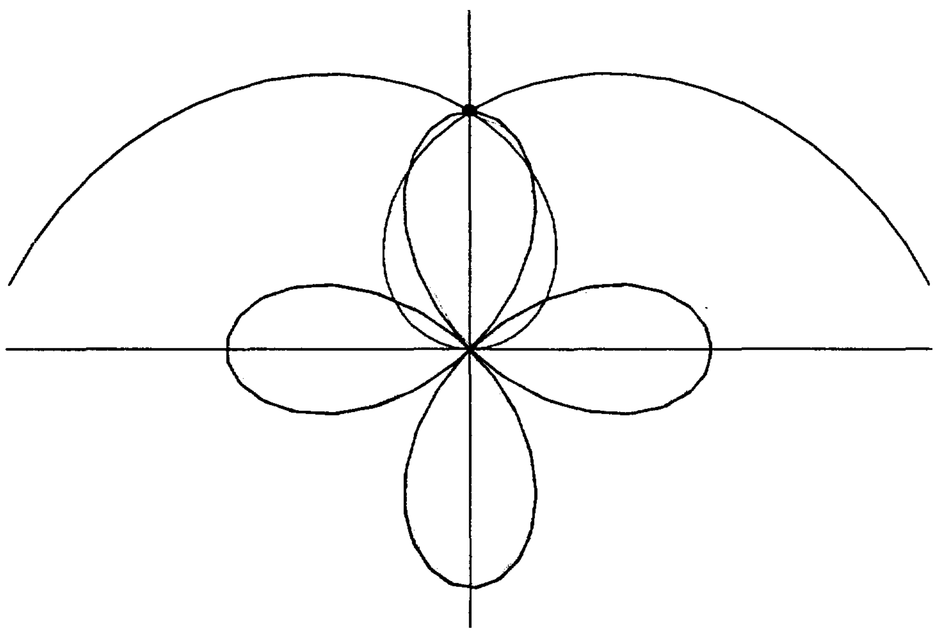

Figure [8]: $r_5(\theta) = \frac{\pi}{2}\cos(2\theta)$.

Example: $$r_6(\theta) = e^{\theta}, \theta \geq 0$$

As one can see from Figure [9], r_6 meets $\hat{r} = \theta, \theta \geq 0$ in infinitely many points, yet none of these points are fixed, eventually fixed, or periodic points since $\theta < e^{\theta} < e^{e^{\theta}} = r^2(\theta)$ and, in general, $\theta < e^{\theta} < r^n(\theta)$.

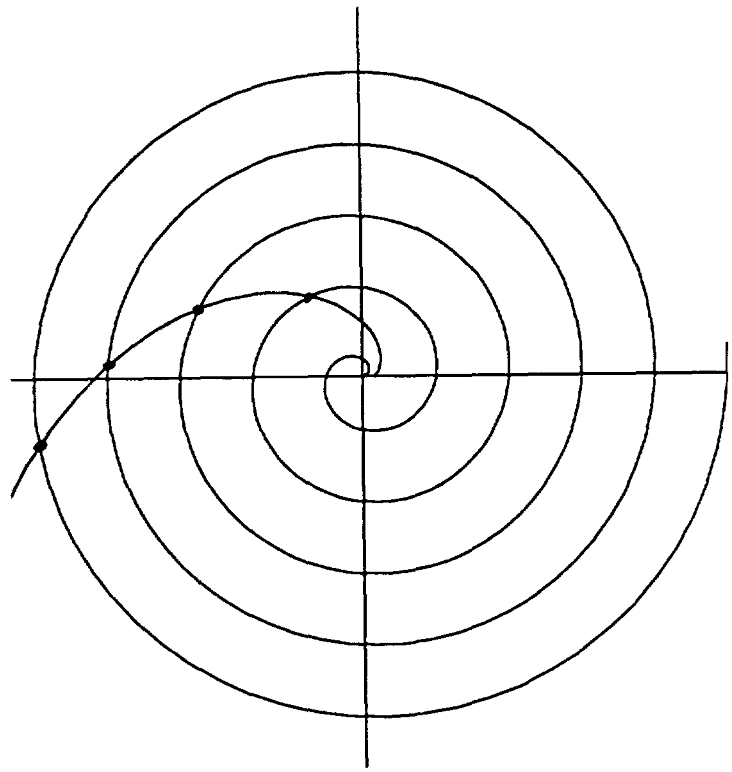

Figure [9]: $r_6(\theta) = e^{\theta}, \theta \geq 0$.

In the rectangular coordinate system a function $y = f(x)$ will meet any vertical line $x = k$ at a maximum of one point. In graphical analysis in polar coordinates, the analog of moving along a vertical line is the movement along a radial line. However, many polar curves will meet a radial line in more than one point, creating a dilemma where one must decide at which point to stop. This choice may not be clear.

Example: $$r_7(\theta) = \frac{\pi}{2} + \pi\cos(\theta)$$

In this example, any radial line drawn in the first or fourth quadrants will meet the graph of r_7 at two different points.

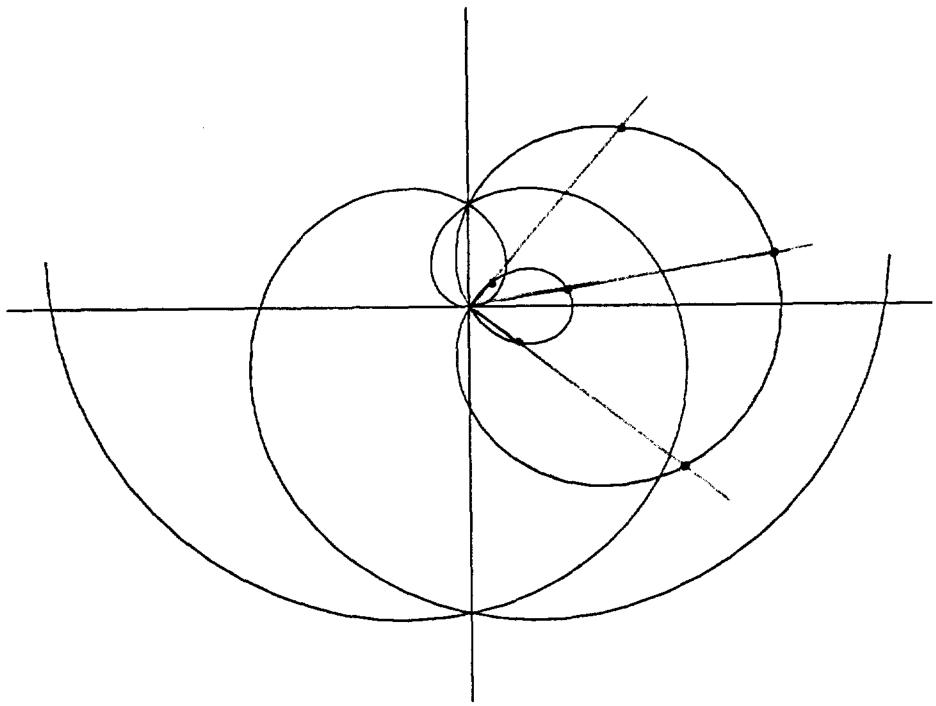

Figure [10]: $r_7(\theta) = \frac{\pi}{2} + \pi\cos(\theta)$.

Some polar curves have fixed points which are impossible to calculate. Also, some fixed points are not easily catagorized as repelling or attracting.

Example: $$r_8(\theta) = \frac{\pi}{2} + \frac{\pi}{2}\sin(\theta)$$

The point $\theta = 0$ is eventually periodic while $\theta = \frac{\pi}{2}$ and $\theta = \pi$ are periodic points of prime period 2. As Figure [11] shows, there is also a fixed point $\theta^* \in (\frac{3\pi}{4}, \pi)$. $r_8'(\theta^*) = \frac{\pi}{2}\cos(\theta^*)$ which can not be evaluated since θ^* can not be determined. In addition, we can not determine, from the graphical analysis, whether θ^* is either repelling or attracting.

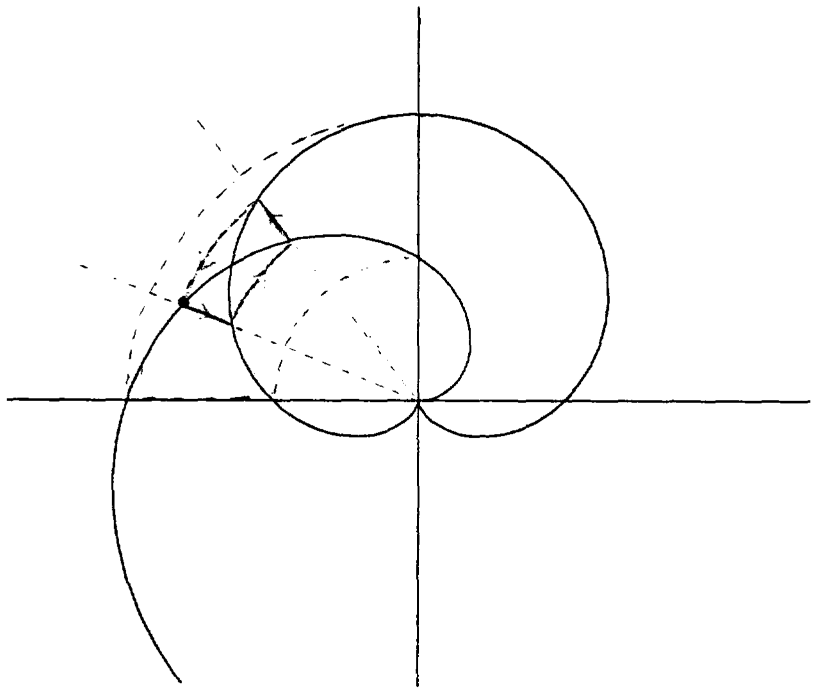

Figure [11]: Graphical analysis of $r_8(\theta) = \frac{\pi}{2} + \frac{\pi}{2}\sin(\theta)$.

Lastly, we observe that even with a limacon, i.e. $r(\theta) = a + b\sin(\theta)$ or $r(\theta) = a + b\cos(\theta)$, which is a fairly simple polar curve, we may not be able to do any analysis at all.

Example: $$r_9(\theta) = 2 + \cos(\theta)$$

It is virtually impossible to solve the equation $2 + \cos(\theta) = \theta$. Hence, we cannot find the fixed points. Also, since we cannot calculate $\cos(3)$, we cannot simplify the forward orbit, $O^+(0) = \{0, 3, \cos(3), 2 + \cos(\cos(3)), \ldots\}$. In fact, no analysis can be done on this simple polar curve. Actually, the reader should note that this problem does not exclusively belong to the polar coordinate system; the rectangular coordinate system has the same failing.

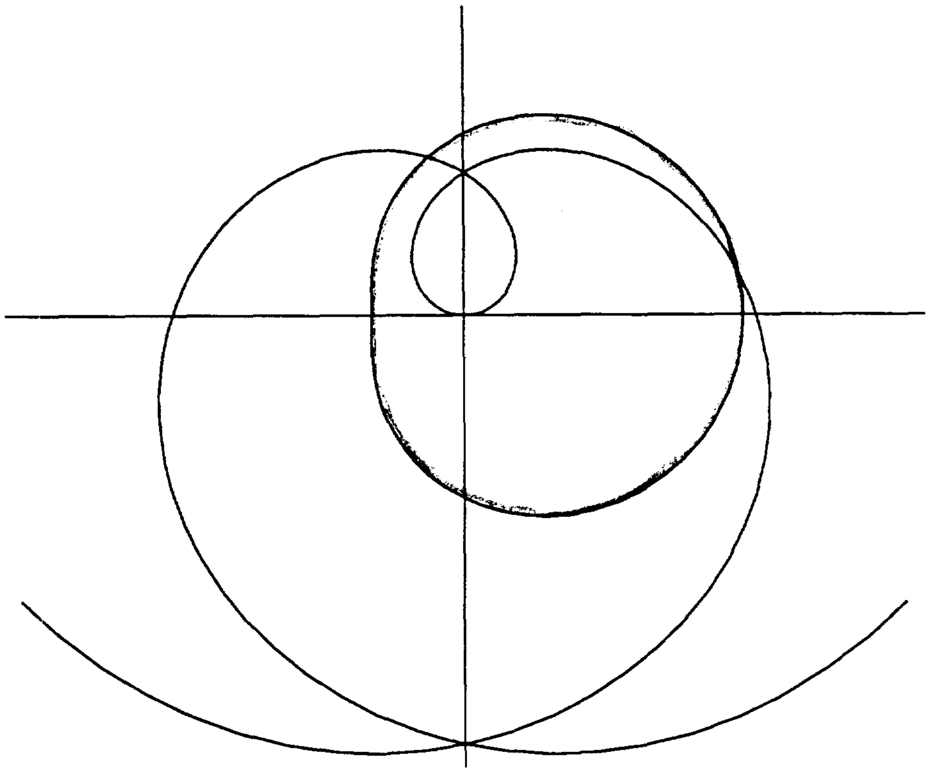

Figure [12]: $r_9(\theta) = 2 + \cos(\theta)$.

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[2] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 1987, Addison-Wesley, Inc., New York.