APPLICATIONS OF THE MONTE CARLO TECHNIQUE IN COMPUTER PROBLEMS

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One can get a fairly good insight into what we mean by Monte Carlo methods just from the name applied to it. We associate the idea of chance and gambling with Monte Carlo and, in general, any calculation which involves the use of random sampling can be referred to as a Monte Carlo calculation.[1] It is used to solve problems which depend in some way upon probability, where physical experimentation is impractical, and it is impossible to create an exact formula for the problem. Sometimes it can be used to find answers to physical questions often having no relation to probability.[2] George R. Stibitz has summed up a description of the method very concisely:

> "Fundamentally the Monte Carlo method replaces a determinate problem by a game of chance, and the solution for the determinate problem by the expected score on playing the game. Each move is dictated by a chance event, such as the draw of a card or inspection of the next digit in a random sequence of digits. The rules of the game are determined by the problem being solved and the scores obtained by playing the game are related to the solution."[3]

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[1] F.L. Alt and M. Rubinoff, ed., <u>Advances in Computers</u>, (New York, The Academic Press Inc, 1964), p. 323.

[2] Daniel D. McCracken, "The Monte Carlo Method", <u>Scientific American</u>, Vol. 192 No. 22, May 1955, p. 95.

[3] George R. Stibitz and Jules A. Larrivee, <u>Mathematics and Computers</u>, (New York, McGraw-Hill Book Co., Inc., 1957), p. 178.

Since the problems are based on randomness and repeated trials are necessary, the Monte Carlo method may be said to depend for feasibility on high speed computers.[4]

The two types of problems handled by Monte Carlo methods are called probabilistic or deterministic according to whether or not they are directly concerned with random processes or not. In the simplest Monte Carlo approach to a probabilistic problem, random numbers that simulate the physical random processes of the problem are observed until the solution can be inferred from their behavior. The second type or deterministic problem has no direct association with random processes, but by studying the problem and using mathematical theory a stochastic process with the same distribution can be constructed so that this problem can be solved numerically by Monte Carlo simulation. To distinguish direct simulation from this second technique, it is sometimes called sophisticated Monte Carlo.[5]

The name and systematic development of Monte Carlo methods started during World War II. Physicists John Von Neumann and Stanislas Ulam were studying how neutrons behave while traveling through various materials. Their solution resembled a roulette wheel and consequently the code name Monte Carlo.[6] Credit for the systematic development

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[4] Stibitz, op. cit., p. 176.

[5] J.M. Hammersley and D.C. Handscomb, Monte Carlo Methods, (New York, John Wiley and Sons, Inc., 1964), pp. 2-4.

[6] McCracken, op. cit., p. 90.

of these ideas is given to Harris and Herman Kohn in 1948.[7] Many isolated incidents of the use of Monte Carlo methods can be found which were previous to 1944, but proper credit is still given to these men as independently rediscovering and publishing their methods. One example is the story of George Louis LeClerc de Buffon, an eighteenth century aristocrat and gambler, who used the fundamental properties of the Monte Carlo technique to find the area of an irregularly shaped lake. He drew the outline of the lake in the center of a measured square of paper and threw needles at random onto it. The proportion of the needles landing in the lake would give the area of the lake in proportion to that of the square, the area of which was already known.[8] More complex Monte Carlo techniques were discovered in a paper by Lord Kelvin on the Boltzmann equation about 1904. But Kelvin was more concerned with results rather than the techniques which apparently seemed obvious to him. With the intensive study of Monte Carlo methods in the 1950's there was an attempt to solve every problem in sight by Monte Carlo. This served eventually to discredit the subject since all of these problems could not be solved efficiently with these methods. But recently, with the recognition of those problems in which it is the best and maybe even the only technique of finding a solution, Monte Carlo methods have come back into

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[7] Hammersley, op. cit., p. 8.

[8] Roger Piper, The Story of Computers, (New York, Harcourt, Brace and World, Inc., 1964), p. 134.

favorable use.[9]

One of the main advantages of the Monte Carlo method is its versatility.[10] Whenever the experiment is easier to perform than the numerical methods, then the use of random sequences is worth while as a practical method of computation.[11] It is characteristic of the method that the error decreases rapidly at first and then very slowly. So the greatest value of the method may be in getting very rough estimates for a solution after which standard methods for improving the solution may be applied or transformations made on the equations and the Monte Carlo method applied again.[12] But much work remains to be done on the method since accuracy of Monte Carlo approximations improve only as the square of the number of trials. In other words, to double the accuracy of an answer, the number of trials must be quadrupled. This could mean that as many as 100 million trials could be necessary for one problem, and this is still impracticable even on the fastest present computers. Another drawback is that results of one Monte Carlo calculation can seldom be extended to another set of conditions.[13]

As we have mentioned, all Monte Carlo computations have one essential process in common - at some point in the

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[9] Hammersley, op. cit., pp. 8-9.

[10] Alt, op. cit., p. 323.

[11] Stibitz, op. cit., p. 177.

[12] Ibid., p. 183.

[13] McCracken, op. cit., p. 96.

calculations a set of actual values must be substituted for a random variable. These values that are substituted are called random numbers.[14] To use the Monte Carlo method successfully on automatic computers a plentiful supply of random numbers must be easily available to the machine when they are needed in computations.[15] Since random numbers are so essential to Monte Carlo methods, they must be defined and understood before solutions can be found. "In theory a random number is one of a long series in which any number is as likely as another to occur without regard to the numbers coming before or after it."[16] The question comes to mind now as to how a computer can produce random numbers as it needs them since randomness suggests anything but a formula or process. Therefore mathematicians define random numbers so that for all practical purposes this sequence of numbers satisfies statistical tests of being random.

A sequence such as this which is statistically random is called pseudorandom. If we let the symbol $\mathcal{E}$ stand for a random number, then we can compute a pseudorandom sequence from a sequence of **positive** integers x_i from the relation $\mathcal{E}_i = x_i/m$ where m is a suitable positive integer.[17] Another

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[14] Hammersley, op. cit., p. 25.

[15] Stibitz, op. cit., p. 188.

[16] Piper, op. cit., p. 135.

[17] Hammersley, op. cit., pp. 26-27.

method for generating pseudorandom sequences was suggested by a mathematician named Lehmer in 1951. It is referred to as the recurrence relation $x_i \equiv ax_{i-1}$ (modulo m) and can be generalized to $x_i \equiv ax_{i-1} + c$ (modulo m). Here m is a large integer determined by the design of the computer (usually a large power of 2 or of 10), and a, c, x are integers between 0 and m-1. The numbers x_i/m are then used as the pseudorandom sequence.

These formulas are called congruential methods of generating pseudorandom numbers. Sequences such as these will repeat themselves after at most m steps and will therefore be periodic. If we choose m=16, a=3, c=1, x_0=2, the sequence of x's generated by the recurrence formula $x_i \equiv ax_{i-1} + c$ (mod m) is 2, 7, 6, 3, 10, 15, 14, 11, 2, 7, . . . so the period is 8. We must be sure that the period is longer than the number of random numbers required in any single experiment, and usually the value of m is large enough to permit this. The full period of m can always be achieved provided:

1) c and m have no common divisors;
2) $a \equiv 1$ (mod p) for every prime factor p of m;
3) $a \equiv 1$ (mod 4) if m is a multiple of 4.

There is another type of sequence called quasirandom that is also used in some kinds of Monte Carlo work. It is really a non-random sequence having only particular statistical properties that concern the problem. It can be used when the violation of some statistical tests will not invalidate the result.[18]

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[18] Hammersley, op. cit., p. 27.

There are several published tables of random numbers available for Monte Carlo work, the best known being M. G. Kendall and B. Babington Smith (1939) Tables of Random Sampling Numbers, Tracts for Computers; and Cambridge University Press and Rand Corporation (1955), A Million Random Digits with 100,000 Normal Deviates., Glencoe, Illinois: Free Press. The Rand tables are also available on punched cards for direct use in the computers.[19]

After a discussion of random numbers we come to the applications of the technique, the most interesting side of a study of Monte Carlo methods. The methods are becoming widely used to solve many different types of problems in all situations. Since Ulam and Von Neumann were among the first to develop the method to any extent, it was used mainly on problems of nuclear physics, such as the diffusion of neutrons, the absorption of gamma rays and atomic pile shielding.[20] Other scientific problems, such as the lifetime of comets, or controlling floodwater can even be efficiently solved with Monte Carlo methods. The military has found it useful in simulating air battle models. Maybe the most promising field which is still new and has much potential is in the area of business and operations research. It has been used in the analysis of storage systems and inventory policy, in the study of bottlenecks and queueing systems in industrial production processes.[21] Actuarial applications of

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[19] Hammersley, op. cit., p. 26.

[20] McCracken, op. cit., p. 96.

[21] Hammersley, op. cit., p. 44.

Monte Carlo methods have found great use in recent years. A closer look at some specific examples of these applications will give a clearer insight into how Monte Carlo methods operate.

Actuarial applications are becoming more prominent in the insurance business and have many fascinating results. The method in one example is used in making mortality studies for a closed group of lives. Basically this technique simulates the mortality outcome of the group with the construction of a model which has the same probabilistic properties. If a life has probability q_x of death, then a random number is given the same risk of being less than or equal to q_x - an event which has the same probability q_x. The solution of the model results in a frequency distribution of claim costs for the group being studied. This technique makes only one assumption, that q_x is the actual probability of death for each life in the group.

This routine of comparing a random number with the probability of death, repeated many times for a group of lives being studied, simulates several trials of the mortality experience for the year. An experiment of this type was programmed on the Datatron 205 with a deck of cards, one card for each life in the group containing the age, sex, and amount of insurance in force for that life, being the input of the program. The output showed the amount of claims for each trial, the average claims for all trials, and finally a frequency distribution of claims.

The group consisted of 306 males aging from 23 to 75 years old with insurance coverage from $2,000 to $10,000. The expected claims for this group were $17,200, based on the 1950-1954 intercompany group mortality experience. After 100 trials were made, the average amount of claims per trial was $17,625, which shows how very close the Monte Carlo technique estimates the expected claims. The following was the frequency distribution of claims for the group:

Amount of claims	Number of trials
0 - 25,000	82
25,000 - 27,000	0
27,000 - 30,000	9
30,000 - 35,000	5
35,000 - 50,000	4
Greater than 50,000	0
Total	100

Thus, there is an 18% chance that claims for the year will exceed $25,000, a 9% chance of exceeding $30,000, a 4% chance of exceeding $35,000, and a very small chance of exceeding $50,000. A further breakdown of claims for the 0 - $25,000 range was not available for this problem.[22]

Monte Carlo techniques are also useful in determining distribution charts that illustrate the probability that with a certain premium formula the insurance company will not suffer a loss which is greater than a specified per cent

[22] Russell M. Collins, Jr., "Actuarial Application of the Monte Carlo Technique", Reprinted from the Transactions of the Society of Actuaries, Vol. XIV, 1962, pp. 365-367.

on X number of policies. It can even be used to find what premium to charge for a policy if X number are sold so that the loss will be less than a certain per cent. The following is an illustration of how Monte Carlo methods could be applied to such problems. The plan of insurance in the problem provided a sum insured for n years with an endowment E payable at the end of the term period. The 2n+1 different margins were constructed from information already known concerning probability of death and withdrawal for the insurance period. (Margins for a policy are ". . . essentially a type of asset share at the time of termination of that policy. . ."). The computer stored the 2n+1 margins and the problem associated with the 2n+1 ways in which a policy can be terminated. For a specific number of policies issued annually, say 15, to obtain a distribution of average margins per $1000 of sum insured a Monte Carlo process determined the losses and gains for the year by selecting a termination year for each of the 15 policies. By repeating this operation 3000 times for the 15 policies, the 3000 average margins were used to draw the graph of the frequency distribution.[23] Different distribution curves for per cent losses according to the number of policies sold could be computed in the same manner.

Many unexpected and fascinating applications of Monte

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[23]John M. Boermeester, "An experiment concerning confidence limits for gross premiums" Part I, Actuarial Research Conference on Risk Theory and Topics of Multivariate Analysis, University of Michigan, 1966.

Carlo techniques are sometimes discovered as Mr. Burton D. Jay illustrated in his article in "The Actuary" newsletter. He described how the Monte Carlo method was applied to an insurance sales contest called GIBINGO. For every "GIB" or Guaranteed Issue Benefit option sold, a letter B, I, N, G, or O was to be awarded randomly according to the last digit of the policy number. The salesman would be awarded $50 when he had collected all five letters for every policy sold below $10,000 or $100 for policies over $10,000. A "wild" card or letter needed was given whenever a salesman issued a new policy as a result of the Guaranteed Insurability option. The Sales Department wanted to know what the costs of the contest would be and what the chances were of a salesman getting a BINGO if he sold five, six, etc. G.I.B.'s. The Monte Carlo method was applied after calculations grew too inefficient beyong eight sales. The contest was cancelled when cost was found to be $10,800 with a standard deviation of just under $2,000 and it was found that the prizes would be too small to provide real incentive for a reasonable volume of sales. But finding new uses of the method for all types of problems can provide not only enjoyment but a valuable knowledge of refinements and practices which are useful for future problems.[24]

Telephone companies are other businesses having to deal

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[24] Burton D. Jay, "Risk Theory: An Interesting Application", The Actuary, The Newsletter of the Society of Actuaries, Vol. I, No. 1, March 1967, p. 4.

with probabilistic problems such as setting up exchanges. Chance arises because the demand for service at any one time depends on individual decisions and is quite unpredictable. The planner must decide how many of the switches that are used in the course of a telephone call should be installed. To do this he uses a suitable probability distribution for random fluctuations to the average calling rate at the busy hours of the day. Using the Monte Carlo method can produce the random fluctuations needed.[25]

Military use of this method in the study of air battle models is another example of determining an outcome for an event by random choice. The computer keeps track of aircrafts and missles at each increment of time. As time moves on, each item is "moved" according to its mission, speed, and other characteristics. Planes are detected, bombs dropped and so on. These events and outcomes are probabilistic in nature so the use of the Monte Carlo technique simulates this aspect.[26]

The use of probability is not so obvious in the next example of Monte Carlo calculations applied to the control of floodwater and the construction of dams on the Nile. It is considered a probabilistic problem because the quantity of water in the river varies randomly from season to season.

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[25] Hammersley, op. cit., p. 3.

[26] Andrew G. Favret, Introduction to Digital Computer Applications, (New York, Reinhold Publishing Corporation, 1965), p. 136.

The data used consisted of records of weather, rainfall, and water levels over a period of 48 years. The problem was to see what would happen to the water if certain dams were built and certain water control policies were exercised. Each combination of dam sites and policies plus certain meterological conditions in extremely dry or wet years and in a typical year all had to be examined. Then each combination had to be evaluated in terms of construction costs and other economic factors. Theoretical mathematics could not be used because of the many practical details such as the characteristics of the river bed and losses by evaporation. But direct Monte Carlo simulation could be used even though it still demanded a large quantity of calculations on a high speed computer.[27]

Another direct simulation problem without too many scientific details is concerned with the lifetime of comets. A long period comet follows a sequence of elliptic orbits with the sun at one focus. Scientists know that the comet's energy is inversely proportional to the length of the semi-major axis of the ellipse and at one short time when the comet passes through the immediate vicinity of the sun and planets on its orbit, the comet loses some of its energy because of the gravitational field of Jupiter and Saturn. The energy is decreased by a random component, thus suggesting use of the Monte Carlo technique. The successive energy losses may be taken as independent random numbers, n_1, n_2,

[27]Hammersley, op. cit., pp. 43-44.

n_3, . . ., drawn from a standardized normal distribution. A comet starting with an energy $-z_0$, has on the next orbits energies of $-z_0$, $-z_1 = -z_0 + n_1$, $-z_2 = -z_1 + n_2$. . . This process continues until z changes sign which means the comet departs on a hyperbolic orbit and is lost from the solar system.

According to Kepler's Third Law, the time taken to describe an orbit with energy -z is $z^{-3/2}$ and the total lifetime of the comet would be:

$$G = \sum_{i=0}^{T-1} z_i^{-3/2},$$ where z_T is the first negative quantity in the sequence z_0, z_1, . . .
G is a random variable and the problem is to determine its distribution for a specific z_0. Theoretical handling of the problem is too difficult because of the -3/2 exponent, but simulation makes the problem easier to solve.

The required n_i can be generated by the formula:

$$n = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_N - \tfrac{1}{2}N$$

(which depends on the central limit theorem) with N = 12. This procedure was repeated a large number of times and the proportion p(g) of those values of G which did not exceed g gave a direct estimate of the required distribution function $P(G \leq g)$.[28]

Applications dealt with in physics are many times too complex to understand without a certain knowledge of nuclear physics, but one illustration of low energy showers can be

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[28] Hammersley, op. cit., pp. 44-45.

briefly examined for the basic principles of the Monte Carlo method used. Because of the statistical nature of these showers they are easily treated by the Monte Carlo method. The material chosen in the article was lead. As a photon is absorbed in the lead, it forms a pair of electrons which in turn radiate photons again. After several stages of this cycle the initial energy becomes divided among many low energy electrons and photons. To simulate this phenomenon, the distance into the lead was divided into intervals of about one millimeter. The electrons or photons were then followed through the intervals and their fate established by spinning a wheel of chance. This wheel of chance was a cylinder which had a family of curves drawn on it relating to properties of electrons and photons. It was tested for randomness and the details of the curves are not necessary in understanding the basic principle of using the Monte Carlo method. The wheel was spun at each interval until the electron had lost all its energy. Then the operator would follow the electron and photons until the whole shower was dead.[29]

A final application, which is closely associated with problems of diffusion processes such as the neutron showers example, deals with percolation processes. They are concerned "with deterministic flow in a random medium, in contrast with diffusion processes which are concerned with

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[29]Robert R. Wilson, "Showers Produced by Low Energy Electrons and Photons", <u>Monte Carlo Method</u>, (U.S. Department of Commerce, National Bureau of Standards Applied Mathematics Series, 12), pp. 1-2.

random flow in a deterministic medium." For most problems of this type, Monte Carlo methods provide the only known way of obtaining quantitative answers.

A typical example of this type of application is called bond percolation on the cubic lattice. The question is to what extent will the interior of a porous material get wet if put in a bucket of water. Some of the interconnecting pores of the material are large enough to convey water and some too small so that they block any passage of water. The situation is idealized by supposing the pores form a cubic lattice. Places where pores interconnect are called sites and two sites which are a unit distance apart are neighbors. Pores are also referred to as bonds and each has probability p of being large enough to transmit water and probability $q=1-p$ of being too small. When we immerse the chunk of material all sites on its surface become wet, and from these sites the water flows through unblocked bonds into the interior.

The percolation probability $P(p)$ stands for the proportion of interior sites which become wet when M (the number of sites along each edge of the chunk) is very large. It is a non-decreasing function of p, such that $P(0) = 0$ and $P(1) = 1$. There exists a number p_0 called the critical probability, such that when the proportion of unblocked pores is less than p_0, the water wets only the skin of the lump. But when the proportion is greater than p_0, the water wets the interior almost uniformly. Symbolically,

$P(p) = 0$ when $0 \leq p < p_0$, while $P(p) > 0$ for $p_0 < p \leq 1$. If we wish to calculate p_0 the critical probability, it's not hard to imagine what an enormous amount of calculation would be involved if direct simulation were used. Random numbers would label each bond as either blocked with probability q or unblocked with probability p, and each site would be examined. If the site were connected to a wet site by an unblocked bond, the examination would continue until no more sites could be made wet. Finally we would count the number of wet sites. But if M is large, say M=200, there are eight million sites and twenty-four million bonds in the chunk. This would require a tremendous amount of storage facility in the computer. To examine all the sites it might require one or two hundred repetitions before no more sites could be wetted. Then to obtain a graph of P(p) we must repeat the work for about fifty different values of p. This would require about 10^{12} or 10^{13} pieces of information and would keep a modern high speed computer busy for about fifty years.

By changing our viewpoint slightly, we can overcome this difficulty. Instead of starting water from all sites along the surface and looking at the inward flow, we can start the water at just one fixed interior site called the source site and follow its outward flow. If $P_N(p)$ represents the probability that the water will wet at least N other sites, then as N approaches infinity, $P_N(p)$ approaches P(p). It turns out that $N \sim 6000$ is a sufficiently large number.

Besides reducing storage requirements, the Monte Carlo experiment needs to be repeated only until either all N sites are wet or fewer than N are wet, but no unblocked bonds lead to a dry site. The total computing time reduces to about one year, still a very large amount of computing to be practical.

One more transformation on the problem can reduce the computing time even further. This time we can calculate P(p) simultaneously for all values of p from only one Monte Carlo experiment instead of repeating the experiment for several values of p to build up the graph of P(p) as in the second version. Before, we labeled each bond as either blocked or unblocked; while now we assign a rectangularly distributed random variable $\mathcal{E}$ independently to each bond. Infinitely many different fluids start from the same source site instead of one single fluid as before. Each fluid will have a number g in $0 \leq g \leq 1$ and will be called the g-fluid. If we say that a particular bond with an assigned random $\mathcal{E}$ will be capable of transmitting the g-fluid if and only if $g \leq \mathcal{E}$, then if the g_0-fluid wets S sites then all g-fluids that are less than g_0 will also wet S sites. If g_N is the maximum value of g such that g-fluids wet N or more sites, it can be proved that $P(g_N \geq 1-p) = P_N(p)$. Therefore all that needs to be calculated is a sample of values for g_N. The proportion of values $g_N \geq 1-p$ will be an estimate of $P_N(p)$ for each p in $0 \leq p \leq 1$.[30]

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[30] Hammersley, op. cit., pp. 134-139.

This bond percolation problem has demonstrated not only the Monte Carlo technique but the value of using transformations on the problems from direct simulation to make the Monte Carlo method more efficient and practicable.

These applications are quite varied and as the Monte Carlo technique is used more and more, new refinements will make it an efficient method for solving many other problems in science and business. With the rise of the computer age, knowledge of the Monte Carlo method is becoming a valuable asset in obtaining the best use of the computers that are available to most companies. Solutions that were once impossible or which involved a prohibitive amount of computation are now being solved with efficiency and accurate results because of man's ever-searching mind and constant desire to make all things possible and easier in the present world.

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