PERIODIC SOLUTIONS OF A CERTAIN SECOND ORDER DIFFERENTIAL EQUATION

SENIOR THESIS

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INTRODUCTION

The study of oscillations in numerous linear and nonlinear physical systems (electric and electronic circuits, machines, satellites, etc.) leads to linear homogeneous differential systems with periodic coefficient matrices. Such systems can not be solved except in trivial cases and to obtain information about the solutions, we must investigate the systems directly. The basic results were obtained by Gaston Floquet (4) in 1883, and are commonly called "Floquet theory." This theory yields a complete description of the possible analytic forms that the solution may have, and shows that the long term behavior of the solutions is determined by the eigen values of a certain matrix.

We consider a system

$$x' = A(t)\cdot x \tag{1.0}$$

where $A(t)$ is continuous for all t and of period $p>0$, that is $A(t + p) = A(t)$ for all t. To clarify the exposition, we must assume that p is the least period of $A(t)$. A monodromy matrix of (1.0) is a nonsingular matrix C associated with a fundamental matrix solution $X(t)$ of (1.0) through the relation

$$X(t + p) = X(t)\cdot C.$$

The eigen values ρ of a monodromy matrix are called characteristic multipliers of (1.0) and any λ such that $\rho = e^{\lambda p}$ is called a characteristic exponent of (1.0). One notices that the characteristic exponents are not uniquely defined but the multipliers are. The real parts of the characteristic exponents are uniquely defined and we can always choose the exponents λ as the eigen values of B, where B is any matrix so that $C = e^{Bp}$. The characteristic multipliers do not depend upon the particular monodromy matrix chosen; that is, the particular fundamental solution used to define the monodromy matrix. In fact, if $X(t)$ is a fundamental matrix solution, $X(t + p) = X(t)\cdot C$ and $Y(t)$ is any other fundamental matrix solution, then there is a non-singular matrix D such that $Y(t) = X(t)\cdot D$. Therefore, $Y(t + T) = X(t + T)\cdot D = X(t)\cdot C\cdot D = Y(t)\cdot D^{-1}\cdot C\cdot D$ and the monodromy matrix for $Y(t)$ is $D^{-1}\cdot C\cdot D$. On the other hand, matrices which are similiar have the same eigen values. It is customary to use the term monodromy matrix for $X(t)$, where $X(t)$, $X(0) = I$, is a fundamental matrix of (1.0) known that there is a periodic solution of (1.0) of period p (or 2p) if and only if there is a multiplier equal to +1 (or −1). The difficulty is that the multipliers are the eigen values of $X(p)$ and the theory offers no way of computing them short of computing the solutions themselves. But if it were possible to compute the solutions (in the sense of finding formulas for them), there would be no need for the theory. This difficulty is very great indeed. It has been, and continues to be, a source of much research. It is to the solution of this problem that we address this paper, not however, in full generality here, suggested at all, but on a far more modest scale to be delineated in the succeding sections.

SECTION II

PERIODIC SOLUTIONS OF A CERTAIN SECOND ORDER DIFFERENTIAL EQUATION

Lemma 1

Let $a(t)$ be a function $\in C^0$ and periodic of period ω.

Let

$$F(t)=\int_0^t a(u)\,du. \qquad (2.1)$$

Then $F(t)$ has period ω iff $F(\omega)=0$.

Proof

Define the function $G(t)=F(t+\omega)$. So, F has period ω iff $G(t)=F(t)$. But,

$$G(t)=\int_0^{t+\omega} a(u)\,du. \qquad (2.2)$$

Hence, by the Fundamental theorem of calculus,

$$G'(t)=a(t+\omega)=a(t)=F'(t).$$

So,

$$G(t)-F(t)=\text{ constant }=c. \qquad (2.3)$$

Setting $t=0$ in (2.3) yields

$$G(0)-F(0)=c.$$

But,

$$F(0)=0.$$

Thus,

$$G(0)=c.$$

Hence,

$$G(t)-F(t)=G(0)=F(\omega).$$

Therefore,

$$G(t) = F(t) \text{ iff } F(\omega) = 0.$$

<u>Lemma 2</u>

Consider the equation

$$\frac{dx}{dt} + a(t)x(t) = b(t), \tag{2.4}$$

where a(t) and b(t) are functions $\in C^0$ and periodic of period ω. Let

$$F(t) = \int_0^t a(u)\, du, \quad \phi(t) = \int_0^t e^{\int_0^s a(u)\, du} \cdot b(s)\, ds.$$

Then, (1), if $F(\omega) \neq 0$, (2.4) has a unique solution of period ω;

(2), if $F(\omega) = 0$ and $\phi(\omega) = 0$, there is no solution of period ω;

(3), if $F(\omega) = 0$ and $\phi(\omega) = 0$, all solutions have period ω.

<u>Proof</u>

We know that, (1), x(t) has period ω, iff $x(0) = x(\omega)$.

But,

$$x(t) = e^{-\int_0^t a(u)\, du}\left[\int_0^t e^{\int_0^s a(u)\, du} \cdot b(s)\, ds + c\right]$$

ie.

$$x(t) = e^{-F(t)}\left[\phi(t) + c\right]. \tag{2.5}$$

So,

$$x(0) = e^{-F(0)}\left[\phi(0) + c\right] = C$$

and

$$x(\omega) = e^{-F(\omega)}\left[\phi(\omega) + c\right].$$

Hence, $x(0) = x(\omega)$ implies

$$C = e^{-F(\omega)}\left[\phi(\omega) + c\right]$$

$$\text{ie. } C(1 - e^{-F(\omega)}) = e^{-F(\omega)}\phi(\omega).$$

Thus,

$$C = \frac{e^{-F(\omega)}\phi(\omega)}{1 - e^{-F(\omega)}}. \tag{2.6}$$

So, the unique periodic solution will be

$$x(t) = e^{-F(t)}\left[\phi(t) + \frac{e^{-F(\omega)}\phi(\omega)}{1 - e^{-F(\omega)}}\right]. \tag{2.7}$$

<u>(2) and (3)</u>

$x(t)$ is periodic with period ω, iff $x(0) = x(\omega)$,

ie. iff $C = e^{-F(\omega)}\left[\phi(\omega) + c\right]$,

ie. iff $C = \phi(\omega) + c$ (in view of $F(\omega) = 0$)

ie. iff $\phi(\omega) = 0$.

Theorem 1

Let $a(t)$ and $f(t)$ be functions $\in C^1$ and periodic with period ω. Let $f(0) = 0$. Now, consider the equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + a'(t)x = f'(t) \qquad (2.8)$$

with $x'(0) = 0$ and $x(0) = 0$. Now,

$$\frac{d^2x}{dt^2} + \frac{d}{dt}(\, a(t)x \,) = f'(t).$$

Hence,

$$\frac{dx}{dt} + a(t)x = c + f(t),$$

where c = constant. Now, $x'(0) = 0$, $x(0) = 0$, and $f(0) = 0$ imply $c = 0$.

Let

$$F(t) = \int_0^t a(u)\, du,$$

and

$$\phi(t) = \int_0^t f(u)\cdot e^{\int_0^u a(y)\, dy}\, du.$$

Then, (1), equation (2.8) has a unique periodic solution if $F(\omega) \neq 0$;

(2), all solutions of (2.8) are periodic if $F(\omega) = 0$ and $\phi(\omega) = 0$;

(3), there is no periodic solutions if $F(\omega) = 0$ and $\phi(\omega) \neq 0$.

Example 1

Consider the differential equation

$$\frac{d^2x}{dt^2} + \sin^2(t)\frac{dx}{dt} + \sin(2t)\cdot x = \sin(2t), \tag{2.9}$$

with $x'(0) = 0$ and $x(0) = 0$. Here,

$$a(t) = \sin^2(t) = \tfrac{1}{2}(1 - \cos(2t))$$

and

$$f(t) = \sin(2t),$$

with $f(0) = 0$. Also, both $a(t)$ and $f(t) \in C^1$ and are periodic with period π.

Let

$$\begin{aligned} F(t) &= \int_0^t a(u)\,du \\ &= \int_0^t \tfrac{1}{2}(1 - \cos(2u))\,du \\ &= \tfrac{1}{2}(u - \tfrac{1}{2}\sin(2u))\Big|_0^t \\ &= \tfrac{1}{2}(t - \tfrac{1}{2}\sin(2t)). \end{aligned}$$

Now, $F(\pi) = \frac{1}{2}\pi \neq 0$. Then (2.9) has a unique periodic solution.

Example 2

Consider the differential equation

$$\frac{d^2x}{dt^2} + \sin(t)\frac{dx}{dt} + \cos(t)\cdot x = \sin(t), \tag{2.10}$$

with $x(0) = x'(0) = 0$. So,

$$a(t) = f(t) = \sin(t)$$

and $a(t)$ and $f(t) \in C^1$ and are periodic of period 2π. Also, $f(0) = 0$.

Let

$$F(t) = \int_0^t a(u)\ du$$

$$= \int_0^t \sin(u)\ du$$

$$= -\ (\cos(u))\Big|_0^t$$

$$= 1 - \cos(t).$$

So, $F(2\pi) = 1 - \cos(2\pi) = 0$. Let

$$\phi(t) = \int_0^t f(u)\cdot e^{\int_0^u a(y)\ dy}du$$

$$= \int_0^t \sin(u)\cdot e^{\int_0^u \sin(y)\ dy}\ du$$

$$= \int_0^t \sin(u)\cdot e^{(1 - \cos(u))}du$$

$$= (e^{1 - \cos(u)})\Big|_0^t$$

$$= (e^{1 - \cos(t)} - 1).$$

Now, $\phi(2\pi) = (e^{1 - \cos(2\pi)} - 1) = (e^0 - 1) = 0$. Thus, all solutions of (2.10) are periodic.

Example 3

Consider the differential equation

$$\frac{d^2x}{dt^2} + \cos(t)\frac{dx}{dt} - \sin(t)\cdot x = 1 - \cos(t) \tag{2.11}$$

with $x(0) = x'(0) = 0$. Now,

$$a(t) = \cos(t) \text{ and } f(t) = 1 - \cos(t).$$

Both $a(t)$ and $f(t) \in C^1$ and are periodic with period 2π.

Also,

$$f(0) = 1 - \cos(0) = 0.$$

Let

$$F(t) = \int_0^t \cos(u)\,du$$
$$= (\sin(u))\Big|_0^t$$
$$= \sin(t) - \sin(0)$$
$$= \sin(t).$$

Hence,

$$F(2\pi) = \sin(2\pi) = 0.$$

Let

$$\phi(t) = \int_0^t f(u)\cdot e^{\int_0^u a(y)\,dy}\,du$$
$$= \int_0^t (1 - \cos(u))\cdot e^{\int_0^u \cos(y)\,dy}\,du$$
$$= \int_0^t (1 - \cos(u))\cdot e^{\sin(u)}\,du$$
$$= \int_0^t e^{\sin(u)}\,du - \int_0^t \cos(u)\cdot e^{\sin(u)}\,du$$
$$= \int_0^t e^{\sin(u)}du - (e^{\sin(u)})\Big|_0^t$$
$$= \int_0^t e^{\sin(u)}du - (e^{\sin(t)} - e^{\sin(0)}).$$

So,

$$\phi(t) = \int_0^t e^{\sin(u)}\,du + (1 - e^{\sin(t)}).$$

Now,

$$\phi(2\pi) = \int_0^{2\pi} e^{\sin(u)}\,du + (1 - e^{\sin(2\pi)})$$
$$= \int_0^{2\pi} e^{\sin(u)}\,du + (1 - 1)$$
$$= \int_0^{2\pi} e^{\sin(u)}\,du.$$

Hence, $\phi(2\pi) \neq 0$ since $e^{\sin(u)} > 0$. Thus, no solution of (2.11) is periodic.

Lemma 3

Let $A(t) = \begin{bmatrix} a_1(t) & 0 \\ 0 & a_2(t) \end{bmatrix}$. Let $a_i(t)$, $i = 1,2$, be continuous and periodic functions of period ω. Let

$$F_i(t) = \int_0^t a_i(u)\, du, \quad i = 1,2.$$

Now, if $F_i(\omega) = 0$, $i = 1,2$, then the fundamental matrix of the differential equation

$$X' = A(t)\cdot X, \tag{2.12}$$

where X is a column vector, is periodic of period ω.

Proof

Let $X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $A(t)$ be as defined above. Then,

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} a_1(t) & 0 \\ 0 & a_2(t) \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Now,

$$\frac{dx}{dt} = a_1(t)\cdot x \tag{2.13}$$

and

$$\frac{dy}{dt} = a_2(t)\cdot y \tag{2.14}$$

Solving (2.13) for x we obtain

$$x(t) = B\cdot e^{\int_0^t a_1(u)\, du}; \tag{2.15}$$

and for (2.14),

$$y(t) = C\cdot e^{\int_0^t a_2(u)\, du}, \tag{2.16}$$

Now, (2.15) can be re-written as

$$x = B \cdot e^{F_1(t)}$$

and (2.16) as

$$y = C \cdot e^{F_2(t)}.$$

Now,

$$F_i(t) = \int_0^t a_i(u)\, du = F_i(t+\omega) = \int_0^{t+\omega} a_i(u)\, du, \quad i = 1,2.$$

Hence, if $F_i(\omega) = 0$, by lemma 1, $F_i(t)$ are periodic with period ω. So,

$$\Phi(t) = \begin{bmatrix} B \cdot e^{F_1(t)} & 0 \\ 0 & C \cdot e^{F_2(t)} \end{bmatrix}$$

is periodic with period ω. We also note that $\Phi(t)$ is the fundamental matrix solution of (2.12) since $\det \Phi(t) \neq 0$. Thus, $\Phi(t)$ is periodic with period ω, and is the fundamental matrix solution.

<u>Example 4</u>

Consider

$$X' = \begin{pmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{pmatrix} \cdot X, \tag{2.17}$$

where X is a column vector. So,

$$a_1(t) = \sin(t)$$
$$a_2(t) = \cos(t)$$

and $a_1(t)$ and $a_2(t)$ are periodic with period 2π.

Let

$$F_1(t) = \int_0^t a_1(u)\, du$$
$$= \int_0^t \sin(u)\, du$$
$$= 1 - \cos(t).$$

Hence, $F_1(2\pi) = 1 - \cos(2\pi) = 0.$

Let

$$F_2(t) = \int_0^t a_2(u)\, du$$

$$= \int_0^t \cos(u)\, du$$

$$= \sin(t).$$

So, $F_2(2\pi) = \sin(2\pi) = 0$. Hence, the fundamental matrix of (2.17) is periodic with period 2π.

Lemma 4

Let $A^{**}(t) = \begin{bmatrix} a_1(t) & 0 \\ a_3(t) & a_2(t) \end{bmatrix}$ where $a_i(t)$, $i = 1,2,3$,

are continuous and periodic functions with period ω.

Let

$$F_i(t) = \int_0^t a_i(u)\,du,\ i = 1,2.$$

Let

$$\psi(t) = \int_0^t a_3(s)\cdot e^{\int_0^s (a_1(u) - a_2(u))\,du}\,ds.$$

If $F_i(\omega) = 0$, $i = 1,2$, and $\psi(\omega) = 0$, then the fundamental matrix of

$$X' = A^{**}(t)\cdot X \tag{2.18}$$

is periodic with period ω.

Proof

Make the transformation

$$X = B(t)\cdot Y \tag{2.19}$$

where

$$B(t) = \begin{bmatrix} 1 & 0 \\ b(t) & 0 \end{bmatrix} \tag{2.20}$$

and $b(t)$ is a solution of

$$\frac{dx}{dt} + (a_1(t) - a_2(t))\cdot x = a_3(t).$$

In view of the hypothesis that $F(\omega) = 0$ and $\psi(\omega) = 0$, $b(t)$ is periodic with period ω by lemma 2. Then $B(t)$ as given in (2.20) is periodic with period ω.

The transformation (2.19) changes (2.18) to

$$Y' = A(t)\cdot Y \tag{2.21}$$

where

$$A(t) = \begin{bmatrix} a_1(t) & 0 \\ 0 & a_2(t) \end{bmatrix}$$

and is periodic with period ω. Again, in view of $F_i(\omega) = 0$, $i = 1,2$, by lemma 3, the fundamental matrix $Y^*(t)$ of (2.21) is periodic with period ω. Then, by (2.19) the fundamental matrix $X^*(t)$ of (2.18) is periodic with period ω.

<u>Example 5</u>

Consider

$$X' = \begin{pmatrix} \cos(t) & 0 \\ \cos(t) - \sin(t) & \sin(t) \end{pmatrix} X,$$

where X is a column vector. Let

$$a_1(t) = \cos(t)$$

$$a_2(t) = \sin(t)$$

$$a_3(t) = \cos(t) - \sin(t).$$

Then $a_i(t)$, $i = 1,2,3$, are periodic with period 2π.

Let

$$\begin{aligned} F_1(t) &= \int_0^t a_1(u)\, du \\ &= \int_0^t \cos(u)\, du \\ &= (\sin(u))\Big|_0^t \\ &= \sin(t) - \sin(0) \\ &= \sin(t). \end{aligned}$$

Hence, $F_1(2\pi) = 0$.

Let

$$F_2(t) = \int_0^t a_2(u)\, du$$

$$= \int_0^t \sin(u)\, du$$

$$= -(\cos(u))\Big|_0^t$$

$$= -(\cos(t) - \cos(0))$$

$$= 1 - \cos(t).$$

Hence, $F_2(2\pi) = 1 - \cos(2\pi) = 0.$

Let

$$F_3(t) = \int_0^t a_3(u)\, du$$

$$= \int_0^t (\cos(u) - \sin(u))\, du$$

$$= (\sin(u) + \cos(u))\Big|_0^t$$

$$= \sin(t) + \cos(t) - 1.$$

Hence, $F_3(2\pi) = \sin(2\pi) + \cos(2\pi) - 1 = 0.$

Let

$$\psi(t) = \int_0^t a_3(s)\, e^{\int_0^s (a_1(u) - a_2(u))\, du}\, ds$$

$$= \int_0^t (\cos(s) - \sin(s))\, e^{\int_0^s (\cos(u) - \sin(u))\, du}\, ds$$

$$= \int_0^t (\cos(s) - \sin(s))\, e^{(\sin(u) + \cos(u))\big|_0^s}\, ds$$

$$= \int_0^t (\cos(s) - \sin(s))\, e^{(\sin(s) + \cos(s) - 1)}\, ds$$

$$= (e^{(\sin(s) + \cos(s) - 1)})\Big|_0^t$$

$$= e^{(\sin(t) + \cos(t) - 1)} - e^{(\sin(0) + \cos(0) - 1)}.$$

So,

$$\psi(t) = e^{\sin(t) + \cos(t) - 1} - 1.$$

Hence,

$$\Psi(2\pi) = e^{\sin(2\pi) + \cos(2\pi) - 1} - 1$$

$$= e^{0 + 1 - 1} - 1$$

$$= e^{0} - 1$$

$$= 1 - 1 = 0.$$

Thus, the fundamental matrix of (2.22) is periodic with period 2π.

SECTION III

CRITIQUE

There are other directions of further possible researches. For example, one can examine the Hill' equation

$$X''(t) + a(t)\, X'(t) + b(t)\, X(t) = 0 \qquad (3.I)$$

where a(t) and b(t) $\varepsilon\ C^{o}$ and are periodic of period ω . Though it apparently appears that the analysis of this theses can be extended to (3.I) a close examination of the present work reveals the great difficulties inherent in (3.I) often calling upon ingeneious technical devices to deal with highly complicated situations. In a subsquent paper, the present author intends to examine (3.I) as well as to explore the scope and limitations of the techniques that have been employed in this thesis.

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