Using Manipulatives in Trigonometry and Calculus

An Honors Thesis (HONRS 499)

by

Melissa K. Taft

Thesis Advisor

Dr. Kay Meeks-Roebuck

Kay d. Meeks Roebuck

Ball State University

Muncie, Indiana

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Purpose of Thesis

This collection of activities uses manipulatives in order to teach trigonometry and calculus. Manipulatives are a way of teaching mathematics with a more hands-on approach. This helps to bring some of the more abstract ideas of mathematics into more of a concrete context. The purpose of these activities is for teachers to help their students to more easily understand the ideas and concepts of trigonometry and calculus with an activity that uses the senses to make it more concrete.

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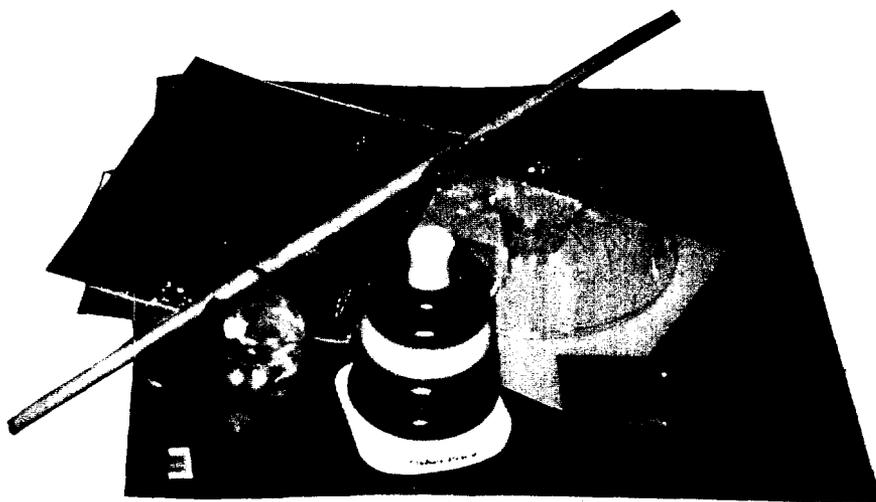
Introduction

Manipulatives in mathematics are often thought of as resources that are used in order to teach elementary level mathematics. Manipulatives can be defined as "objects that can be touched and moved by students to introduce or reinforce a mathematical concept" (U.S. Dept. of Ed. 1). This definition does not limit itself to those topics only at the elementary level. Mathematics, especially at the higher levels, is often thought of as something that is abstract. It makes sense then that, in order for students to better understand, there needs to be a connection made between the abstract and the concrete. This is the role that manipulatives can play in mathematics. As ideas become more abstract, it would seem that there would be a greater need for these strategies. The National Council of Teachers of Mathematics emphasizes the view that lasting understanding of the concepts and strategies of mathematics is a necessity and is strengthened by the use of manipulatives (Bippert 386). When students are using manipulatives, they are using several senses at one time. Therefore, things that are concrete will have more of an effect on a person than that which is abstract.

Research has been done about the use of manipulatives in mathematics. Unfortunately, most of this has been done at primarily the elementary level. It has been shown that manipulatives "promote active learning, build motivation, and counteract boredom" (Kober 30). It has also been found that the long-term use of manipulatives is more effective than short-term use (U.S. Dep. of Ed. 1). Even though the research is very much in favor of the use of manipulatives in order to teach mathematics, many teachers are still not using them. Reasons for this could include that teachers do not know enough about them and they do not feel they have the resources. When it comes to manipulatives, everyday items and things that can be

made are often used. There has not been a lot of research done in how manipulatives are used at the high school level. This is probably true because they are not used nearly as much as they are at the elementary, or even the middle school, levels.

The manipulatives that are used in secondary mathematics are often used in classes such as pre-algebra, algebra, and geometry. There is not much research that is being done on how manipulatives are being used in trigonometry or calculus. This paper, therefore, is a compilation of activities in which manipulatives are used in these subject areas to either introduce ideas or reinforce those that have already been taught. All of these strategies for teaching different topics give abstract ideas more of a concrete meaning. They use everyday materials that can easily be made or found. These activities will be helpful in teaching new ideas and reinforcing old ones, along with fostering more ideas of how many of the abstract ideas of higher level mathematics can be brought to more of a concrete level.

Subject: Trigonometric Functions
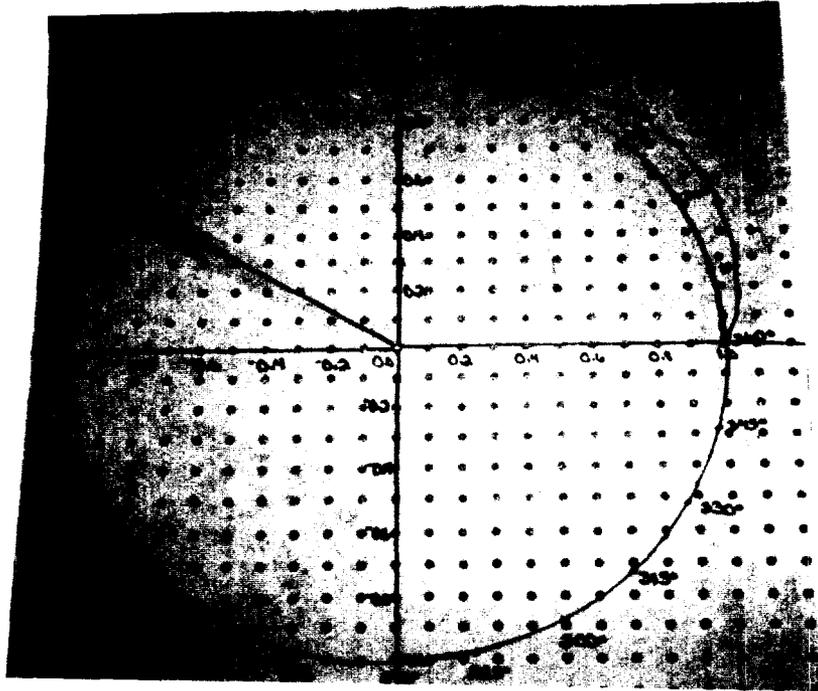
Trigonometric functions can be looked at with the help of a unit circle. This can be done because a radius drawn from the origin to a point (x,y) on the circle forms the hypotenuse of a right triangle. This triangle will have legs of length x and y, and a hypotenuse of length one. By constructing a unit circle board consisting of a 2 feet by 2 feet piece of pegboard with x- and y-axes marked, students will be able to see what the trigonometric function are and how they relate to one another.

Proficiency Indicators

1)   Understand the relationship between trigonometric and circular functions.

Procedures

1)   Use a piece of 2 feet by 2 feet pegboard and mark the x- and y-axes.

2)   Make sure that the scale of the two axes extends from -1 to +1, in which each

scale division of 0.1 value stands for one inch.

3) When this is finished, the end points of +1 and -1 will represent lengths of +10 inches and -10 inches.

4) The circle will then have a radius of 10 inches and the circumference should be labeled at 15 degree increments.

5) Attach a clear plastic rectangle, with a line printed on it to represent the radius of the unit circle on a Cartesian coordinate system with the center at the origin, so that it can rotate freely.

6) Have the students determine what part of the triangle that the radius is, in which the radius is drawn to the point (x,y) on the unit circle. (They should realize the radius of the circle is the hypotenuse of the triangle.)

7) Position the line on the plane on the plastic rectangle on the 15 degree mark and estimate the x- and y-values for the point on the unit circle at the end of the radius.

8) Ask the students what the position of the x- and y-values are in terms of a triangle where the radius forms the hypotenuse of the right triangle of length 1 and the legs are the lengths x and y.

9) Have the students come up with the trigonometric functions in terms of the radius, x, and y.

10) Have the students write these values on a sheet of paper.

11) Have the students then find the x- and y-values for all values from 30 to 360 degrees by 15 degree increments.

12) Ask the students to use the information they have found in order to determine the sign of the x-value, the y-value, the sine of the angle, the cosine of the angle, and the tangent of the angle in each quadrant.

13) From the relationships they have found for sine, cosine, and tangent, have them calculate these with their data using a calculator.

14) Compare the x, y, and the calculated tangent values to those they got when they calculated the cosine, sine, and tangent with their calculators.

15) Have the students then choose several arbitrary angles and use their calculators to find the sine, cosine, and tangent of each angle.

<u>Source</u>

Center for Occupational Research and Development. Applied Mathematics: <u>Unit 22: Using Trigonometric Functions</u>. Waco, Texas: Center for Occupational Research and Development, 1989.

Example Activity

	x	y	sin	cos	tan
15°	.95	.25	.25	.95	.26
30°	.85	.50	.50	.85	.59
45°	.70	.70	.70	.70	1.00
60°	.50	.88	.88	.50	1.76
75°	.25	.95	.95	.25	3.80
90°	0.00	1.00	1.00	0.00	undefined

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$$\sin = y/\text{rad.} = y/1 = y$$

$$\cos = x/\text{rad.} = x/1 = x$$

$$\tan = y/x$$

$$\sec = 1/\cos = 1/x$$

$$\csc = 1/\sin = 1/y$$

$$\cot = 1/\tan = x/y$$

Subject: Radian Measurements

The Radian Walk is a practice game that allows the students to review and exercise what they have learned about the relationship between the coordinates, radian measurements, and degree measurements of a unit circle. This allows the students to take part and not just memorize these types of measurement. The game uses the same idea as a cake walk with a little twist.

Proficiency Indicators

1)   Understand the relationship between trigonometric and circular functions.

2)   Locate points in the plane using both rectangular and polar coordinates.

Procedures

1)   Use electrical tape or masking tape to make a unit circle with a diameter of 12 feet on the floor.

2)   Make the x- and y-axes to mark the 90 degree angles and label them so students know the location of 0 radians.

3)   Mark the 30, 45, and 60 degree angles in each quadrant.

4)   Place a spinner at the origin and turn on a tape player and have the students walk around the circle until the music stops.

5)   At that point each student must be on one of the marked angles of the unit circle.

6)   Spin the spinner, which indicates the students who must name his or her coordinates, radian measurement, and degree measurements.

7)   If a students makes a mistake, s/he is eliminated from the game and the student remaining at the end wins.

Source

Kusy-Seuerson, Jean. "The Radian Walk". <u>Mathematics Teacher</u> 85 (1992): 726-27.

The unit circle with radian measures and coordinate points:

- $(0, 1)$ at $\pi/2$
- $(-1/2, \sqrt{3}/2)$ at $2\pi/3$
- $(1/2, \sqrt{3}/2)$ at $\pi/3$
- $(-\sqrt{2}/2, \sqrt{2}/2)$ at $3\pi/4$
- $(\sqrt{2}/2, \sqrt{2}/2)$ at $\pi/4$
- $(-\sqrt{3}/2, 1/2)$ at $5\pi/6$
- $(\sqrt{3}/2, 1/2)$ at $\pi/6$
- $(-1, 0)$ at π
- $(1, 0)$ at 2π
- $(-\sqrt{3}/2, -1/2)$ at $7\pi/6$
- $(\sqrt{3}/2, -1/2)$ at $11\pi/6$
- $(-\sqrt{2}/2, -\sqrt{2}/2)$ at $5\pi/4$
- $(\sqrt{3}/2, -\sqrt{3}/2)$ at $7\pi/4$
- $(-1/2, -\sqrt{3}/2)$ at $4\pi/3$
- $(1/2, -\sqrt{3}/2)$ at $5\pi/3$
- $(0, -1)$ at $3\pi/2$
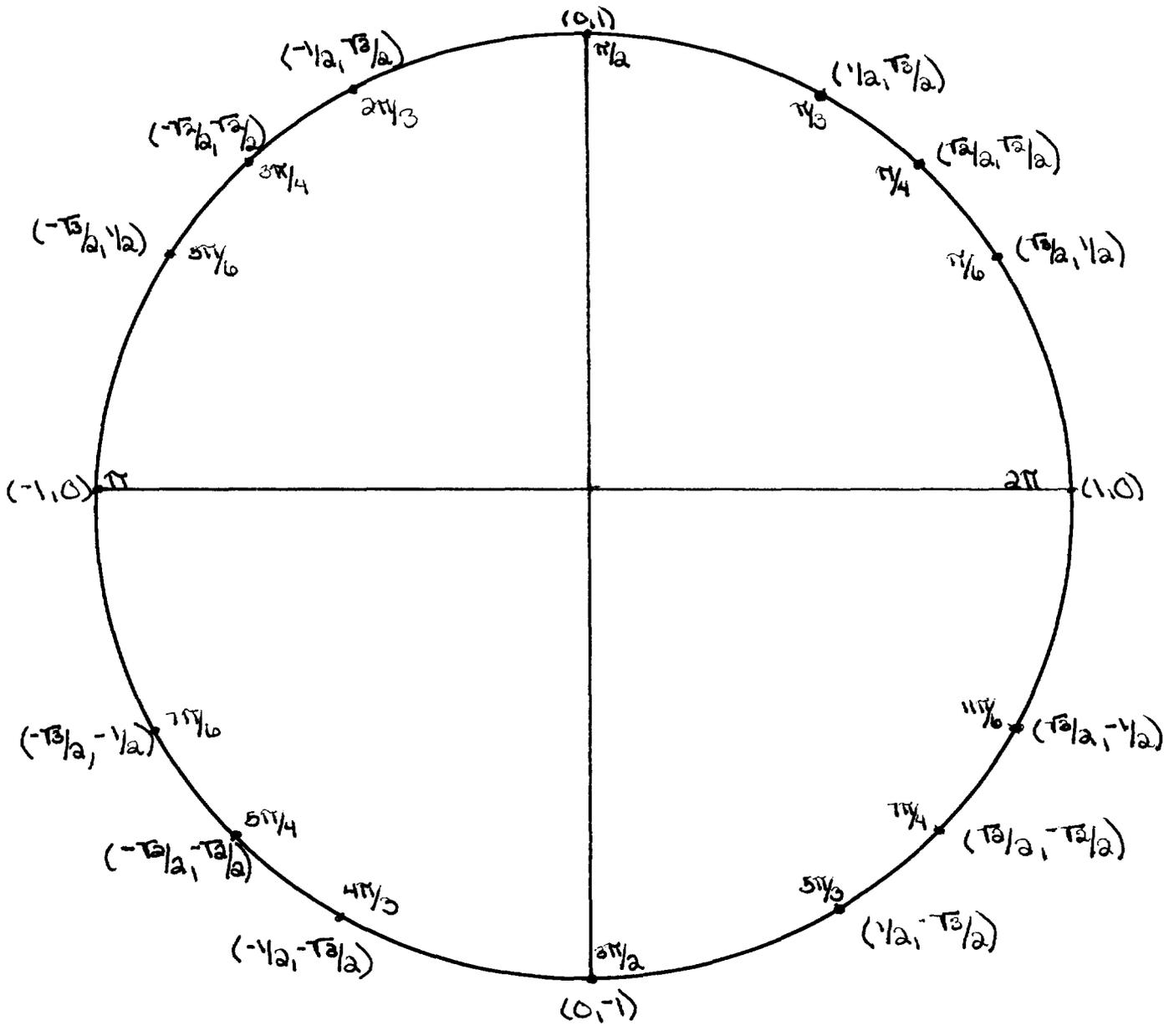
Subject: Reference Angle

The reference angle is a part of trigonometry with which many students have difficulty. The definition of a reference angle is "the positive acute angle formed between the terminal ray and the x-axis". One way of making this more clear for the students, besides simply defining it, is to construct a Cartesian plane with a terminal ray in order for students to be able to visualize and better understand the definition.

Proficiency Indicators

Understand the relationship between trigonometric and circular functions.

Procedures

1) Use a Cartesian plane of heavy black poster board, which is in the shape of a square with an x- and y-axis drawn on the board and a brightly colored terminal ray that freely swings around the quadrant.

2) Explain that the reference angle is the angle formed between the x-axis and the colored ray.

3) Call out different angles of rotation and have a student volunteer rotate the ray about the plane to the approximate location of the angle.

4) First ask for the measure of the reference angle.

5) Then explain that a reference triangle is a triangle formed with an altitude between the x-axis and the tip of the terminal ray.

6) Have the volunteer draw, with chalk, the altitude. (Be careful to use the visual effect so that the students do not perceive the reference angle as being located anywhere but between the x-axis and the terminal ray.)

7)   Have the students construct a small version of the Cartesian plane for their own use. (Whenever students are having problems with concept of a reference angle, ask them to explain the idea in terms of their model.)

Source

Stanulonis, Joanne. "Where Is My Reference Angle?" <u>Mathematics Teacher</u> 85 (1992): 537.
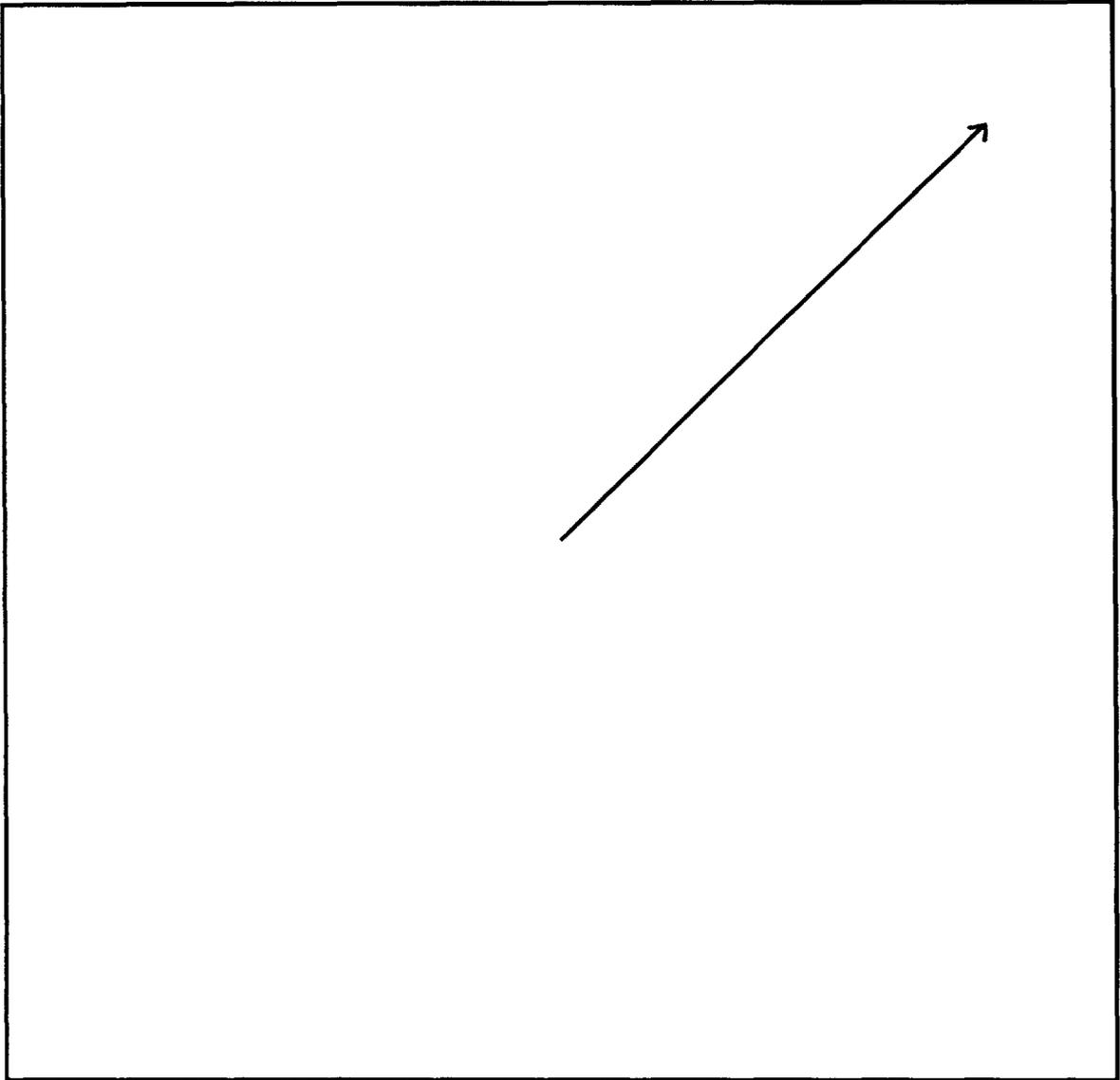
Subject: Exponential Growth

This activity combines two ways of looking at exponential growth. The first part of this activity will involve the students folding paper. The other will have the students tossing cubes. In both of these, they will be doing activities and collecting data in order to graph their information and compare them to the graphs of the exponential functions, $y=2^X$ and $y=(1/2)^X$.

<u>Proficiency Indicators</u>

1) Represent and analyze relationships using tables, verbal rules, graphs, and equations.

2) Describe real world phenomena with tables, graphs, equations, and rules.

3) Make translations among tables, equations, verbal rules, and graphical representations of functions.

Procedures

1) The students should first break up into pairs in order to do these activities.

2) Have the students then make a chart with the columns: Number of Folds, Number of Regions Formed on Paper, and Area of Smallest Region (as demonstrated in the example).

3) Each pair should collect this data by taking different sized pieces of paper and folding them, beginning the chart with zero folds and continuing until the paper can no longer be folded.

4) Have each student make a graph with the number of regions on the y-axis and the number of folds on the x-axis.

5) Ask them to then graph $y=2^x$ and explain how these graphs compare with the other two graphs in each pair.

6) Explain that these are characteristics of the graph of exponential functions.

7) Then have each student make a graph with the area of the smallest region on the y-axis and the number of folds on the x-axis.

8) Ask the students how this graph compares to their first graph and how they think they could change the equation $y=2^x$ to make a graph that would closer resemble their graphs.

9) They should figure this to be $y=2^{-x}$ or $y=(1/2)^x$.

10) Have the pairs of students then take cubes of some sort that have one side that differs from the other sides. (The kind used here has one side with a point sticking out.)

11) One student should start with 100 cubes in a small paper or plastic cup and cover the cup and shake it.

12) Then s/he spills the cubes on a flat surface, making sure they form only a

single layer.

13) S/he should remove those with the distinctive side facing up in the air.

14) The other student in each pair should record the information on the chart.

15) The students should then continue the process for fifteen trials, recording the information on the table.

16) Have the pairs graph this information with the number of the trial on the x-axis and the number left on the y-axis.

17) Have the pairs determine how this graph relates to exponential growth in terms of the graphs they drew earlier.

Source

Meeks, Kay I. and Melissa A. Warden. EXCUSe ME: EXperiment, Connect and Understand Science and Mathematics Enjoyably. Central Regional Conference of the National Council of Teachers of Mathematics, 24-26 March 1994.

Example Activity - Paper Folding   (This example uses an 8.5 inches by 11 inches sheet of paper)

Number of Folds	Number of Regions Formed on Paper	Area of Smallest Region
0	1	93.5
1	2	46.75
2	4	23.375
3	8	11.6875
4	16	5.84375
5	32	2.921875
6	64	1.4609375
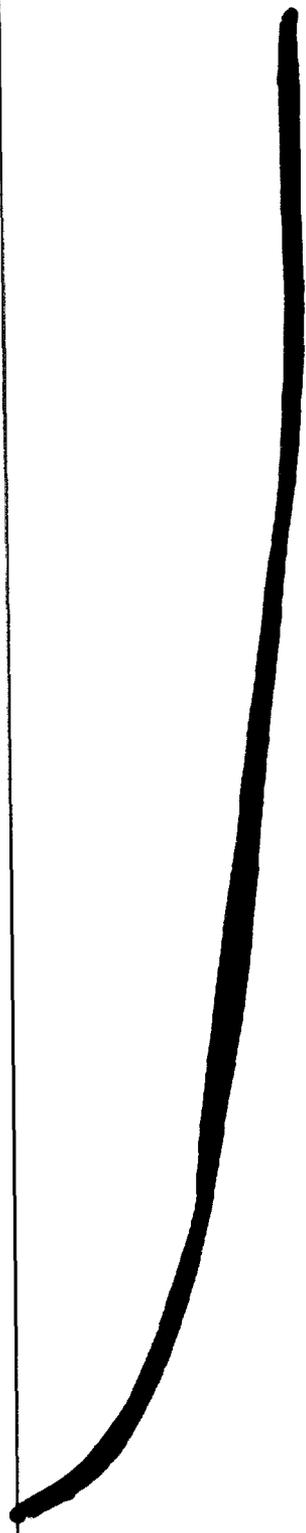
$y = 2^x$
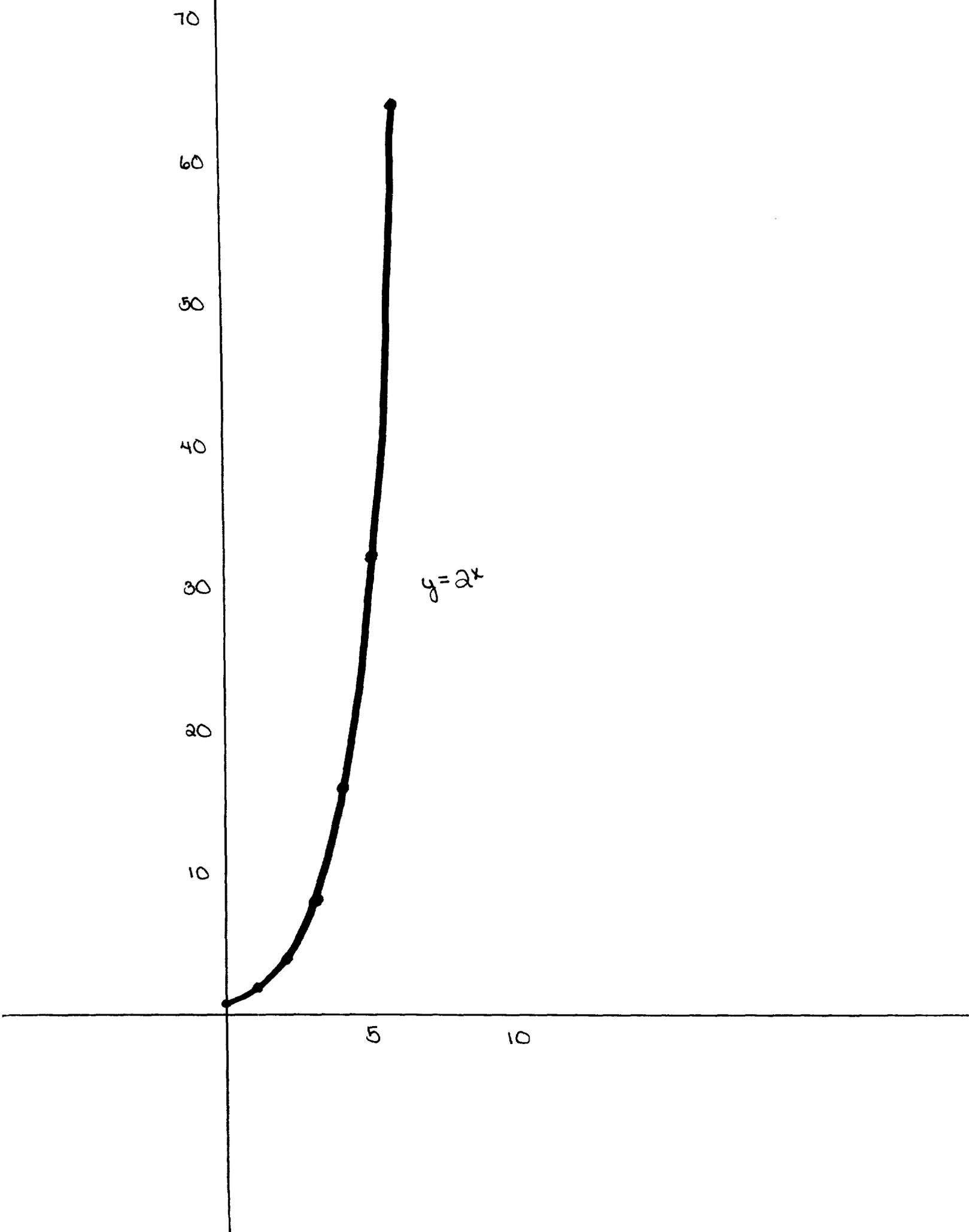
Example Activity - Tossing Cubes

Trial	Number Started With	Number Removed	Number Left
1	100	11	89
2	89	16	73
3	73	11	62
4	62	13	49
5	49	10	39
6	39	4	35
7	35	5	30
8	30	7	23
9	23	3	20
10	20	5	15
11	15	3	12
12	12	2	10
13	10	2	8
14	8	4	4
15	4	1	3
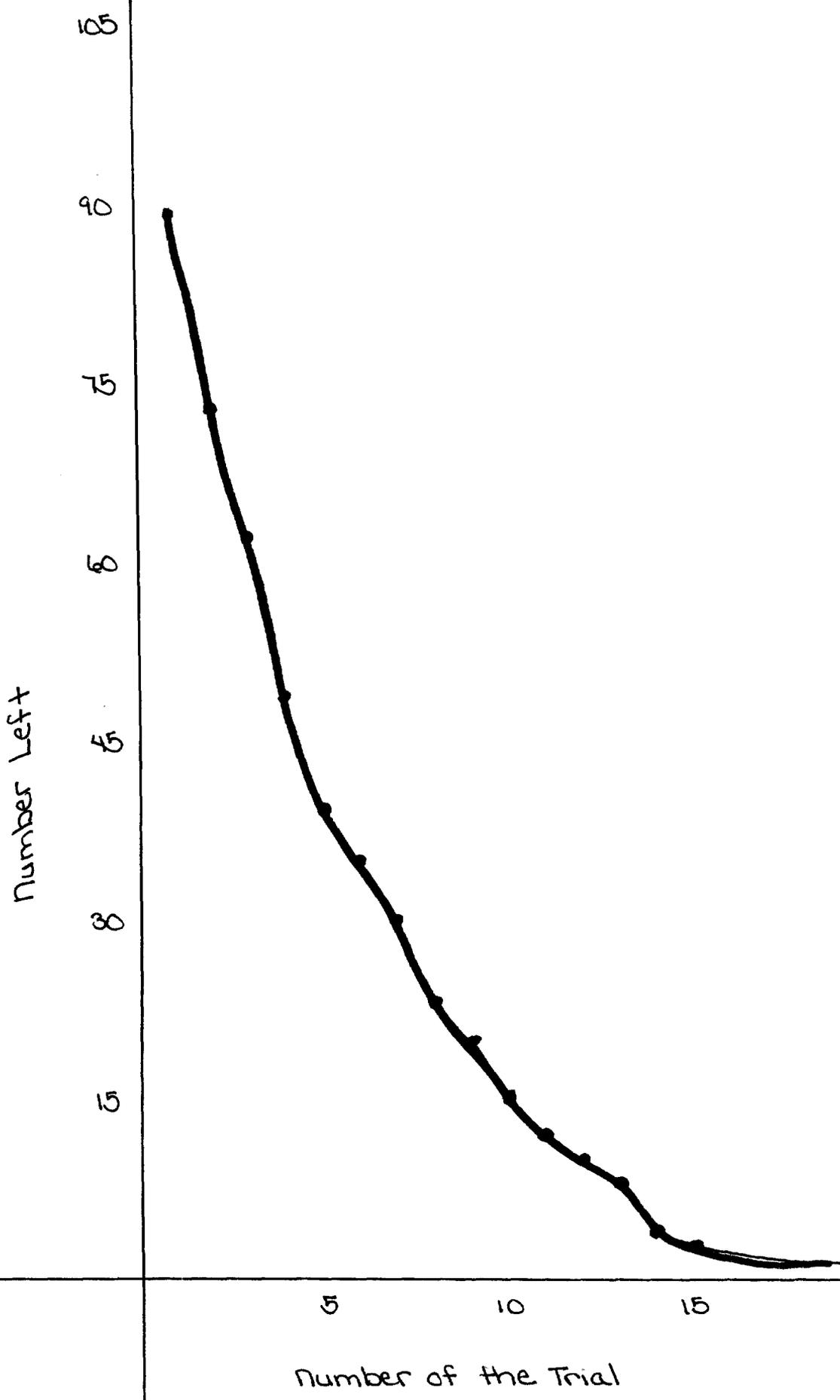
Subject: Infinite Series

One way to model an infinite series is to use a bouncing ball, such as a racquet ball. By collecting data on the rebound height, the ideas and concepts behind an infinite series can be better illustrated, even though in reality the ball will bounce a finite number of times. Upon examining a rebound ratio, students will be able to see in theory what an infinite series is.

Proficiency Indicators

1)   Find the limit at a finite point and at infinity, if either or both exist, of selected algebraic functions.

Procedures

1)   Have the students break up into groups of three and place yard sticks against a wall in order to measure the rebound height of a ball.

2)   They should then make a chart with the columns labeled: Trial, 60 Inch Drop, Time, 30 Inch Drop, Time, 15 Inch Drop, and Time, to record the rebound height for one bounce and the time it takes for the ball to stop bouncing.

3)   The students then should drop the ball from these heights, marking the height at the top of the ball on the rebound.

4)   Another student should measure the time it takes for the ball to stop bouncing for each case using a stopwatch.

5)   The groups should do three trials for each height that the ball is dropped from and take an average of the rebound heights and the time for each drop in order to get more accurate data.

6)   The students should then take the average of the trials for each set of

information.

7) Have the students then take the ratio of the rebound height over the original height for each drop in order to find what is known as the rebound ratio, which can be labeled r.

8) Explain that in theory, the next bounce at each drop should have the height of $r(rH) = r^2H$.

9) Each height after this should be the rebound ratio multiplied by the height of the rebound before.

10) Have the students then find the heights for five bounces at each drop, making a chart with the columns: Bounce, Equation, 60 Inch Drop, 30 Inch Drop, and 15 Inch Drop (the equation should be rH, r^2H,...).

11) Ask the students what would happen if they continued multiplying by the rebound ratio.

12) They should understand that this would go on infinitely, but that in reality the ball does stop bouncing.

13) After they have figured this information, explain that the ball must travel both up and down after the initial drop and have them find an equation that would find the total distance traveled by the ball.

14) They will find this to be $H+2rH+2r^2H+2r^3H+...$ and have them use their data in this equation.

15) Have the students then write their series in Σ notation, in order to find the total distance traveled, which would be $H + \sum_{i=1}^{\infty} 2r^i H = H + 2H\sum_{i=1}^{\infty} r^i$.

16) Explain how it can be written in the following way:

$H+2Hr+2Hr^2+...$

$=-H+2H+2Hr+...$

$$= -H + \sum_{i=0}^{n} 2Hr^i.$$

17)   Explain that if this was an infinite series, then it would be written $-H + \sum_{i=0}^{\infty} 2Hr^i$.

Source

Decker, Robert and John Williams. <u>Bringing Calculus to Life</u>. Lab 19: 67-69.

Example Activity

Trial	60 Inch Drop	Time	30 Inch Drop	Time	15 Inch Drop	Time
1	$39\frac{1}{2}$ inches	9 sec.	$21\frac{1}{4}$ inches	6 sec.	$11\frac{1}{4}$ inches	5 sec.
2	$38\frac{1}{2}$ inches	9 sec	$20\frac{1}{2}$ inches	7 sec.	$11\frac{3}{4}$ inches	$5\frac{1}{2}$ sec.
3	$39\frac{1}{2}$ inches	9 sec.	$21\frac{1}{4}$ inches	7 sec.	11 inches	5 sec.
Average	$39\frac{1}{12}$ inches	9 sec.	21 inches	$6\frac{2}{3}$ sec.	$11\frac{1}{3}$ inches	$5\frac{1}{6}$ sec.

$$r = \frac{39\frac{1}{12}}{60}$$

$$= \frac{\frac{469}{12}}{60}$$

$$= \frac{469}{720}$$

$$r = \frac{21}{30}$$

$$= \frac{7}{10}$$

$$= \frac{7}{10}$$

$$r = \frac{11\frac{1}{3}}{15}$$

$$= \frac{34}{45}$$

$$= \frac{34}{45}$$

Bounce	Equation	60 Inch Drop		30 Inch Drop		15 Inch Drop	
1	rH	39.5	inches	21	inches	11.3	inches
2	r^2H	25.458	inches	14.7	inches	8.563	inches
3	r^3H	16.583	inches	10.29	inches	6.470	inches
4	r^4H	10.802	inches	7.203	inches	4.888	inches
5	r^5H	7.036	inches	5.042	inches	3.693	inches

$$-H + \sum_{i=0}^{5} 2Hr^i$$

$$= -H + \sum_{i=0}^{5} 2Hr^i$$

$$= -H + 2H\sum_{i=0}^{5} r^i$$

60 inch drop: $-60 + 2(60)\left[\left(\frac{469}{720}\right)^0 + \left(\frac{469}{720}\right)^1 + \left(\frac{469}{720}\right)^2 + \left(\frac{469}{720}\right)^3 + \left(\frac{469}{720}\right)^4 + \left(\frac{469}{720}\right)^5\right]$

≈ 377.9276

30 inch drop: $-30 + 2(30)\left[\left(\frac{7}{10}\right)^0 + \left(\frac{7}{10}\right)^1 + \left(\frac{7}{10}\right)^2 + \left(\frac{7}{10}\right)^3 + \left(\frac{7}{10}\right)^4 + \left(\frac{7}{10}\right)^5\right]$

$= 146.4702$

15 inch drop: $-15 + 2(15)\left[\left(\frac{34}{45}\right)^0 + \left(\frac{34}{45}\right)^1 + \left(\frac{34}{45}\right)^2 + \left(\frac{34}{45}\right)^3 + \left(\frac{34}{45}\right)^4 + \left(\frac{34}{45}\right)^5\right]$

≈ 84.8955

Subject: Volume of Solid Revolutions

One way to show the cylindrical-shell method of integration in finding the volume of a solid revolution is to use a bundt cake. By cutting this into pieces, students can find a parabolic equation which represents the outline of the cake, then use the cylindrical-shell method to find the volume.

Proficiency Indicators

1) Apply the integrals of algebraic, exponential, logarithmic, trigonometric, and inverse trigonometric functions to real world problems.

2) Using graphing utilities, explore the concepts of calculus.

Procedures

1) Have the students draw the area bounded by $y = 4x-2x^2$ and $y=0$ about the y-axis.

2) Have the students then explain what this will look like when rotated about the y-axis.

3) Show them a picture and/or a model of what this will look like and explain how it resembles a bundt cake.

4) Make a bundt cake and cut the cake into pieces after showing the figure to the students.

5) Trace the outline of one piece of the cake on a piece of graph paper.

6) The students should then observe, after some attempts with other functions, that in order to approximate the cake outline, a parabolic equation could best fit.

7) Make an x- and y-axis on the graph paper and approximate what a parabola would look like with the tracing as an approximation.

8) They should come up with something like $y-k=a(x-h)^2$ with the vertex (h,k) and one x-intercept, with the hole at the center having a 2 inch diameter.

9) After taking the values from their graph of y, x, and h, have them solve for a.

10) They should then take this value for a and the value for h in order to solve for y in terms of x.

11) With this equation, f(x), using the cylindrical-shell method, they should integrate to find the volume by using $\int_m^n 2pi\ x\ f(x)$, with m and n being the x-values of the x-intercepts.

<u>Source</u>

Rahn, James R. "Giving Meaning to Volume in Calculus." <u>Mathematics Teacher</u> 84 (1991): 110-112.

Example Activity

From the graph, the following information is known:

$$y=0 \qquad k=^{15}/_4 \qquad x=^1/_2 \qquad h=^5/_2$$

Therefore:

$$y-k = a(x-h)^2$$

$$0-^{15}/_4 = a(^1/_2 - ^5/_2)^2$$

$$-^{15}/_4 = a(-2)^2$$

$$-^{15}/_4 = a(4)$$

$$-^{15}/_{16} = a$$

$$y - ^{15}/_4 = -^{15}/_{16} (x- ^5/_2)^2$$

$$y - ^{15}/_4 = -^5/_{16} (x^2 - 5x + ^{25}/_4)$$

$$y - ^{15}/_4 = -^{15}/_{16} x^2 + ^{75}/_{16} x - ^{375}/_{64}$$

$$y = -^{15}/_{16} x^2 + ^{75}/_{16} x - ^{135}/_{64}$$

$$\int_m^n 2 \, pi \, x \, f(x)$$

$$= \int_{1/2}^{9/2} 2 \, pi \, x \, (^{-15}/_{16} x^2 + ^{75}/_{16} x - ^{135}/_{64})$$

$$= 2 \, pi \int_{1/2}^{9/2} (^{-15}/_{16} x^3 + ^{75}/_{16} x^2 - ^{135}/_{64} x)$$

$$= 2 \, pi \left[^{-15}/_{64} x^4 + ^{25}/_{16} x^3 - ^{135}/_{128} x^2 \right]_{1/2}^{9/2}$$

$$= 2\pi\left[\left(\frac{-98415}{1024} + \frac{18225}{128} - \frac{10935}{512}\right) - \left(\frac{-15}{1024} + \frac{25}{128} - \frac{135}{512}\right)\right]$$

$$= 2\pi\left[\left(\frac{-98415}{1024} + \frac{145800}{1024} - \frac{21870}{1024}\right) - \left(\frac{-15}{1024} + \frac{200}{1024} - \frac{270}{1024}\right)\right]$$

$$= 2\pi\left(\frac{25515}{1024} + \frac{85}{1024}\right)$$

$$= 2\pi\left(\frac{25600}{1024}\right)$$

$$\approx 157.0796 \text{ in}^3$$

$(h,k) = (2.5, 3.75)$

$(0.5, 0)$

$(4.5, 0)$
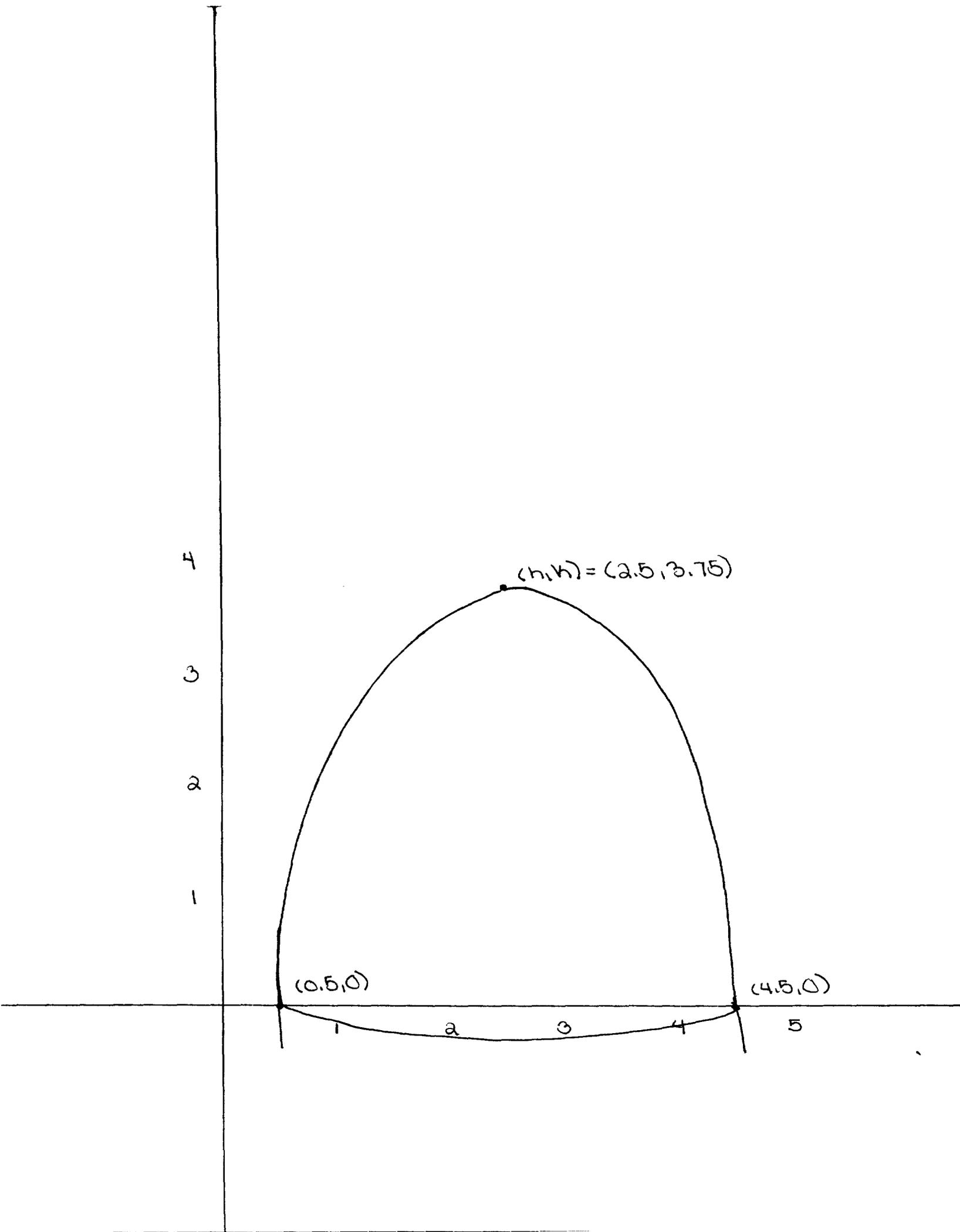
Subject: Volume
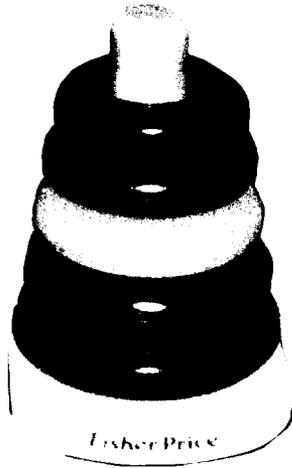
By using a simple child's stacking toy, students can better understand and visualize finding the volume of a solid of revolution using the disc method of integration. Since students are familiar with the toy, and since it can be separated into distinct discs, it is a useful illustration of the initial steps of the integration method. They will take steps from finding the volume of each part to coming up with a way of finding the volume of a solid of revolution using integration.

Proficiency Indicators

1)   Demonstrate the understanding of the limit and continuity of a function.

2)   Use various methods to find the integrals of algebraic, exponential, logarithmic, trigonometric, and inverse trigonometric functions.

Procedures

1) Explain to the students that this toy should be a representation of an area that is rotated about an axis, which would be the center of the toy with the equation $x=g(y)$.

2) Have the students draw a two-dimensional picture of the toy using a white background and a flashlight to trace.

3) Have them draw a vertical line down the center of the picture to illustrate the y-axis.

4) Explain that the region being rotated in the drawing is the region on one side of the y-axis.

5) Have them make a chart labeled: Ring, Diameter, Radius (R), Height (h), and Volume (V) for each ring of the toy.

6) They should then measure the diameter and the height in millimeters, as closely as possible, in order to figure the radius and height to find the volume for each ring, which should be approximated to four decimal places.

7) Ask the students how they would find the volume for each separate disk of the toy. They should realize to use the equation $V=piR^2h$. (Have them think of each as a solid, instead of a doughnut-like figure).

8) Have the students then find the total volume for the rings of the figure by adding up the volumes of each separate ring.

9) Explain that this can be done another way using integration.

10) Begin with $V=piR^2h$ for each and write as the summation, $V \approx \sum_{i=1}^{n} pi[R(y_i)]^2 \triangle y$ and show what $R(y_i)$ and $\triangle y$ represent on the picture.

11) Take the limit in order to find the integral form, $V=pi\int_a^b[R(y_i)]^2\triangle y$ and label a and b on the picture.

12)   Explain that if the equation is known for the region being rotated, the volume can be found using this integration method also.

Source

Larson, Roland E., Robert P. Hostelter, and Bruce H. Edwards. <u>Calculus with Analytic Geometry</u>. Lexington, Massachusetts: D.C. Heath and Company, 1990.
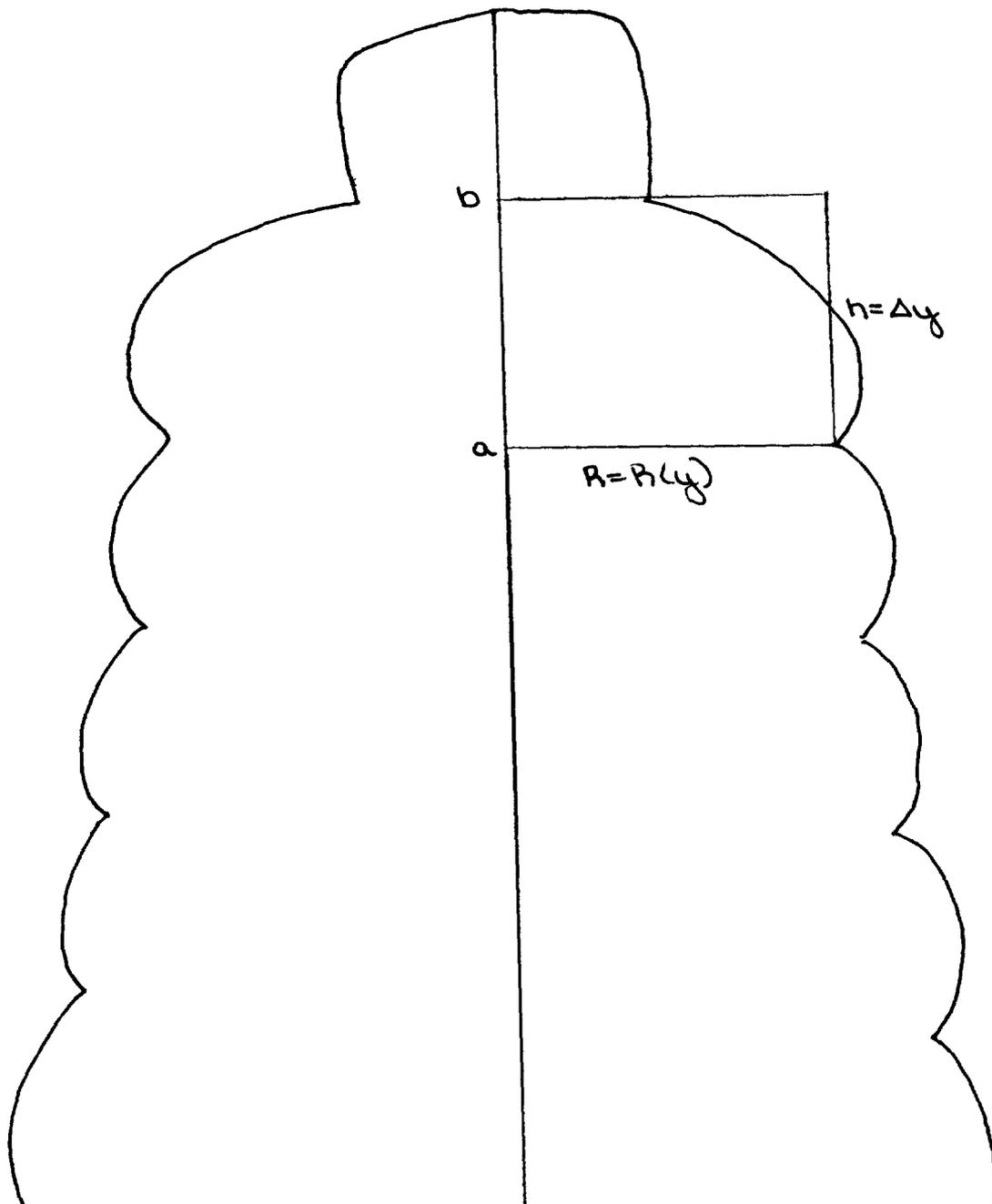
b

$h = \Delta y$

a

$R = R(y)$

Example Activity

Ring	Diameter	Radius (R)	Height (h)	Volume (V)
Red	90 mm	45 mm	26 mm	165404.8532 mm
Orange	97 mm	48.5 mm	26 mm	192135.0943 mm
Yellow	105 mm	102.5 mm	28 mm	24252.4130 mm
Green	114 mm	57 mm	29 mm	296004.0014 mm
Blue	124 mm	62 mm	27 mm	326059.6189 mm

Total Volume = V1 + V2 + V3 + V4 + V5 = 1222055.9800 mm

$V = \text{pi } R2 \, h$

$\approx \sum_{i=1}^{n} \text{pi } [R(yi)]2 \triangle y$

$= \text{pi} \sum_{i=1}^{n} [R(yi)]2 \triangle y$

$= \lim_{n \to \infty} \text{pi} \sum [R(yi)]2 \triangle y$

$= \text{pi} \int_{a}^{b} [R(yi)]2 \, dy$

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