Reflection Groups in Three Dimensions

An Honors Thesis (HONRS 499)

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Purpose of Thesis

This honor's thesis discusses finite reflection groups. The *real individual research work* that took place was:

A. Finding a complete list of all the different "words" associated with each of these finite reflection groups;

B. finding the matrices that describe the transformations associated to each word; and

C. finding a geometric description of all the transformations (for A3 and B3). This work is contained in the tables at the end of this thesis.

This thesis concerns reflection groups in dimension three. Before describing our research, we provide some definitions and background information. We use [Humphreys] as a general reference for this background.

Let V denote R^3 (three dimensional space). Suppose that P is a plane through the origin and L is the line through the origin that is perpendicular to P. The "reflection across P" is defined to be the unique linear transformation r of V such that r(**v**) = **v** when **v** is an element of P, and r(**v**) is -**v** when **v** is an element of L. There is a simple formula for r(**v**). Let **b** be either of the two unit vectors (vectors of length one) in L; then r(**v**) = **v**-2*(**v**.**b**)**b**. Often we think of the reflection as given by a 3X3 matrix.

Now suppose we are given a collection of unit vectors **b1**, **b2**, ... , **bn** in V. We can consider the reflection ri associated to each bi. We then form all possible "words" in the "letters" r1, ... , rn. The "reflection group generated by r1, ... , rn" is defined to be the collection of linear transformations of V (or equivalently, 3X3 matrices) associated to all possible words. (Some of the words can produce reflections which were not already in the list of generators; however, most words do not correspond to reflections.)

There are two complications that need to be considered:

First, two different words may produce equal elements of the reflection group. For example, the words r1 r1 and r2 r2 are different words, but both give rise to the same matrix, the identity. Short of computing matrices explicitly, it is a difficult problem to determine when two words produce the same matrix. We discuss this in several examples in this thesis.

Second, most of the time, the reflection group contains infinitely many elements. This paper treats the (rare) cases when the group has only finitely many elements. In fact, there are only three "interesting" examples, which have been given names A3, B3, and H3. The groups have 24, 48, and 120 elements, respectively. (There are further examples of reflection groups which arise from considering separate reflection groups in a plane and a perpendicular line, but these are comparatively easy and we do not consider them here.)

The finite reflection groups are best investigated using an object called a "root system" as a tool. Given a finite reflection group G, we consider all the elements of G which happen to be reflections. To each of these reflections we can associate the two unit vectors on the line perpendicular to the plane fixed by the reflection. The collection of all the vectors obtained in this way is called the "root system associated to the finite reflection group." A3 contains 12 roots, B3 contains 18 roots, and H3 contains 30 roots.

Root systems have been extensively studied by mathematicians. For example, using known information, we were able to construct physical models of the root systems A3, B3, and H3. Also, our work relied on several important theorems about root systems, which we now describe.

In any of the above root systems, one can find a collection of three vectors **b1**, **b2**, **b3** which form a "base" of the root system. This means: the three vectors are a basis of V, and also, given any vector **b** in the root system, when we express **b** as a linear combination of **b1**, **b2**, and **b3** (**b**=x1 **b1** + x2 **b2** + x3 **b3**) then either all three x's are greater than or equal, or all three are less than or equal to zero.

The base is important, because it is known that every element of the reflection group is obtained from a word that only involves the corresponding "simple reflections" r1, r2, and r3 (so reflections with respect to the other roots in the root system are not needed to generate the reflection group). Moreover, information about r1, r2, and r3 already is enough to predict when two different words in r1, r2, and r3 produce the same element of the reflection group. This information is encoded in diagrams associated to the three root systems:

A3

```
o----------o----------o
```

B3

```
    4
o----------o----------o
```

H3

```
    5
o----------o----------o  .
```

The circles represent elements of the base of the root system and the number above the sections indicates how many times the product of the two reflections must be multiplied with itself to produce the identity. If there is no number above the bar, then the number is understood to be three. Also, if there is no bar between two circles, this signifies that the number is two. It has been proven that the groups can be generated from the three reflections perpendicular to these basis vectors, and the relations among the generators are all consequences of the above relations, and the fact that any reflection multiplied with itself is the identity.

Using A3 as an example, it is easy to see how to use these diagrams to list all elements of the reflection groups. For illustration purposes, let r1 be a reflection across the plane perpendicular to the first basis vector, and let r2 and r3 be defined similarly. The diagram tells us that

r1 r2 r1 r2 r1 r2 = I.

Any reflection is its own inverse. Therefore, if one were to multiply both sides of the above equation by r2 on the right, the following would be produced:

r1 r2 r1 r2 r1 = r2.

With more matrix multiplication on the right in the same fashion it is easy to show that

I. r1 r2 r1 = r2 r1 r2.

Likewise,

II. r2 r3 r2 = r3 r2 r3 and III. r1 r3 = r3 r1.

These are the three fundamental relations for A3. For B3 the three fundamental relations are

I. r1 r2 r1 r2 = r2 r1 r2 r1

II. r2 r3 r2 = r3 r2 r3

III. r1 r3 = r3 r1.

For H3 the three fundamental relations are

I. r1 r2 r1 r2 r1 = r2 r1 r2 r1 r2

II. r2 r3 r2 = r3 r2 r3

III. r1 r3 = r3 r1.

Using these properties one can indeed find what all of the "words" are that describe the finite reflection groups. This is easier said than done, and here is an illustration of the difficulty. One theorem that is of note is that in any of these groups there will always be one unique element that has the most number of simple reflections in its shortest expression. For example, if A3, the long element is r3 r2 r3 r1 r2 r3. However, this element can be written in many ways as a product of six simple reflections, for example, as r1 r2 r1 r3 r2 r1. We show how the three fundamental relations can be used to show the two expressions are equal:

r3 r2 r3 r1 r2 r3	= r2 r3 r2 r1 r2 r3	with expression II
	= r2 r3 r1 r2 r1 r3	with expression I
	= r2 r1 r3 r2 r1 r3	with expression III
	= r2 r1 r3 r2 r3 r1	with expression III
	= r2 r1 r2 r3 r2 r1	with expression II
	= r1 r2 r1 r3 r2 r1	with expression I

The *real individual research work* that took place during this honors thesis was:

A. Finding a complete list of all the different "words" associated with each of these finite reflection groups;

B. finding the matrices that describe the transformations associated to each word; and

C. finding a geometric description of all the transformations (for A3 and B3). This work is contained in the tables at the end of this thesis.

Here is an overview of the work in part A. Here we will take A3 and show how the words of up to length three are developed. We start off with the identity matrix and add one of the three reflections onto the right side. This produces one word with no reflections and three words with only one reflections. In writing out these words, they would have lengths of zero and one, respectively. One of each of the three reflections is then added to the words of length one. In doing so, one has to be careful not to keep something in the list that can be written either in a shorter or in a different way. This means specifically that r1 r3 and r3 r1 would not both be kept on the list of words, and r1 r1, r2 r2, and r3 r3 would not be kept because each simplifies down to be the identity matrix.

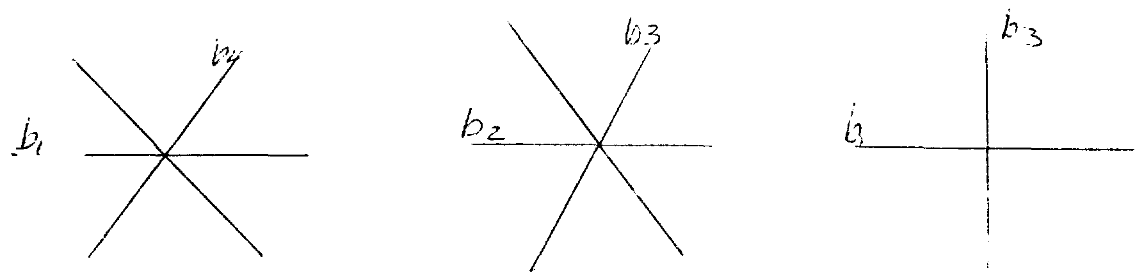
After the words of length two are found, then words of length three must be determined. One starts with the five distinct elements of the reflection group that have length two, and multiplies on the right by r1, r2, or r3, producing fifteen words. However, some of these coincide with elements of length one, and some of the words of length three produce elements of the reflection group which are equal. For example, in the first case, when one adds on r1 onto the right side of r2 r1 the element is reducible down to simply r2. In the end, the six words of length three giving distinct elements of the reflection group are r1 r2 r1, r1 r2 r3, r2 r1 r3, r2 r3 r2, r3 r2 r1, and r1 r3 r2. The three words of length three that can be written as other words are r2 r1 r2, r2 r3 r1, and r3 r2 r3. Finally, the six words with three letters that are reducible are r1 r2 r2, r2 r1 r1, r2 r3 r3, r3 r2 r2, r1 r3 r1, and r1 r3 r3, so these can also be written as words of length one. It is a considerable amount of work to eliminate the redundant expressions for words, especially as their lengths grow large, as the above example with the long element of A3 already indicated. Here is a table to show how many words of any given lengths were in each of the three reflection groups.

	A3	B3	H3
0	1	1	1
1	3	3	3
2	5	5	5
3	6	7	7
4	5	8	9
5	3	8	11
6	1	7	12
7		5	12
8		3	12
9		1	12
10			11
11			9
12			7
13			5
14			3
15			1
longest element	r3 r2 r3 r1 r2 r3	r2 r1 r2 r3 r1 r3 r2 r1 r3	r1 r2 r3 r2 r1 r2 r3 r1 r2 r1 r3 r2 r1 r3 r2

It is possible to see from the above table that each of the reflection groups has a symmetrical pattern as to the number of elements of each word length.

We now indicate how Part B was accomplished. We first found the matrices for the simple reflections , as discussed below. We then used Mathematica (a computer algebra program) to compute the matrices for all the "words". We remark that there was a choice as to take the usual (x,y,z) coordinates, or coordinates with respect to the base **b1**, **b2**, **b3**. The latter is easier, so it was adopted.

This work requires knowing the dot products of all the elements of each base. Since the base vectors are unit vectors, each dot product is the cosine of the angle between the two vectors. We used the physical models to determine these angles. Given any two base vectors, all the root vectors lying in the same plane form a symmetrical pattern. For example, in A3, the patterns are:



from which we determine b1.b2 = cosine (120°) = -1/2,

b2.b3 = cosine (120°) = -1/2,

b1.b3 = cosine (90°) = 0.

Similarly, for B3,

b1.b2 = cosine (135°) = $-\sqrt{2}/2$

b2.b3 = cosine (120°) = -1/2,

b1.b3 = cosine (90°) = 0,

and for H3,

$$b1.b2 = \text{cosine}\,(144^\circ) = -(1+\sqrt{5})/4$$

$$b2.b3 = \text{cosine}\,(120^\circ) = -1/2,$$

$$b1.b3 = \text{cosine}\,(90^\circ) = 0.$$

In order to find the matrix that represents a reflection perpendicular to **b1**, one must find where the three basis vectors would go with this reflection. In doing so, the following formula was used:

$$r1(\mathbf{v}) = \mathbf{v} - 2(\mathbf{v} \,.\, \mathbf{b1})\mathbf{b1}\,.$$

Using this formula, one can see that **b1** goes to **-b1+0b2+0b3**, **b2** goes to **b1+b2+0b3**, and **b3** stays on 0**b1**+0**b2**+**b3**. Each of these facts can be used to make up a column of the matrix for the reflection. The resulting matrix is

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

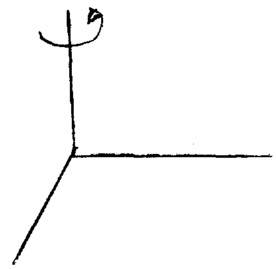
The rest of the matrices for the simple reflections can be found similarly.

Finally, we described how we accomplished Part C. Note from the tables in the back of this thesis is that all of the determinants are either 1 or -1. This comes from the fact that any matrix that is a reflection has a determinant of -1. Since Det[AB] = Det[A]Det[B], the determinant of the matrix for any word will be 1 or -1, depending on whether it is the product of an even or odd number of reflections. Each of the two cases has to be examined separately because they describe very different circumstances.

If the determinant of the matrix is 1, then it is just a rotation around a vector. In appropriate coordinates, this can be represented by the following matrix,

$$A = \begin{pmatrix} \text{Cos theta} & -\text{Sin theta} & 0 \\ \text{Sin theta} & \text{Cos theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A visual representation of this would be

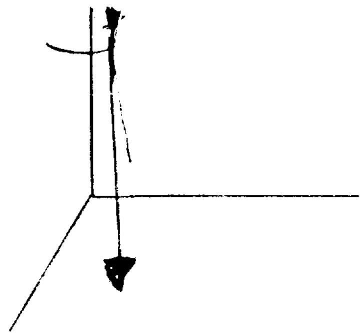


The characteristic polynomial of the above matrix is $(x-1)(x^2-2\cos(\theta)x+1)$, which has roots 1, cos(theta)+isin(theta), and cos(theta)-isin(theta). These are the eigenvalues of A. A vector that is fixed by this rotation is any vector **v** such that A**v**=**v**, that is, a null vector of the matrix A-I. One either finds the nullspace of A-I and eigenvalues of the matrix A by hand or has a computer compute this. The method used in this thesis was to utilize a computer. Any vector in the nullspace gives a vector on the axis of rotation, and this was produced by the computer. Also, the computer produced the three eigenvalues: 1, Cos(theta)+iSin(theta), and Cos(theta)-iSin(theta). These values could be used in the complex plane to help to determine an angle theta between 0 degrees and 180 degrees. The physical models were then used to see if the angle in question was clockwise or counterclockwise around the given vector on the axis of rotation.

If the determinant of the matrix is -1, then the transformation is the product of a rotation around an axis and a reflection across the plane perpendicular to that axis. It is very possible that the rotation is zero, making the transformation nothing more than a reflection. In this case the matrix being used would be

$$A=\begin{pmatrix} \text{Cos theta} & -\text{Sin theta} & 0 \\ \text{Sin theta} & \text{Cos theta} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A visual representation of this is



The characteristic polynomial of the above matrix is $(x+1)(x^2-2\cos(\theta)x+1)$, which has roots -1, cos(theta)+isin(theta), and cos(theta)-isin(theta). These are the eigenvalues of A. With this case a vector on the axis is one that solves the equation Av=-v, that is, a nullvector for the matrix A+I. The eigenvalues and nullspace are found in much the same way as the determinant equal 1 case. The physical models are also used to show if the rotation is clockwise or counterclockwise again.

It is important to note that anything with an absolute value of less than 10 to the negative seventh power should be considered as zero in the tables.

Bibliography

Anton, H., Elementary Linear Algebra, 6th edition, John Wiley and Sons, New York, 1991.

Grove, L., and C. Benson, Finite Reflection Groups, 2nd edition, Springer-Verlag, New York, 1985.

Herstein, I., Topics in Algebra, 2nd edition, Xerox College Publishing, Toronto, 1975.

Humphreys, J., Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1990.

```
(*              A3                *)

(* This is the complete set of matrices and reflections that are needed
  to develop the matrices for A3. The symbol a stands for a reflection
  perpendicular to b1, the symbol b stands for a reflection perpendicular to b2,
 and the symbol c stands for a reflection perpendicular to b3. *)

(* Length 0 *)

(* This is the identity matrix *)

MatrixForm[IdentityMatrix[3]]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* Length 1 *)

(* This is a reflection across b1 *)

MatrixForm[a]
```

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* This is a reflection across b2*)

MatrixForm[b]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* This is a reflection across b3 *)

MatrixForm[c]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

```
(* Length 2 *)

(* This is a rotation of 120 degrees about b1Xb2 *)
```

```
MatrixForm[a . b]
```

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* This is a rotation of -120 degrees about b1Xb2 *)

MatrixForm[b . a]
```

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* This is a rotation of 180 degrees about b1Xb3 *)

MatrixForm[a . c]
```

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

```
(* This is a rotation of 120 degrees about b2Xb3 *)

MatrixForm[b . c]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

```
(* This is a rotation of -120 degrees about b2Xb3 *)

MatrixForm[c . b]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

```
(* Length 3 *)

(* This is a reflection across b1+b2 *)
```

```
MatrixForm[a . b . a]
```

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* This is a reflection across b2+b3 *)

MatrixForm[b . c . b]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

```
(* This is a rotation of 90 degrees around b1+b3 *)

MatrixForm[a . b . c]
```

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

```
(* This is a rotation of 90 degrees around b3-b1 *)

MatrixForm[a . c . b]
```

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

```
(* This is a rotation of -90 degrees around b1+b3 *)

MatrixForm[c . b . a]
```

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

```
(* This is a rotation of -90 degrees around b3-b1 *)
```

```
MatrixForm[b . a . c]
```

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

```
(* Length 4 *)

(* This is a rotation of -120 degrees around -b1+2 b2+b3 *)

MatrixForm[a . b . a . c]
```

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

```
(* This is a rotation of -120 degrees around -b1-2 b2+b3*)

MatrixForm[b . c . b . a]
```

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

```
(*  This is a rotation of 120 degrees around -b1-2 b2+b3*)

MatrixForm[a . b . c . b]
```

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

```
(*  This is a rotation of 120 degrees around -b1+2 b2+b3*)

MatrixForm[a . c . b . a]
```

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

```
(* This is a rotation of 180 degrees around b1+b3 *)
```

```
MatrixForm[b . a . c . b]
```

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

```
(* Length 5 *)

(* This is a reflection and rotation of (-90) degrees about b1+2 b2+b3 *)

MatrixForm[b . a . b . c . b]
```

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

```
(* This is a reflection and rotation of 90 degrees about b1+2 b2+b3 *)

MatrixForm[b . a . c . b . a]
```

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

```
(* This is a reflection across b1+b2+b3 *)

MatrixForm[a . b . c . b . a]
```

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

```
(* Length 6 *)

(* This is a rotation of 180 degrees around -b1+b3 *)

MatrixForm[b . a . b . c . b . a]
```

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

```
(*              B3                  *)

(* This is the complete set of matrices and reflections that are needed
  to develop the matrices for B3. The symbol a stands for a reflection
  perpendicular to b1, the symbol b stands for a reflection perpendicular to b2,
 and the symbol c stands for a reflection perpendicular to b3. Also,
 1.41421 is the square root of two. *)

(* Length 0 *)

(* This is the identity matrix *)

MatrixForm[IdentityMatrix[3]]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* Length 1 *)

(* This is a reflection across b1 *)

MatrixForm[a]
```

$$\begin{pmatrix} -1 & 1 & 1.41421 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* This is a reflection across b2 *)

MatrixForm[b]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* This is a reflection across b3 *)

MatrixForm[c]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1.41421 & 0 & -1 \end{pmatrix}$$

```
(* Length 2 *)

(* This is a rotation of 120 degrees around b1Xb2 *)
```

MatrixForm[a . b]

$$\begin{pmatrix} 0. & -1. & 1.41421 \\ 1. & -1. & 0. \\ 0. & 0. & 1. \end{pmatrix}$$

(* This is a rotation of -90 degrees around b1Xb3 *)

MatrixForm[a . c]

$$\begin{pmatrix} 1. & 1. & -1.41421 \\ 0. & 1. & 0. \\ 1.41421 & 0. & -1. \end{pmatrix}$$

(* This is a rotation of -120 degrees around b1Xb2 *)

MatrixForm[b . a]

$$\begin{pmatrix} -1. & 1. & 1.41421 \\ -1. & 0. & 1.41421 \\ 0. & 0. & 1. \end{pmatrix}$$

(* This is a rotation of 180 degrees around b2Xb3 *)

MatrixForm[b . c]

$$\begin{pmatrix} 1. & 0. & 0. \\ 1. & -1. & 0. \\ 1.41421 & 0. & -1. \end{pmatrix}$$

(* This is a rotation of 90 degrees around b1Xb3 *)

MatrixForm[c . a]

$$\begin{pmatrix} -1. & 1. & 1.41421 \\ 0. & 1. & 0. \\ -1.41421 & 1.41421 & 1. \end{pmatrix}$$

(* Length 3 *)

(* This is a reflection across b1+b2 *)

MatrixForm[a . b . a]

$$\begin{pmatrix} 0. & -1. & 1.41421 \\ -1. & 0. & 1.41421 \\ 0. & 0. & 1. \end{pmatrix}$$

(* This is a rotation of (-60) degrees and a reflection across -(2^.5) b2+b3 *)

MatrixForm[a . b . c]

$$\begin{pmatrix} 2. & -1. & -1.41421 \\ 1. & -1. & 0. \\ 1.41421 & 0. & -1. \end{pmatrix}$$

(* This is a reflection across (2^.5) b1+b3 *)

MatrixForm[a . c . a]

$$\begin{pmatrix} -1. & 2. & 6.66134\times10^{-16} \\ 0. & 1. & 0. \\ -1.41421 & 1.41421 & 1. \end{pmatrix}$$

(* This is a rotation of (-60) degrees and a reflection across (2^.5) b2+b3 *)

MatrixForm[b . a . c]

$$\begin{pmatrix} 1. & 1. & -1.41421 \\ 1. & 0. & -1.41421 \\ 1.41421 & 0. & -1. \end{pmatrix}$$

(* This is a rotation of 60 degrees and a reflection across -(2^.5) b2+b3 *)

MatrixForm[b . c . a]

$$\begin{pmatrix} -1. & 1. & 1.41421 \\ -1. & 0. & 1.41421 \\ -1.41421 & 1.41421 & 1. \end{pmatrix}$$

(* This is a rotation of 60 degrees and a reflection across (2^.5) b2+b3 *)

```
MatrixForm[c . a . b]
```

0.	-1.	1.41421
1.	-1.	0.
0.	-1.41421	1.

```
(*  This is a reflection across (2^.5) b3+b1 *)

MatrixForm[c . a . c]
```

1.	1.	-1.41421
0.	1.	0.
6.66134×10^{-16}	1.41421	-1.

```
(* Length 4 *)

(* This is a rotation of 90 degrees around (2^.5) b1+b3 *)

MatrixForm[b . a . c . b]
```

2.	-1.	-1.41421
1.	0.	-1.41421
1.41421	0.	-1.

```
(*  This is a rotation of 180 degrees around -b1-b2-(2^.5) b3 *)

MatrixForm[a . b . c . a]
```

-2.	1.	1.41421
-1.	0.	1.41421
-1.41421	1.41421	1.

```
(* This is a rotation of 90 degrees around b3 *)

MatrixForm[a . c . a . b]
```

1.	-2.	6.66134×10^{-16}
1.	-1.	0.
0.	-1.41421	1.

```
(* This is a rotation of 180 degrees around (2^.5) b1+(2^.5) b2+b3 *)
```

MatrixForm[a . c . a . c]

$$\begin{pmatrix} -1. & 2. & -6.66134\times10^{-16} \\ 0. & 1. & 0. \\ 6.66134\times10^{-16} & 1.41421 & -1. \end{pmatrix}$$

(* This is a rotation of -90 degrees around b3 *)

MatrixForm[b . a . c . a]

$$\begin{pmatrix} -1. & 2. & 6.66134\times10^{-16} \\ -1. & 1. & 6.66134\times10^{-16} \\ -1.41421 & 1.41421 & 1. \end{pmatrix}$$

(* This is a rotation of 90 degrees around (2^.5) b1+b3*)

MatrixForm[b . c . a . b]

$$\begin{pmatrix} 0. & -1. & 1.41421 \\ -1. & 0. & 1.41421 \\ 0. & -1.41421 & 1. \end{pmatrix}$$

(* This is a rotation of 120 degrees around (2^1.5) b1+(2^.5) b2+b3 *)

MatrixForm[b . c . a . c]

$$\begin{pmatrix} 1. & 1. & -1.41421 \\ 1. & 0. & -1.41421 \\ 6.66134\times10^{-16} & 1.41421 & -1. \end{pmatrix}$$

(* This is a rotation of -120 degrees around (2^1.5) b1+(2^.5) b2+b3*)

MatrixForm[c . a . c . b]

$$\begin{pmatrix} 2. & -1. & -1.41421 \\ 1. & -1. & 0. \\ 1.41421 & -1.41421 & -1. \end{pmatrix}$$

(* Length 5 *)

(* This is a rotation of (-90) degrees and a reflection across (2^.5) b1+b3 *)

```
MatrixForm[c . a . c . b . a]
```

$$\begin{pmatrix} -2. & 1. & 1.41421 \\ -1. & 0. & 1.41421 \\ -1.41421 & 6.66134\times 10^{-16} & 1. \end{pmatrix}$$

```
(* This is a rotation of 60 degrees and a reflection across (2^1.5) b1+(2^.5) b2+b3 *)

MatrixForm[a . b . a . c . a]
```

$$\begin{pmatrix} -2. & 1. & 1.41421 \\ -1. & 1. & 6.66134\times 10^{-16} \\ -1.41421 & 1.41421 & 1. \end{pmatrix}$$

```
(* This is a rotation of (-60) degrees and a reflection across (2^1.5) b1+
  (2^.5) b2+b3 *)

MatrixForm[a . b . c . a . b]
```

$$\begin{pmatrix} -1. & -1. & 1.41421 \\ -1. & 0. & 1.41421 \\ 0. & -1.41421 & 1. \end{pmatrix}$$

```
(* This is a rotation of 90 degrees and a reflection across (2^.5) b1+b3 *)

MatrixForm[a . b . c . a . c]
```

$$\begin{pmatrix} 4.44089\times 10^{-16} & 1. & -1.41421 \\ 1. & 0. & -1.41421 \\ 6.66134\times 10^{-16} & 1.41421 & -1. \end{pmatrix}$$

```
(* This is a rotation of 90 degrees and a reflection across b3 *)

MatrixForm[a . c . a . b . c]
```

$$\begin{pmatrix} 1. & -2. & -6.66134\times 10^{-16} \\ 1. & -1. & 0. \\ 1.41421 & -1.41421 & -1. \end{pmatrix}$$

```
(* This is a reflection across (2^.5) b1+(2^.5) b2+b3 *)
```

```
MatrixForm[b . a . c . a . b]
```

$$\begin{pmatrix} 1. & -2. & 6.66134\times10^{-16} \\ 0. & -1. & 6.66134\times10^{-16} \\ 0. & -1.41421 & 1. \end{pmatrix}$$

```
(* This is a rotation of (-90) degrees and a reflection across b3 *)

MatrixForm[b . a . c . a . c]
```

$$\begin{pmatrix} -1. & 2. & -6.66134\times10^{-16} \\ -1. & 1. & -6.66134\times10^{-16} \\ 6.66134\times10^{-16} & 1.41421 & -1. \end{pmatrix}$$

```
(* This is a reflection across b1+b2+(2^.5) b3 *)

MatrixForm[b . c . a . b . c]
```

$$\begin{pmatrix} 2. & -1. & -1.41421 \\ 1. & 0. & -1.41421 \\ 1.41421 & -1.41421 & -1. \end{pmatrix}$$

```
(* Length 6 *)

(* This is a rotation of -120 degrees around (2^.5) b2+b3 *)

MatrixForm[c . a . c . b . a . b]
```

$$\begin{pmatrix} -1. & -1. & 1.41421 \\ -1. & 0. & 1.41421 \\ -1.41421 & -6.66134\times10^{-16} & 1. \end{pmatrix}$$

```
(* This is a rotation of 180 degrees around b1+b2 *)

MatrixForm[c . a . b . c . a . c]
```

$$\begin{pmatrix} 4.44089\times10^{-16} & 1. & -1.41421 \\ 1. & 0. & -1.41421 \\ 6.66134\times10^{-16} & 6.66134\times10^{-16} & -1. \end{pmatrix}$$

```
(* This is a rotation of 180 degrees around b1+(2^.5) b3 *)
```

MatrixForm[a . b . a . c . a . b]

$$\begin{pmatrix} -1. & -1. & 1.41421 \\ 0. & -1. & 6.66134\times10^{-16} \\ 0. & -1.41421 & 1. \end{pmatrix}$$

(* This is a rotation of 120 degrees around (2^.5) b2+b3 *)

MatrixForm[a . b . a . c . a . c]

$$\begin{pmatrix} 4.44089\times10^{-16} & 1. & -1.41421 \\ -1. & 1. & -6.66134\times10^{-16} \\ 6.66134\times10^{-16} & 1.41421 & -1. \end{pmatrix}$$

(* This is a rotation of 120 degrees around (-(2^.5)) b2+b3 *)

MatrixForm[a . b . c . a . b . c]

$$\begin{pmatrix} 1. & -1. & -1.41421 \\ 1. & 0. & -1.41421 \\ 1.41421 & -1.41421 & -1. \end{pmatrix}$$

(* This is a rotation of 180 degrees around (2^.5) b1+b3 *)

MatrixForm[b . a . c . a . b . c]

$$\begin{pmatrix} 1. & -2. & -6.66134\times10^{-16} \\ 9.42055\times10^{-16} & -1. & -6.66134\times10^{-16} \\ 1.41421 & -1.41421 & -1. \end{pmatrix}$$

(* This is a rotation of (-120) degrees around (-(2^.5)) b2+b3*)

MatrixForm[c . a . b . a . c . a]

$$\begin{pmatrix} -2. & 1. & 1.41421 \\ -1. & 1. & 6.66134\times10^{-16} \\ -1.41421 & 6.66134\times10^{-16} & 1. \end{pmatrix}$$

(* Length 7 *)

(* This is a rotation of 60 degrees and a reflection across .648886 b1+ .324443 b2+.6882476 b3 *)

MatrixForm[c . a . c . b . a . b . c]

$$\begin{pmatrix} 1. & -1. & -1.41421 \\ 1. & 0. & -1.41421 \\ 1.33227\times10^{-15} & -6.66134\times10^{-16} & -1. \end{pmatrix}$$

(* This is a rotation of -60 degrees and a reflection across .648886 b1+ .324443 b2+ .6882476 b3*)

MatrixForm[c . a . b . c . a . c . a]

$$\begin{pmatrix} -4.44089\times10^{-16} & 1. & -1.41421 \\ -1. & 1. & 6.66134\times10^{-16} \\ -6.66134\times10^{-16} & 1.33227\times10^{-15} & -1. \end{pmatrix}$$

(* This is a rotation of (-90) degrees and a reflection across (2^.5) b1+ (2^.5) b2+b3 *)

MatrixForm[b . a . b . c . a . c . b]

$$\begin{pmatrix} 1. & -1. & -1.41421 \\ 8.88178\times10^{-16} & -1. & -6.66134\times10^{-16} \\ 1.41421 & -1.41421 & -1. \end{pmatrix}$$

(* This is a reflection across 2 b1+(2^.5) b2+b3 *)

MatrixForm[a . b . c . a . b . c . a]

$$\begin{pmatrix} -1. & 4.44089\times10^{-16} & 8.88178\times10^{-16} \\ -1. & 1. & 6.66134\times10^{-16} \\ -1.41421 & 6.66134\times10^{-16} & 1. \end{pmatrix}$$

(* This is a rotation of 90 degrees and a reflection across (2^.5) b1+(2^.5) b2+b3 *)

MatrixForm[c . a . b . a . c . a . b]

$$\begin{pmatrix} -1. & -1. & 1.41421 \\ 0. & -1. & 6.66134\times10^{-16} \\ -1.41421 & -6.66134\times10^{-16} & 1. \end{pmatrix}$$

(* Length 8 *)

(* This is a rotation of 180 degrees around b2 *)

```
MatrixForm[a.b.c.a.b.c.a.c]
```

$$\begin{pmatrix} -1. & 4.44089\times10^{-16} & -8.88178\times10^{-16} \\ -1. & 1. & -6.66134\times10^{-16} \\ 6.66134\times10^{-16} & 6.66134\times10^{-16} & -1. \end{pmatrix}$$

```
(* This is a rotation of 180 degrees around b1 *)

MatrixForm[c.a.b.a.c.a.c.b]
```

$$\begin{pmatrix} 1. & -1. & -1.41421 \\ 8.88178\times10^{-16} & -1. & -6.66134\times10^{-16} \\ 1.33227\times10^{-15} & -6.66134\times10^{-16} & -1. \end{pmatrix}$$

```
(* This is a rotation of 180 degrees around b3 *)

MatrixForm[b.a.b.c.a.c.b.a]
```

$$\begin{pmatrix} -1. & 4.44089\times10^{-16} & 8.88178\times10^{-16} \\ -8.88178\times10^{-16} & -1. & 5.8994\times10^{-16} \\ -1.41421 & 6.66134\times10^{-16} & 1. \end{pmatrix}$$

```
(* Length 9 *)

(* This is the negative identity matrix *)

MatrixForm[b.a.b.c.a.c.b.a.c]
```

$$\begin{pmatrix} -1. & 4.44089\times10^{-16} & -8.88178\times10^{-16} \\ -5.38771\times10^{-17} & -1. & -5.8994\times10^{-16} \\ 6.66134\times10^{-16} & 6.66134\times10^{-16} & -1. \end{pmatrix}$$

```
(*              H3                  *)

(* This is the complete set of matrices and  reflections that are needed
  to develop the matrices for H3. The symbol a stands for a reflection
  perpendicular to b1, the symbol b stands for a reflection perpendicular to b2,
 and the symbol c stands for a reflection perpendicular to b3. Also,
 1.618034 is twice the cosine of 144 degrees. *)


(* Length 0 *)

MatrixForm[IdentityMatrix[3]]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* Length 1 *)

MatrixForm[a]
```

$$\begin{pmatrix} -1 & 1.618034 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
MatrixForm[b]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1.618034 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
MatrixForm[c]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

```
(* Length 2 *)

MatrixForm[a . b]
```

$$\begin{pmatrix} 1.61803 & -1.61803 & 1.61803 \\ 1.61803 & -1. & 1. \\ 0. & 0. & 1. \end{pmatrix}$$

```
MatrixForm[a . c]
```

$$\begin{pmatrix} -1. & 1.61803 & 0. \\ 0. & 1. & 0. \\ 0. & 1. & -1. \end{pmatrix}$$

MatrixForm[b . a]

$$\begin{pmatrix} -1. & 1.61803 & 0. \\ -1.61803 & 1.61803 & 1. \\ 0. & 0. & 1. \end{pmatrix}$$

MatrixForm[b . c]

$$\begin{pmatrix} 1. & 0. & 0. \\ 1.61803 & 0. & -1. \\ 0. & 1. & -1. \end{pmatrix}$$

MatrixForm[c . b]

$$\begin{pmatrix} 1. & 0. & 0. \\ 1.61803 & -1. & 1. \\ 1.61803 & -1. & 0. \end{pmatrix}$$

(* Length 3 *)

MatrixForm[a . b . a]

$$\begin{pmatrix} -1.61803 & 1. & 1.61803 \\ -1.61803 & 1.61803 & 1. \\ 0. & 0. & 1. \end{pmatrix}$$

MatrixForm[a . b . c]

$$\begin{pmatrix} 1.61803 & 0. & -1.61803 \\ 1.61803 & 0. & -1. \\ 0. & 1. & -1. \end{pmatrix}$$

MatrixForm[c . b . c]

$$\begin{pmatrix} 1. & 0. & 0. \\ 1.61803 & 0. & -1. \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[a . c . b]

$$\begin{pmatrix} 1.61803 & -1.61803 & 1.61803 \\ 1.61803 & -1. & 1. \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[b . a . b]

$$\begin{pmatrix} 1.61803 & -1.61803 & 1.61803 \\ 1. & -1.61803 & 2.61803 \\ 0. & 0. & 1. \end{pmatrix}$$

MatrixForm[b . a . c]

$$\begin{pmatrix} -1. & 1.61803 & 0. \\ -1.61803 & 2.61803 & -1. \\ 0. & 1. & -1. \end{pmatrix}$$

```
MatrixForm[c . b . a]
```

$$\begin{pmatrix} -1. & 1.61803 & 0. \\ -1.61803 & 1.61803 & 1. \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

```
(* Length 4 *)

MatrixForm[a . b . a . b]
```

$$\begin{pmatrix} 8.14065\times 10^{-8} & -1. & 2.61803 \\ 1. & -1.61803 & 2.61803 \\ 0. & 0. & 1. \end{pmatrix}$$

```
MatrixForm[a . b . a . c]
```

$$\begin{pmatrix} -1.61803 & 2.61803 & -1.61803 \\ -1.61803 & 2.61803 & -1. \\ 0. & 1. & -1. \end{pmatrix}$$

```
MatrixForm[c . a . b . c]
```

$$\begin{pmatrix} 1.61803 & 0. & -1.61803 \\ 1.61803 & 0. & -1. \\ 1.61803 & -1. & 0. \end{pmatrix}$$

```
MatrixForm[c . b . c . a]
```

$$\begin{pmatrix} -1. & 1.61803 & 0. \\ -1.61803 & 2.61803 & -1. \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

```
MatrixForm[a . c . b . a]
```

$$\begin{pmatrix} -1.61803 & 1. & 1.61803 \\ -1.61803 & 1.61803 & 1. \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

```
MatrixForm[b . a . b . a]
```

$$\begin{pmatrix} -1.61803 & 1. & 1.61803 \\ -1. & 8.14065\times 10^{-8} & 2.61803 \\ 0. & 0. & 1. \end{pmatrix}$$

```
MatrixForm[b . a . b . c]
```

$$\begin{pmatrix} 1.61803 & 0. & -1.61803 \\ 1. & 1. & -2.61803 \\ 0. & 1. & -1. \end{pmatrix}$$

MatrixForm[b . a . c . b]

$$\begin{pmatrix} 1.61803 & -1.61803 & 1.61803 \\ 2.61803 & -2.61803 & 1.61803 \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[c . b . a . b]

$$\begin{pmatrix} 1.61803 & -1.61803 & 1.61803 \\ 1. & -1.61803 & 2.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

(* Length 5 *)

MatrixForm[a . b . a . b . a]

$$\begin{pmatrix} -8.14065\times 10^{-8} & -1. & 2.61803 \\ -1. & 8.14065\times 10^{-8} & 2.61803 \\ 0. & 0. & 1. \end{pmatrix}$$

MatrixForm[a . b . a . b . c]

$$\begin{pmatrix} 8.14065\times 10^{-8} & 1.61803 & -2.61803 \\ 1. & 1. & -2.61803 \\ 0. & 1. & -1. \end{pmatrix}$$

MatrixForm[a . b . c . a . b]

$$\begin{pmatrix} 2.61803 & -2.61803 & 1. \\ 2.61803 & -2.61803 & 1.61803 \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[c . a . b . c . a]

$$\begin{pmatrix} -1.61803 & 2.61803 & -1.61803 \\ -1.61803 & 2.61803 & -1. \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

MatrixForm[c . b . c . a . b]

$$\begin{pmatrix} 1.61803 & -1.61803 & 1.61803 \\ 2.61803 & -2.61803 & 1.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[a . c . b . a . b]

$$\begin{pmatrix} 8.14065\times 10^{-8} & -1. & 2.61803 \\ 1. & -1.61803 & 2.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[b . a . b . c . a]

$$\begin{pmatrix} -1.61803 & 2.61803 & -1.61803 \\ -1. & 2.61803 & -2.61803 \\ 0. & 1. & -1. \end{pmatrix}$$

MatrixForm[b . c . a . b . c]

$$\begin{pmatrix} 1.61803 & 0. & -1.61803 \\ 2.61803 & -1. & -1.61803 \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[b . a . c . b . a]

$$\begin{pmatrix} -1.61803 & 1. & 1.61803 \\ -2.61803 & 1.61803 & 1.61803 \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

MatrixForm[c . b . a . b . a]

$$\begin{pmatrix} -1.61803 & 1. & 1.61803 \\ -1. & 8.14065\times10^{-8} & 2.61803 \\ -1. & 8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[c . b . a . b . c]

$$\begin{pmatrix} 1.61803 & 0. & -1.61803 \\ 1. & 1. & -2.61803 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

(* Length 6 *)

MatrixForm[a . b . a . b . c . a]

$$\begin{pmatrix} -8.14065\times10^{-8} & 1.61803 & -2.61803 \\ -1. & 2.61803 & -2.61803 \\ 0. & 1. & -1. \end{pmatrix}$$

MatrixForm[a . b . a . b . c . b]

$$\begin{pmatrix} 2.61803 & -1.61803 & -1. \\ 2.61803 & -1. & -1.61803 \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[a . b . a . c . b . a]

$$\begin{pmatrix} -2.61803 & 1.61803 & 1. \\ -2.61803 & 1.61803 & 1.61803 \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

MatrixForm[c . a . b . c . a . b]

$$\begin{pmatrix} 2.61803 & -2.61803 & 1. \\ 2.61803 & -2.61803 & 1.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[c . b . c . a . b . a]

$$\begin{pmatrix} -1.61803 & 1. & 1.61803 \\ -2.61803 & 1.61803 & 1.61803 \\ -1. & 8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[c . b . c . a . b . c]

$$\begin{pmatrix} 1.61803 & 0. & -1.61803 \\ 2.61803 & -1. & -1.61803 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[c . b . a . b . a . b]

$$\begin{pmatrix} 8.14065\times 10^{-8} & -1. & 2.61803 \\ -1. & -8.14065\times 10^{-8} & 2.61803 \\ -1. & -8.14065\times 10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[a . c . b . a . b . c]

$$\begin{pmatrix} 8.14065\times 10^{-8} & 1.61803 & -2.61803 \\ 1. & 1. & -2.61803 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[b . a . b . c . a . b]

$$\begin{pmatrix} 2.61803 & -2.61803 & 1. \\ 3.23607 & -2.61803 & 8.14065\times 10^{-8} \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[b . c . a . b . c . a]

$$\begin{pmatrix} -1.61803 & 2.61803 & -1.61803 \\ -2.61803 & 3.23607 & -1.61803 \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

MatrixForm[b . a . c . b . a . b]

$$\begin{pmatrix} 8.14065\times 10^{-8} & -1. & 2.61803 \\ 1.31719\times 10^{-7} & -1.61803 & 3.23607 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[c . b . a . b . a . c]

$$\begin{pmatrix} -1.61803 & 2.61803 & -1.61803 \\ -1. & 2.61803 & -2.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

(* Length 7 *)

MatrixForm[a . b . a . b . c . a . b]

$$\begin{pmatrix} 2.61803 & -1.61803 & -1. \\ 3.23607 & -2.61803 & 8.14065\times 10^{-8} \\ 1.61803 & -1. & 0. \end{pmatrix}$$

MatrixForm[a . b . a . b . c . b . a]

$$\begin{pmatrix} -2.61803 & 2.61803 & -1. \\ -2.61803 & 3.23607 & -1.61803 \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

MatrixForm[a . b . a . c . b . a . b]

$$\begin{pmatrix} 1.31719\times10^{-7} & -1.61803 & 2.61803 \\ 1.31719\times10^{-7} & -1.61803 & 3.23607 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[c . a . b . c . a . b . a]

$$\begin{pmatrix} -2.61803 & 1.61803 & 1. \\ -2.61803 & 1.61803 & 1.61803 \\ -1. & 8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[b . c . a . b . c . a . b]

$$\begin{pmatrix} 2.61803 & -2.61803 & 1. \\ 2.61803 & -3.23607 & 1.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[c . b . c . a . b . a . b]

$$\begin{pmatrix} 8.14065\times10^{-8} & -1. & 2.61803 \\ 1.31719\times10^{-7} & -1.61803 & 3.23607 \\ -1. & -8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[c . b . c . a . b . a . c]

$$\begin{pmatrix} -1.61803 & 2.61803 & -1.61803 \\ -2.61803 & 3.23607 & -1.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[c . b . a . b . a . b . c]

$$\begin{pmatrix} 8.14065\times10^{-8} & 1.61803 & -2.61803 \\ -1. & 2.61803 & -2.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[a . c . b . a . b . c . b]

$$\begin{pmatrix} 2.61803 & -1.61803 & -1. \\ 2.61803 & -1. & -1.61803 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[b . a . b . a . c . b . a]

$$\begin{pmatrix} -2.61803 & 1.61803 & 1. \\ -3.23607 & 2.61803 & 8.14065\times10^{-8} \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

MatrixForm[b.a.c.b.a.b.c]

$$\begin{pmatrix} 8.14065\times10^{-8} & 1.61803 & -2.61803 \\ 1.31719\times10^{-7} & 1.61803 & -3.23607 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[c.b.a.b.a.c.b]

$$\begin{pmatrix} 2.61803 & -2.61803 & 1. \\ 3.23607 & -2.61803 & 8.14065\times10^{-8} \\ 1.61803 & -1.61803 & 8.14065\times10^{-8} \end{pmatrix}$$

(* Length 8 *)

MatrixForm[a.b.a.b.c.a.b.a]

$$\begin{pmatrix} -2.61803 & 2.61803 & -1. \\ -3.23607 & 2.61803 & 8.14065\times10^{-8} \\ -1.61803 & 1.61803 & 0. \end{pmatrix}$$

MatrixForm[a.b.a.b.c.b.a.b]

$$\begin{pmatrix} 1.61803 & -2.61803 & 1.61803 \\ 2.61803 & -3.23607 & 1.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[a.b.a.c.b.a.b.a]

$$\begin{pmatrix} -1.31719\times10^{-7} & -1.61803 & 2.61803 \\ -1.31719\times10^{-7} & -1.61803 & 3.23607 \\ -1. & 8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[a.b.a.c.b.a.b.c]

$$\begin{pmatrix} 1.31719\times10^{-7} & 1. & -2.61803 \\ 1.31719\times10^{-7} & 1.61803 & -3.23607 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[c.a.b.c.a.b.a.c]

$$\begin{pmatrix} -2.61803 & 2.61803 & -1. \\ -2.61803 & 3.23607 & -1.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[b.c.a.b.c.a.b.a]

$$\begin{pmatrix} -2.61803 & 1.61803 & 1. \\ -2.61803 & 1. & 1.61803 \\ -1. & 8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[b . c . a . b . c . a . b . c]

$$\begin{pmatrix} 2.61803 & -1.61803 & -1. \\ 2.61803 & -1.61803 & -1.61803 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[c . b . c . a . b . a . c . b]

$$\begin{pmatrix} 2.61803 & -2.61803 & 1. \\ 2.61803 & -3.23607 & 1.61803 \\ 1.61803 & -1.61803 & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[b . a . b . a . c . b . a . b]

$$\begin{pmatrix} 1.31719\times10^{-7} & -1.61803 & 2.61803 \\ 1. & -2.61803 & 2.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[b . a . c . b . a . b . c . a]

$$\begin{pmatrix} -8.14065\times10^{-8} & 1.61803 & -2.61803 \\ -1.31719\times10^{-7} & 1.61803 & -3.23607 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[c . b . a . b . a . c . b . a]

$$\begin{pmatrix} -2.61803 & 1.61803 & 1. \\ -3.23607 & 2.61803 & 8.14065\times10^{-8} \\ -1.61803 & 1. & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[c . b . a . b . a . c . b . c]

$$\begin{pmatrix} 2.61803 & -1.61803 & -1. \\ 3.23607 & -2.61803 & -8.14065\times10^{-8} \\ 1.61803 & -1.61803 & -8.14065\times10^{-8} \end{pmatrix}$$

(* Length 9 *)

MatrixForm[a . b . a . b . c . a . b . a . b]

$$\begin{pmatrix} 1.61803 & -2.61803 & 1.61803 \\ 1. & -2.61803 & 2.61803 \\ 1. & -1.61803 & 1.61803 \end{pmatrix}$$

MatrixForm[a . b . a . c . b . c . a . b . a]

$$\begin{pmatrix} -1.61803 & 2.13125\times10^{-7} & 1.61803 \\ -2.61803 & 1. & 1.61803 \\ -1. & 8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[a . b . a . b . c . b . a . b . c]

$$\begin{pmatrix} 1.61803 & -1. & -1.61803 \\ 2.61803 & -1.61803 & -1.61803 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[a . b . a . c . b . a . b . a . c]

$$\begin{pmatrix} -1.31719\times10^{-7} & 1. & -2.61803 \\ -1.31719\times10^{-7} & 1.61803 & -3.23607 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[c . a . b . c . a . b . c . a . b]

$$\begin{pmatrix} 1.61803 & -2.61803 & 1.61803 \\ 2.61803 & -3.23607 & 1.61803 \\ 1.61803 & -1.61803 & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[b . c . a . b . c . a . b . a . b]

$$\begin{pmatrix} 1.31719\times10^{-7} & -1.61803 & 2.61803 \\ -1. & -1. & 2.61803 \\ -1. & -8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[b . c . a . b . c . a . b . c . a]

$$\begin{pmatrix} -2.61803 & 2.61803 & -1. \\ -2.61803 & 2.61803 & -1.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[b . c . b . a . b . c . a . b . a]

$$\begin{pmatrix} -2.61803 & 1.61803 & 1. \\ -2.61803 & 1. & 1.61803 \\ -1.61803 & 1. & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[c . b . c . a . b . a . c . b . c]

$$\begin{pmatrix} 2.61803 & -1.61803 & -1. \\ 2.61803 & -1.61803 & -1.61803 \\ 1.61803 & -1.61803 & -8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[b . a . b . c . a . b . a . b . c]

$$\begin{pmatrix} 1.31719\times10^{-7} & 1. & -2.61803 \\ 1. & 8.14065\times10^{-8} & -2.61803 \\ 1. & 0. & -1.61803 \end{pmatrix}$$

MatrixForm[c . b . a . b . c . a . b . a . b]

$$\begin{pmatrix} 1.31719\times10^{-7} & -1.61803 & 2.61803 \\ 1. & -2.61803 & 2.61803 \\ 8.14065\times10^{-8} & -1. & 1. \end{pmatrix}$$

MatrixForm[a.c.b.a.b.a.c.b.a]

$$\begin{pmatrix} -2.61803 & 2.61803 & -1. \\ -3.23607 & 2.61803 & 8.14065\times10^{-8} \\ -1.61803 & 1. & 8.14065\times10^{-8} \end{pmatrix}$$

(* Length 10 *)

MatrixForm[a.b.a.b.c.a.b.a.b.a]

$$\begin{pmatrix} -1.61803 & 5.0312\times10^{-8} & 1.61803 \\ -1. & -1. & 2.61803 \\ -1. & 8.14065\times10^{-8} & 1.61803 \end{pmatrix}$$

MatrixForm[c.b.a.c.b.a.c.b.a.b]

$$\begin{pmatrix} 1.31719\times10^{-7} & -1.61803 & 2.61803 \\ -1. & -1. & 2.61803 \\ 8.14065\times10^{-8} & -1. & 1. \end{pmatrix}$$

MatrixForm[a.b.a.c.b.c.a.b.a.c]

$$\begin{pmatrix} -1.61803 & 1.61803 & -1.61803 \\ -2.61803 & 2.61803 & -1.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[a.b.a.c.b.a.b.a.c.b]

$$\begin{pmatrix} 1.61803 & -1. & -1.61803 \\ 2.61803 & -1.61803 & -1.61803 \\ 1.61803 & -1.61803 & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[a.b.c.b.a.b.c.a.b.a]

$$\begin{pmatrix} -1.61803 & 2.13125\times10^{-7} & 1.61803 \\ -2.61803 & 1. & 1.61803 \\ -1.61803 & 1. & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[b.c.a.b.c.a.b.a.b.c]

$$\begin{pmatrix} 1.31719\times10^{-7} & 1. & -2.61803 \\ -1. & 1.61803 & -2.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[b.c.a.b.c.a.b.c.a.b]

$$\begin{pmatrix} 1.61803 & -2.61803 & 1.61803 \\ 1.61803 & -2.61803 & 1. \\ 1.61803 & -1.61803 & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[c . b . a . c . b . a . c . b . a . c]

$$\begin{pmatrix} -2.61803 & 2.61803 & -1. \\ -2.61803 & 2.61803 & -1.61803 \\ -1.61803 & 1. & -8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[c . b . a . b . a . c . b . a . b . c]

$$\begin{pmatrix} 1.31719\times10^{-7} & 1. & -2.61803 \\ 1. & 8.14065\times10^{-8} & -2.61803 \\ 8.14065\times10^{-8} & 8.14065\times10^{-8} & -1. \end{pmatrix}$$

MatrixForm[c . b . a . b . c . a . b . a . b . a]

$$\begin{pmatrix} -1.31719\times10^{-7} & -1.61803 & 2.61803 \\ -1. & -1. & 2.61803 \\ -8.14065\times10^{-8} & -1. & 1. \end{pmatrix}$$

MatrixForm[c . b . a . b . c . a . b . a . b . c]

$$\begin{pmatrix} 1.31719\times10^{-7} & 1. & -2.61803 \\ 1. & 8.14065\times10^{-8} & -2.61803 \\ 8.14065\times10^{-8} & 8.14065\times10^{-8} & -1. \end{pmatrix}$$

(* Length 11 *)

MatrixForm[a . b . a . b . c . a . b . a . b . a . c]

$$\begin{pmatrix} -1.61803 & 1.61803 & -1.61803 \\ -1. & 1.61803 & -2.61803 \\ -1. & 1.61803 & -1.61803 \end{pmatrix}$$

MatrixForm[c . b . a . c . b . a . c . b . a . b . c]

$$\begin{pmatrix} 1.31719\times10^{-7} & 1. & -2.61803 \\ -1. & 1.61803 & -2.61803 \\ 8.14065\times10^{-8} & 8.14065\times10^{-8} & -1. \end{pmatrix}$$

MatrixForm[a . b . a . c . b . c . a . b . a . c . b]

$$\begin{pmatrix} 1. & -1.61803 & 2.13125\times10^{-7} \\ 1.61803 & -2.61803 & 1. \\ 1.61803 & -1.61803 & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[a . b . c . b . a . b . c . a . b . a . b]

$$\begin{pmatrix} -1.61803 & -2.13125\times10^{-7} & 1.61803 \\ -1. & -1. & 2.61803 \\ 8.14065\times10^{-8} & -1. & 1. \end{pmatrix}$$

```
MatrixForm[a.b.c.b.a.b.c.a.b.a.c]
```

$$\begin{pmatrix} -1.61803 & 1.61803 & -1.61803 \\ -2.61803 & 2.61803 & -1.61803 \\ -1.61803 & 1. & -8.14065\times10^{-8} \end{pmatrix}$$

```
MatrixForm[b.c.a.b.c.a.b.a.b.c.b]
```

$$\begin{pmatrix} 1.61803 & -1. & -1.61803 \\ 1.61803 & -1.61803 & -1. \\ 1.61803 & -1.61803 & -8.14065\times10^{-8} \end{pmatrix}$$

```
MatrixForm[b.a.c.b.a.c.b.a.c.b.a]
```

$$\begin{pmatrix} -1.61803 & 2.13125\times10^{-7} & 1.61803 \\ -1.61803 & 2.63437\times10^{-7} & 1. \\ -1.61803 & 1. & 8.14065\times10^{-8} \end{pmatrix}$$

```
MatrixForm[c.b.a.c.b.a.c.b.a.c.b]
```

$$\begin{pmatrix} 1.61803 & -2.61803 & 1.61803 \\ 1.61803 & -2.61803 & 1. \\ 2.13125\times10^{-7} & -1. & 1. \end{pmatrix}$$

```
MatrixForm[c.b.a.c.b.a.c.b.a.b.c]
```

$$\begin{pmatrix} 1.31719\times10^{-7} & 1. & -2.61803 \\ -1. & 1.61803 & -2.61803 \\ 8.14065\times10^{-8} & 8.14065\times10^{-8} & -1. \end{pmatrix}$$

```
(* Length 12 *)
```

```
MatrixForm[a.b.a.b.c.a.b.a.b.a.c.b]
```

$$\begin{pmatrix} 1. & -1.61803 & 5.0312\times10^{-8} \\ 1.61803 & -1.61803 & -1. \\ 1.61803 & -1.61803 & 8.14065\times10^{-8} \end{pmatrix}$$

```
MatrixForm[c.b.a.c.b.a.c.b.a.b.c.b]
```

$$\begin{pmatrix} 1.61803 & -1. & -1.61803 \\ 1.61803 & -1.61803 & -1. \\ 2.13125\times10^{-7} & -8.14065\times10^{-8} & -1. \end{pmatrix}$$

```
MatrixForm[a.b.c.a.b.c.a.b.c.a.b.a]
```

$$\begin{pmatrix} -1. & 2.13125\times10^{-7} & 2.13125\times10^{-7} \\ -1.61803 & 2.63437\times10^{-7} & 1. \\ -1.61803 & 1. & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[a.b.c.b.a.b.c.a.b.a.c.b]

$$\begin{pmatrix} 1. & -1.61803 & 2.13125\times10^{-7} \\ 1.61803 & -2.61803 & 1. \\ 2.13125\times10^{-7} & -1. & 1. \end{pmatrix}$$

MatrixForm[b.c.a.b.c.a.b.a.b.c.b.a]

$$\begin{pmatrix} -1.61803 & 1.61803 & -1.61803 \\ -1.61803 & 1. & -1. \\ -1.61803 & 1. & -8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[c.b.a.c.b.a.c.b.a.c.b.a]

$$\begin{pmatrix} -1.61803 & 2.13125\times10^{-7} & 1.61803 \\ -1.61803 & 2.63437\times10^{-7} & 1. \\ -2.13125\times10^{-7} & -1. & 1. \end{pmatrix}$$

MatrixForm[c.b.a.b.c.a.b.a.b.c.b.a]

$$\begin{pmatrix} -1.61803 & 1.61803 & -1.61803 \\ -1. & 1.61803 & -2.61803 \\ -2.13125\times10^{-7} & 2.63437\times10^{-7} & -1. \end{pmatrix}$$

(* Length 13 *)

MatrixForm[a.b.a.b.c.a.b.a.b.a.c.b.a]

$$\begin{pmatrix} -1. & 2.94532\times10^{-7} & 5.0312\times10^{-8} \\ -1.61803 & 1. & -1. \\ -1.61803 & 1. & 8.14065\times10^{-8} \end{pmatrix}$$

MatrixForm[c.b.a.c.b.a.c.b.a.b.c.b.a]

$$\begin{pmatrix} -1.61803 & 1.61803 & -1.61803 \\ -1.61803 & 1. & -1. \\ -2.13125\times10^{-7} & 2.63437\times10^{-7} & -1. \end{pmatrix}$$

MatrixForm[a.b.c.b.a.b.c.a.b.a.c.b.a]

$$\begin{pmatrix} -1. & 2.13125\times10^{-7} & 2.13125\times10^{-7} \\ -1.61803 & 2.63437\times10^{-7} & 1. \\ -2.13125\times10^{-7} & -1. & 1. \end{pmatrix}$$

MatrixForm[a.b.c.b.a.b.c.a.b.a.c.b.c]

$$\begin{pmatrix} 1. & -1.61803 & -2.13125\times10^{-7} \\ 1.61803 & -1.61803 & -1. \\ 2.13125\times10^{-7} & -8.14065\times10^{-8} & -1. \end{pmatrix}$$

```
MatrixForm[b.c.a.b.c.a.b.a.b.c.b.a.b]
```

$$\begin{pmatrix} 1. & -1.61803 & 2.13125\times10^{-7} \\ 4.2625\times10^{-7} & -1. & 2.63437\times10^{-7} \\ 2.13125\times10^{-7} & -1. & 1. \end{pmatrix}$$

```
(* Length 14 *)

MatrixForm[a.b.c.b.a.b.c.a.b.a.c.b.a.c]
```

$$\begin{pmatrix} -1. & 4.2625\times10^{-7} & -2.13125\times10^{-7} \\ -1.61803 & 1. & -1. \\ -2.13125\times10^{-7} & 2.63437\times10^{-7} & -1. \end{pmatrix}$$

```
MatrixForm[b.c.a.b.c.a.b.a.b.c.b.a.b.c]
```

$$\begin{pmatrix} 1. & -1.61803 & -2.13125\times10^{-7} \\ 4.2625\times10^{-7} & -1. & -2.63437\times10^{-7} \\ 2.13125\times10^{-7} & -8.14065\times10^{-8} & -1. \end{pmatrix}$$

```
MatrixForm[b.c.a.b.c.a.b.a.b.c.b.a.b.a]
```

$$\begin{pmatrix} -1. & 2.13125\times10^{-7} & 2.13125\times10^{-7} \\ -4.2625\times10^{-7} & -1. & 2.63437\times10^{-7} \\ -2.13125\times10^{-7} & -1. & 1. \end{pmatrix}$$

```
(* Length 15 *)

MatrixForm[a.b.c.b.a.b.c.a.b.a.c.b.a.c.b]
```

$$\begin{pmatrix} -1. & -4.2625\times10^{-7} & 2.13125\times10^{-7} \\ 4.2625\times10^{-7} & -1. & 2.63437\times10^{-7} \\ 2.13125\times10^{-7} & -2.63437\times10^{-7} & -1. \end{pmatrix}$$