# EFFECTIVE SOLUTIONS FOR A CERTAIN SECOND ORDER DIFFERENTIAL EQUATION@

An Honors Thesis (ID 499)

by

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@The point at $\infty$ is an irregular singular point of our differential equation.

# I. INTRODUCTION

For the past five decades, considerable progress has been made in studying the behaviour of solutions of equations of the form

$$\frac{dz}{dx} = x^{r-1} \left( \sum_{k=0}^{\infty} A_k x^{-k} \right) z \tag{1.1}$$

where r, the rank of this system, is a non-negative integer, [4] and [5]. The unknown z(x) and the constant coefficients $A_k$ are $n^{th}$ order square matrices and the indicated series is presumed to converge for all sufficiently large values of $|x|$.

G. D. Birkoff in two papers in 1909 and 1913, [1] and [2], attempted to terminate the infinite series in (1) and produce a new reduced equation of the form

$$\frac{dy}{dx} = x^{r-1} \sum_{j=0}^{s} C_j x^{-j} y \tag{1.2}$$

by using one of two types of transformations:

Type I. $z = B(x)y$

where the $n^2$ elements in matrix B are all analytic in some neighborhood of $x = \infty$ and the determinant $B(\infty) \neq 0$.

Type II. $z = v(x)y$

where v(x) can be represented by a convergent series of the form

$$v(x) = x^g \sum_{j=0}^{\infty} v_j x^{-j}$$

where the lead matrix $v_0 \neq 0$; $g$ is an integer and the determinant of $v(x)$ is nonvanishing in some neighborhood of infinity, $r_1 \leq |x| < \infty$. The determinant of the lead matrix $v_0$ may, however, be zero.

G. D. Birkhoff asserted that, if an appropriate transformation of Type I were used, the s in (2) need not exceed r. However, in 1953, Gantmacher, [3], produced a counter-example showing that Birkhoff had made an error.

In a paper in 1963, H. L. Turrittin [6] attempted to make clear to what extent Birkhoff was right and to what extent he was wrong. For instance, there is always some cutoff, i.e., form (2) can always be reached by a suitable transformation of Type I for some sufficiently large finite value of s.

If r = 0, and this is the case covered by the counter-example of Gantmacher, Birkhoff would have been right if he had made his s one unit larger and if the more general transformation of Type II were used; Birkhoff is still correct provided the characteristics of $A_0$ are all distinct.

When $A_0$ has multiple roots, the situation is complicated and obscure.

In this Senior Honors thesis, an attempt is successfully made to effectively solve differential equations of the form

$$y'' + a(z)y' + b(z)y = 0$$

when

$$a(z) = \sum_{i=0}^{\infty} a_i z^{-i} , \quad b(z) = \sum_{i=0}^{\infty} b_i z^{-i}, \quad a_0^2 \neq 4b_0$$

and both power series converge for $|z| > R$.

## II. NATURE OF THE PRESENT PROBLEM

Consider the differential equation

$$y'' + a(z)y' + b(z)y = 0 \tag{2.1}$$

where $a(z) = \sum_{i=0}^{\infty} a_i z^{-i}$

$$b(z) = \sum_{i=0}^{\infty} b_i z^{-i}. \tag{2.2}$$

Here the variable z is complex as the coefficients $a_i$, $b_i$ $(i = 0, 1, 2, \ldots)$ with $a^2 \neq 4\, b_o$. (2.3)

The two power series in (2) converge for $|z| > R$. In the language of Fuch's theory, the differential equation (2.1) will have an irregular singular point at $z = \infty$. We effectively solve the differential equation (2.1) as follows:

We first make the transformation

$$y = y_1$$

and $y_1' = y_2$

Then (2.1) can be written as

$$y_1' = y_2$$

$$y_2' = -a(z)y_2 - b(z)y_1 .$$

This can be put in the matrix form

$$y' = A(z)y \tag{2.4}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $A(z) = \begin{pmatrix} 0 & 1 \\ -b(z) & -a(z) \end{pmatrix}$.

If $A(z) = A_o + \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots. \qquad |z| > R,$

where $A_i$ $(i = 0, 1, 2, \ldots.)$ are 2 x 2 constant matrices, we find that

$$A_o = \begin{pmatrix} 0 & 1 \\ -b_o & -a_o \end{pmatrix}$$

$$A_i = \begin{pmatrix} 0 & 0 \\ -b_i & -a_i \end{pmatrix}, \quad i = 1, 2, 3, \ldots.$$

Let $\lambda_1$ and $\lambda_2$ be the eigen values of the matrix $A_o$. Then $\lambda_1$ and $\lambda_2$ are roots of the characteristic equation

$$\det (A - \lambda I) = 0 ;$$

i.e. $\begin{vmatrix} -\lambda & 1 \\ -b_o & -a_o - \lambda \end{vmatrix} = 0$, which can be written as

$$\lambda(\lambda + a_o) + b_o = 0$$

or $\lambda^2 + a_o\lambda + b_o = 0.$

Thus, if $\lambda_1 = \dfrac{-a_o + \sqrt{a_o^2 - 4 b_o}}{2}$

and $\lambda_2 = \dfrac{-a_o - \sqrt{a_o^2 - 4 b_o}}{2}$.

In view of (2.3), we observe that $\lambda_1$ and $\lambda_2$ are distinct.

Next we diagonalize the matrix $A_o$. To do this, we make the transformation

$$y = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} w \quad \text{in} \tag{2.4}$$

Then, $y' = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} w' = A(z) \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} w$.

This can be written as

$$w' = B(z)\, w, \tag{2.5}$$

where $B(Z) = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} A(z) \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$.

Recalling that $A(z)$ has the representation

$$A_o + \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots, \qquad |z| > R$$

we obtain that

$$B(z) = B_o + \frac{B_1}{z} + \frac{B_2}{z^2} + \ldots, \qquad |z| > R \tag{2.6}$$

where $B_o = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$B_i = \begin{pmatrix} c_i & d_i \\ -c_i & -d_i \end{pmatrix}$$

$$\left.\begin{aligned} \text{with}\quad c_i &= \left[\frac{1}{\lambda_2 - \lambda_1}\right]\left[ b_i + a_i\lambda_1\right] \\ d_i &= \left[\frac{1}{\lambda_2 - \lambda_1}\right]\left[ b_i + a_i\lambda_2\right] \end{aligned}\right\} i = 1, 2, \ldots$$

Next, we make the transformation

$$w = T(z) \quad H \quad \text{in} \tag{2.5}$$

Then,

$$w' = T(z) \; H' + T'(z) \; H = B(z) \; T(z) \; H$$

which can put in the form

$$H' = c(z) \; H, \tag{2.6}$$

where $c(z) = T^{-1}(z)\left[ B(z)\, T(z) - T'(z)\right]$.

We will choose $T(z)$ such that $c(z)$ is given by this truncated representation:

$$c(z) = c_o + \frac{c_1}{z}, \text{ where } c_o = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } c_1 = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}.$$

Let $T(z)$ have the form given by

$$T(z) = I_2 + \frac{T_o}{z} + \frac{T_1}{z^2} + \ldots \qquad , |z| > R$$

where $T_0$, $T_1$, . . . are constant 2 x 2 matrices.

Now $\left[T(z)^{-1}\right] = \left(I_2 + \frac{T_1}{z} + \frac{T_2}{z^2} + \frac{T_3}{z^3} + \cdots\right)^{-1}$

$$= I_2 - \left(\frac{T_1}{z} + \frac{T_2}{z^2} + \frac{T_3}{z^3} + \cdots\right) + \left(\frac{T_1}{z} + \frac{T_2}{z^2} + \frac{T_3}{z^3} + \cdots\right)^2$$

$$- \left(\frac{T_1}{z} + \frac{T_2}{z^2} + \frac{T_3}{z^3} + \cdots\right)^3 + \cdots$$

So, $\left[T(z)^{-1}\right] = I_2 - \frac{T_1}{z} + \frac{1}{z^2}(T_1^2 - T_2) + \frac{1}{z^3}(-T_3 + 2T_1T_2 - T_1^3) + \cdots$

$$T'(z) = -\frac{T_o}{z^2} - \frac{2T_1}{z^3} - \frac{3T_2}{z^4} - \cdots - \frac{(n-1)}{z^n} T_{n-2} - \cdots$$

$$B(z)\,T(z) = \left(\frac{B_o}{1} + \frac{B_1}{z} + \frac{B_2}{z^2} + \cdots\right)\left(I_2 + \frac{T_1}{z} + \frac{T_2}{z^2} + \cdots\right)$$

$$= B + \frac{1}{z}(B_1 + B_oT_1) + \frac{1}{z^2}(B_2 + B_1T_1 + B_oT_2) + \cdots$$

$$+ \frac{1}{z^n}(B_n + B_{n-1}T_1 + \cdots + B_oT_n) + \cdots$$

So, $B(z)\,T(z) - T'(z) = B_o + \frac{1}{z}(B_1 + B_oT_1) + \frac{1}{z^2}(B_2 + B_1T_1 + B_oT_2 - T_o)$

$$+ \cdots + \frac{1}{z^n}\left(B_n + B_{n-1}T_1 + \cdots + B_oT_n - (n-1)\,T_{n-2}\right)$$

$$+ \cdots$$

Thus,

$$C(z) = \left[ I_2 - \frac{T_1}{z} + \frac{1}{z^2} (T_1^2 - T_2) + \frac{1}{z^3} (-T_3 + 2T_1T_2 - T_1^3) + \dots \right] \times$$

$$\left[ B_o + \frac{1}{z} (B_1 + B_oT_1) + \frac{1}{z^2} (B_2 + B_1T_1 + B_oT_2 - T_o) + \dots \right.$$

$$\left. + \frac{1}{z^n} \left( B_n + B_{n-1} T_1 + \dots + B_oT_n - (n-1) T_{n-2} \right) + \dots \right]$$

$$= B_o + \frac{1}{z} \left[ B_1 + B_oT_1 - T_1B_o \right]$$

$$+ \frac{1}{z^2} \left[ B_2 + B_1T_1 + B_oT_2 - T_o - T_1 (B_1 + B_oT_1) + (T_1^2 - T_2) B_o \right]$$

$$+ \frac{1}{z^3} \left[ B_3 + B_2T_1 + B_1T_2 + B_oT_3 - 2T_1 - T_1 (B_2 + B_1T_1 + B_oT_2 - T_o) + \right.$$

$$\left. (T_1^2 - T_2) (B_o - B_1T) + (-T_3 + 2T_1T_2 - T_1^3) B_o \right] + \dots$$

$$= C_o + \frac{C_1}{z} .$$

In order to get the desired truncated form we have to choose $T_1, T_2, T_3 \dots$ so that

$$B_2 + B_1T_1 + B_oT_2 - T_o - T_1(B_1 + B_oT_1) + (T_1^2 - T_2) B_o = 0 .$$

$$B_3 + B_2T_1 + B_1T_2 + B_oT_3 - 2T_1 - T_1(B_2 + B_1T_1 + B_o T_2 - T_o)$$

$$+ (T_1^2 - T_2) (B_o + B_1T) + (-T_3 + 2 T_1T_2 - T_1^3) B_o = 0$$

and so on.

We now make the transformation

$$H = e^{\lambda_1 z} I \quad \text{in (2.6) to obtain}$$

$$e^{\lambda_1 z} I' + \lambda_1 e^{\lambda_1 z} I = c(z) e^{\lambda_1 z} I \quad \text{which}$$

reduces to

$$I' = \left[ c(z) - \lambda_1 I_2 \right] I$$

$$= \left[ D_o + \frac{D_1}{z} \right] I .$$

Here, $D_o = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 - \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$ , with $P = \lambda_2 - \lambda_1$

and $D_1 = C_1 + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ .

Next, we make the transformation

$$I = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} J \quad \text{in (2.7)} .$$

We get

$$I' = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} J' \tag{2.8}$$

$$= \left[ \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} + \frac{1}{z} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \right] \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} J .$$

Hence

$$J' = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} \left[ \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} + \frac{1}{z} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \right] \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} J$$

$$= \left( E_o + \frac{E_1}{z} \right) J,$$

Where

$$E_o = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} 0 & -bP \\ 0 & P \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P_1 & bP_1 + P_2 \\ P_3 & bP_3 \quad P_4 \end{pmatrix}$$

$$= \begin{pmatrix} P_1 - bP_3 & bP_1 + P_2 - bP(bP_3 + P_4) \\ P_3 & bP_3 + P_4 \end{pmatrix}.$$

Choose b so that $b^2 P_3 + B ( P_4 - P_1 ) - P_2 = 0$

Then $$E_1 = \begin{pmatrix} P_1 - bP_3 & 0 \\ P_3 & bP_3 + P_4 \end{pmatrix}$$

$$= \begin{pmatrix} r_1 & 0 \\ P_3 & r_2 \end{pmatrix}$$

where $r_1 = P_1 - bP_3$ and $r_2 = bP_3 + P_4$ .

Finally, make the transformation

$$J' = z^{r_1} \quad k \quad \text{in} \qquad (2.8).$$

We get

$$J' = z^{r_1} k' + r_1 z^{r_1 - 1} k = \left( E_o + \frac{E_1}{z} \right) z^{r_1} k$$

which reduces to

$$k' = \left( E_o + \frac{E_1}{z} - \frac{r_1}{z} I_2 \right) k$$

$$= F_o + \frac{F_1}{z} \quad k \qquad (2.9) .$$

Here, $$F_o = E_o = \begin{pmatrix} 0 & -bP \\ 0 & P \end{pmatrix} .$$

$$F_1 = E_1 - r_1 I_2 = \begin{pmatrix} 0 & 0 \\ P_3 & r_2 - r_1 \end{pmatrix} .$$

We now show that (2.9) is equivalent to the confluent hypergeometric function:

$$\frac{d^2w}{dt^2} + \left(\frac{q}{t} - 1\right)\frac{dw}{dt} - \frac{m}{t}\,w = 0 \qquad (2.10) .$$

Let $t = A^*z$, where $A^*$ is a complex constant .

$$\frac{dw}{dt} = \frac{dw}{dz}\,\frac{dz}{dt} = \frac{1}{A^*}\,\frac{dw}{dz} + \frac{1}{A^*}\,w',$$

$$\frac{d^2w}{dt^2} = \frac{1}{(A^*)^2}\,w''.$$

(2.10) becomes

$$\frac{1}{(A^*)^2}\,w'' + \left(\frac{q}{A^*z} - 1\right)\frac{1}{A^*}\,w' - \frac{m}{A^*z}\,w = 0\,;$$

i.e. $$w'' + \left(\frac{q}{z} - A^*\;w'\right) - \frac{mA^*}{z}\,w = 0 .$$

Now, let $w = c^*k_1$

and $w' = k_2$

Then $w' = c^*k_1' = k_2$

$$w'' = k_2' = \left(A^* - \frac{q}{z}\right)k_2 + \frac{mA^*}{z}\,k_1\,c^*.$$

This can be put in the form

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{c^*} \\ \frac{mA^*c^*}{z} & A^* - \frac{q}{z} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} .$$

If $k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, then we have

$$k' = \left[ \begin{pmatrix} 0 & \frac{1}{c^*} \\ 0 & A^* \end{pmatrix} + \frac{1}{z} \begin{pmatrix} 0 & 0 \\ mA^*c^* & -q \end{pmatrix} \right] K$$

which is the form (2.9).

## BIBLIOGRAPHY OF SOURCES

1. Birkhoff, G. D. "Singular Points of Ordinary Differential Equations." Trans. Amer. Math. Soc., 10 (1909), 463-470.

2. Birkhoff, G. D. "Equivalent Singular Points of Ordinary Linear Differential Equations." Math. Ann., 74 (1913), 134-139.

3. Gantmacher, F. R. The Theory of Matrices, Vol. 2. New York: Chelsea, 1959.

4. Hukuhara, M. "Sur les Points Singuliers des Équations Différentielles Linéaires, III." Mem. Fac. Sci, 2 (1942), 127-37.

5. Tritzinsky, W. J. "Analytic Theory of Linear Differential Equations." Act. Math., 62 (1934), 167-227.

6. Turrittin, H. L. "Reducation of Ordinary Differential Equations to the Birkhoff Canonical Form." Trans. Amer. Math. Soc., 107 (1963), 485-507.