The Finite Radon Transform

An Honors Thesis

by

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Abstract

The Radon transform (first considered by J. Radon in 1917) is an integral transform achieved by integrating a function over a set of lines on the domain of the function, and is used in x-ray tomography to produce pictures of unseen objects. In the Radon transform, one may think of the domain as the body of a person and the function as a density function on the body. In this case, the Radon transform produces an average density along various lines through the body. Physically, these averages may be obtained by measuring the decrease in intensity of a beam of radiation passed through the body along various lines, so one may compute the image of the Radon transform without knowing the original density function explicitly. In x-ray tomography, one produces a picture of the body by reconstructing the original density function. The question "When it possible to reconstruct a picture of the body?" which is of great importance in x-ray tomography, translates into the mathematical question "When is it possible to invert the Radon transform?" In a variety of settings, this latter mathematical question remains unsolved. In order to shed light on these inversion questions, we will investigate the finite Radon transform. In this case, the body consists of finitely many points and is crossed by finitely many lines (each containing the same finite number of points). In this project, we will explore two instances of the finite Radon transform: the complete graph on n vertices (the vertices are the `body' and the edges are the `lines') and finite dimensional vector spaces over finite fields (points in the vector space form the `body' and one-dimensional subspaces, along with their translations, form the `lines').

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1. Introduction

Tomography is used to investigate the interior structure of the human body, rocket motors, rocks, snow packs on the Alps, and violins and other bowed instruments. The objective of tomography is to see what is inside an object without opening it up. This goal is similar to classic problems in mathematics: determine an unknown using given information. In some situations, the unknown quantity could be a real number x expressed in a relation, and in other cases the unknown quantity could be a function with some given information about its behavior. In tomography, the given information is a set of x-ray projections of some unknown object, and the solution is a representation of the unknown object.

In 1917 an Austrian mathematician named Johann Radon showed that this could be done when the total density of every line through the object is known. The *Radon Transform* is used to describe the collection of x-ray projections of an object through all possible lines [1]. The problem of determining an object via its x-ray projections can be devised as: Given the Radon transform of an unknown object, find the object.

Let X be a finite set and Y be a collection of subsets of X. $C(X)$ and $C(Y)$ denote complex-valued functions on X and Y, respectively. We define the finite Radon transform $R: C(X) \rightarrow C(Y)$ as:

$$Rf(y) = \sum_{x \in y} f(x)$$

The main question that needs to be answered is given Rf, when can we invert R to recover f? In order to answer this question, we need to answer several associated questions:

- When is R injective?
- When is R bijective?
- Can we characterize the kernel of R? Can we characterize the image of R?

In this paper, we will present conditions under which R is injective. Then, given that R is injective, we will consider subsets $Y' \subseteq Y$ so that the associated transform $C(X) \rightarrow C(Y')$ is bijective, hence invertible. In this case, Y' is called an admissible set of lines.

2. Some Examples of Finite Radon Transforms

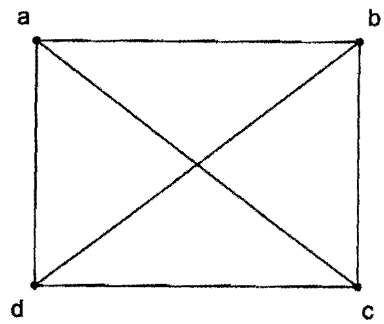
Throughout this paper, we discuss several different examples of types of transforms: the k-set transform, the affine k-plane transform, and the projective k-plane transform.

<u>The *k*-set Transform</u>

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y =$ subsets of X containing k elements. In this example, we can identify (X, Y) when $k = 2$ with the complete graph on n vertices, K_n. An example of a k-set transform when $k = 2$ is shown in Figure 1. Let $X = \{a, b, c, d\}$ and $Y = \{\{a, b\},$

{a, c}, {a, d}, {b, c}, {b, d}, {c, d}}. Note that X is the set of vertices and Y is the set of edges in Figure 1.

Figure 1



Here is an example of a k-set transform when $k = 3$: Let X = {a, b, c, d} and Y = {{a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}}. Let f(a) = 2, f(b) = -1, f(c) = 4, and f(d) = -2. Then $Rf(y_1) = 2 + (-1) + 4 = 5$, where y_1 = {a, b, c}.

<u>The Affine k-plane Transform</u>

Let F denote a finite field, $X = F_d$ = d-dimensional vector space over $F = \{(a_1, a_2, \ldots, a_d) \mid a_j \in F\}$, and Y = affine k-dimensional spaces in X, or $\{\vec{a} + V \mid \vec{a} \in X$ and V = k-dimensional subspace of $X\}$. Here's an example of an affine k-plane transform, where $F = \mathbb{Z}_3$, $d = 2$, and $k = 1$: Let

$$X = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\},$$

and $Y = \{l_1, l_2, \ldots, l_{12}\}$, where

$$\begin{aligned}
l_1 &= F(1, 0) = \{(0, 0), (1, 0), (2, 0)\}\\
l_2 &= F(0, 1) = \{(0, 0), (0, 1), (0, 2)\}\\
l_3 &= F(1, 1) = \{(0, 0), (1, 1), (2, 2)\}\\
l_4 &= F(2, 1) = \{(0, 0), (2, 1), (1, 2)\}\\
l_5 &= (0, 1) + l_1\\
l_6 &= (0, 2) + l_1\\
l_7 &= (1, 0) + l_2\\
l_8 &= (2, 0) + l_2\\
l_9 &= (0, 1) + l_3\\
l_{10} &= (0, 2) + l_3\\
l_{11} &= (0, 1) + l_4\\
l_{12} &= (0, 2) + l_4
\end{aligned}$$

Let $f((i,j)) = i+j$. Then $Rf(l_3) = 0+2+4= 6$.

The Projective k-plane Transform

Let F be a finite field and F_d the d-dimensional vector space over F. We put

$$X = P(F^d) = \{\text{one-dimensional subspaces over } F^d\}$$

and

$$Y = \{(k+1)\text{-dimensional subspaces of } F_d\}.$$

For example, when $k = 1$ we have $X = \{\text{lines through the origin}\}$ and $Y = \{\text{planes through the origin}\}$. Let $F = \mathbb{Z}_2$ and $d = 3$. Then $X = \{\text{lines through } \vec{0} \text{ in } \mathbb{Z}_2^3\}$ and $Y = \text{planes through } \vec{0} \text{ in } \mathbb{Z}_2^3\}$. In this case, the lines are:

$$\{(0, 0, 0), (1, 0, 0)\}$$
$$\{(0, 0, 0), (0, 1, 0)\}$$
$$\{(0, 0, 0), (0, 0, 1)\}$$
$$\{(0, 0, 0), (1, 1, 0)\}$$
$$\{(0, 0, 0), (0, 1, 1)\}$$
$$\{(0, 0, 0), (1, 0, 1)\}$$
$$\{(0, 0, 0), (1, 1, 1)\}$$

The planes in this case are:

$$\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)\}$$
$$\{(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 0, 1)\}$$
$$\{(0, 0, 0), (1, 0, 0), (0, 1, 1), (1, 1, 1)\}$$
$$\{(0, 0, 0), (0, 1, 0), (0, 1, 1), (0, 0, 1)\}$$
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$$\{(0, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 1)\}$$

3. Injectivity and Block Conditions

Now that we have given several examples of the finite Radon transform, we want to describe conditions on X and Y given by E. Bolker [2] that will guarantee a finite Radon transform is injective. However, we first need to introduce some background material.

Radon Adjoints:

We need to establish adjoint to R, given that an inner product of two functions $f, g \in C(X)$ (and similarly for $C(Y)$) is defined as

$$(f, g) = \sum_{a \in X} f(a)\overline{g(a)}$$

First set $G_x = \{y \mid x \in y\}$ (G_x is all elements of Y that contain x). We can identify x with G_x. Then define a transform S: $C(Y) \rightarrow C(X)$ as:

$$Sg(x) = \sum_{y \in G_x} g(y)$$

<u>Lemma 1</u>: S is an adjoint to R.

Proof: We must show that for $f \in C(X)$ and $g \in C(Y)$, $(Rf, g) = (f, Sg)$. Note

$$(Rf, g) = \sum_{y \in Y}\left[\left(\sum_{x \in y} f(x)\right)\overline{g(y)}\right]$$

and

$$(f, Sg) = \sum_{x \in X}\left[f(x)\overline{\left(\sum_{y \in G_x} g(y)\right)}\right].$$

For example, let $X = \{a, b, c\}$, and $Y = \{\{a, b\}, \{b, c\}, \{a\}, \{a, b, c\}\}$, where: $y_1 = \{a, b\}$, $y_2 = \{b, c\}$, $y_3 = \{a\}$, $y_4 = \{a, b, c\}$. In this example, $G_a = (y_1, y_2, y_4)$, $G_b = \{y_1, y_2, y_3\}$, and $G_c = \{y_2, y_4\}$

$$\begin{aligned}(Rf, g) &= \sum_{y \in Y}\left[\left(\sum_{x \in y} f(x)\right)\overline{g(y)}\right]\\
&= [f(a) + f(b)]\overline{g(y_1)} + [f(b) + f(c)]\overline{g(y_2)} + [f(a)]\overline{g(y_3)} + [f(a) + f(b) + f(c)]\overline{g(y_4)}\\
&= f(a)\overline{g(y_1)} + f(b)\overline{g(y_1)} + f(b)\overline{g(y_2)} + f(c)\overline{g(y_2)} + f(a)\overline{g(y_3)}\\
&\quad + f(a)\overline{g(y_4)} + f(b)\overline{g(y_4)} + f(c)\overline{g(y_4)}\end{aligned}$$

and

$$\begin{aligned}(f, Sg) &= \sum_{x \in X}\left[f(x)\overline{\left(\sum_{y \in G_x} g(y)\right)}\right]\\
&= f(a)\left[\overline{g(y_1)} + \overline{g(y_3)} + \overline{g(y_4)}\right] + f(b)\left[\overline{g(y_1)} + \overline{g(y_2)} + \overline{g(y_4)}\right] + f(c)\left[\overline{g(y_2)} + \overline{g(y_4)}\right]\\
&= f(a)\overline{g(y_1)} + f(a)\overline{g(y_3)} + f(a)\overline{g(y_4)} + f(b)\overline{g(y_1)} + f(b)\overline{g(y_2)} + f(b)\overline{g(y_4)}\\
&\quad + f(c)\overline{g(y_2)} + f(c)\overline{g(y_4)}\ .\end{aligned}$$

A direct comparison of terms shows that (Rf, g) and (f, Sg) are equal in this case. In general, to see that $(Rf, g) = (f, Sg)$, note that $f(x)\overline{g(y)}$ is a term of (Rf, g) if and only if $x \in y$, which in turn is true if and only if $y \in G_x$, so $f(x)\overline{g(y)}$ is a term of (f, Sg). ■

<u>The Point-Line Incidence Matrix</u>:

We now define a matrix that can be used to represent a finite Radon transform.

<u>Definition 2</u>: When $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, for $1 \le i \le m$ and $1 \le j \le n$, put

$$a_{ij} = \begin{cases} 0 & x_j \notin y_i \\ 1 & x_j \in y_i \end{cases}$$

The mxn matrix $A = (a_{ij})$ is the <u>point-line incidence matrix</u>.

For example, let $X = \{a, b, c\}$ and $Y = \{\{a, b\}, \{b, c\}, \{a\}, \{a, b, c\}\}$. Then

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

<u>Lemma 3</u>: Given a Radon transform R, the associated point-line matrix A represents R as a linear transformation.

Proof: Suppose $R: C(X) \to C(Y)$. We can identify $C(X)$ with $\mathbb{C}^n$, where $|X| = n$: If $X = \{x_1, \ldots, x_n\}$ and $f(x_k) = b_k$, then $f \in C(x)$ can be identified with $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{C}^n$. Similarly, we can identify $C(Y)$ with $\mathbb{C}^m$. Now observe

$$A\vec{b} = A\begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} f(x_j) \\ \vdots \\ \sum_{j=1}^{n} a_{mj} f(x_j) \end{bmatrix} = \begin{bmatrix} \sum_{x \in y_1} f(x) \\ \vdots \\ \sum_{x \in y_m} f(x) \end{bmatrix} = \begin{bmatrix} Rf(y_1) \\ \vdots \\ Rf(y_m) \end{bmatrix} = Rf(y).$$

Therefore, A represents R.. ■

<u>Block Conditions and Injectivity</u>:

We will now introduce conditions under which the finite Radon transform is injective. Injectivity of the Radon transform is important because it allows us to reconstruct a function f from its image Rf.

Given X and Y, Bolker [2] declares the 'block' conditions to be satisfied if there exist positive numbers α and β such that

$$|G_x| = \alpha \quad \forall x \in X$$
$$|G_x \cap G_{x'}| = \beta \quad \forall x, x' \in X \text{ with } x \neq x' \text{ and } \alpha \neq \beta.$$

Theorem 4: (Bolker) When the block conditions are satisfied, the corresponding Radon transform R is injective.

Proof 1: We will show R is injective by showing that A^TA is invertible. Showing that A^TA is invertible implies that $x \mapsto A^T Ax$ is 1-1, meaning that $x \mapsto Ax$ is 1-1. Because A represents R, this will prove that R is injective.

Consider the entry in row k, column k in A^TA. Note

$$a_{1k}^2 + a_{2k}^2 + \ldots + a_{mk}^2 = a_{1k} + a_{2k} + \ldots + a_{mk}$$

because a_{ij} is either a 0 or a 1. Then this entry is counting the number of elements of Y that contain x_k, which is $|G_{x_k}| = \alpha$

Now consider the entry in row k, column l in A^TA, where $k \neq l$: $a_{1k}a_{1l} + a_{2k}a_{2l} + \ldots + a_{mk}a_{ml}$. Like the first entry, this entry is counting the number of $y \in Y$ that contain x_1 and x_2, which is $|G_x \cap G_x'| = \beta$. It follows that A^TA can be expressed in terms of α and β:

$$A^TA = \begin{bmatrix} \alpha & \beta & \cdots & \beta \\ \beta & \alpha & \cdots & \beta \\ & & \vdots & \\ \beta & \beta & \cdots & \alpha \end{bmatrix} = \beta \begin{bmatrix} w & 1 & \cdots & 1 \\ 1 & w & \cdots & 1 \\ & & \vdots & \\ 1 & 1 & \cdots & w \end{bmatrix},$$

where $w = \dfrac{\alpha}{\beta}$.

By adding all of the rows of A^TA together and replacing the first row with that sum, we get

$$\left| \beta \begin{bmatrix} w & 1 & \cdots & 1 \\ 1 & w & \cdots & 1 \\ & & \vdots & \\ 1 & 1 & \cdots & w \end{bmatrix} \right| = \beta^n \begin{vmatrix} w+(n-1) & w+(n-1) & \cdots & w+(n-1) \\ 1 & w & \cdots & 1 \\ & & \vdots & \\ 1 & 1 & \cdots & w \end{vmatrix}$$

$$= \beta^n \left(w + (n-1) \right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & 1 \\ & & \vdots & \\ 1 & 1 & \cdots & w \end{vmatrix}.$$

Then, by replacing every row by the top row by minus that row, we get:

$$\text{Det}(A^TA)= \beta^n\left(w+(n-1)\right)\begin{vmatrix}1 & 1 & \cdots & 1\\ 0 & w-1 & \cdots & 0\\ & & \vdots & \\ 0 & 0 & \cdots & w-1\end{vmatrix}.$$

Because the matrix is upper triangular, we obtain

$$\begin{aligned}\text{Det}(A^TA)&= \beta^n\left(w+(n-1)\right)(w-1)^{n-1}.\\ &=\beta^n\left(\frac{\alpha}{\beta}+n-1\right)\left(\frac{\alpha}{\beta}-1\right)^{n-1}\\ &=\beta^{n-1}\left(\alpha+\beta\,(n-1)\right)\left(\frac{\alpha}{\beta}-1\right)^{n-1}\\ &=\left(\alpha+\beta\,(n-1)\right)\left(\alpha-\beta\right)^{n-1}.\end{aligned}$$

Since we're assuming the block conditions are in place, we know that $\alpha \neq \beta$ and that α, β are both positive, so $\left(\alpha+\beta\,(n-1)\right)\left(\alpha-\beta\right)^{n-1}$ will never be 0. Therefore, when the block conditions hold, A^TA is invertible, implying that R is injective. ■

We include a second proof of the theorem in which we construct a left inverse for the Radon transform. First, a lemma:

Lemma 5: If R has a left inverse $\hat{R}$, then R is 1-1.

Proof. To show R is 1-1, we need to show that $Rf = 0$ implies that $f = 0$. By hypothesis, $f=\hat{R}Rf= \hat{R}(0)$. $\hat{R}(0) = 0$ because $\hat{R}$ is linear. Therefore, when a left inverse exists, R is 1-1.

Proof 2 of Theorem 4: We will construct a left inverse for R, thus showing that R is invertible.
This proof uses S, the adjoint to R. If $z\in X$, then

$$\left(R\delta_z\right)y=\sum_{x\in y}\delta_z(x)=\begin{cases}0 & \text{if } z\notin y\\ 1 & \text{if } z\in y\end{cases},$$

where δ denotes the characteristic function, so $R\delta_z=\delta_{G_z}$, where $G_z=\left\{y\,|\,z\in y\right\}$.
Then

$$SR\delta_z(x)=S\left(\delta_{G_z}\right)(x)=\sum_{y\in G_x}\delta_{G_z}(y)=\begin{cases}|G_z|=\alpha & \text{if } x=z\\ |G_x\cap G_z|=\beta & \text{if } x\neq z\end{cases}.$$

Therefore, $SR\left(\delta_{G_z}\right)=\left(\alpha-\beta\right)\delta_z+\beta$. By solving for δ_z, we get $\delta_z=\frac{SR\delta_z-\beta}{\alpha-\beta}$. Here, we use the fact that $\alpha\neq\beta$ to ensure that the denominator is not 0. Since any function $\phi\in C(x)$ can be written as $\varphi=\sum_{x\in X}\varphi(x)\delta_x$, we can substitute the previous δ_z to get

$$(*)\qquad \varphi=\frac{1}{\alpha-\beta}SR\varphi-\frac{\beta}{\alpha-\beta}\sum_{x\in X}\varphi(x).$$

Because $\sum_{x\in X}\varphi(x)$ is some constant, we will call this constant $\mu(\varphi)$. Observe that

$$\mu(\varphi)=\left(\varphi,\delta_x\right)=\sum_{x\in X}\varphi(x)=\sum_{x\in X}\varphi(x)\cdot 1=\sum_{x\in X}\varphi(x)\delta_X(x)=\left(\varphi,\delta_x\right).$$

Also, since $\left(S\delta_Y\right)(x)=\sum_{y\in G_x}\delta_Y(y)=\sum_{y\in G_x}1=|G_x|=\alpha$, we have $S\delta_Y=\alpha\delta_x$. We can use these facts, together with the fact that R and S are adjoints, to obtain

$$\mu(\varphi)=\left(\varphi,\frac{1}{\alpha}\left(\alpha\delta_X\right)\right)=\left(\varphi,\frac{1}{\alpha}S\left(\delta_Y\right)\right)=\left(\varphi,S\left(\frac{1}{\alpha}\delta_Y\right)\right)=\left(R\varphi,\frac{1}{\alpha}\delta_Y\right)$$
$$=\sum_{y\in Y}R\varphi(y)\frac{1}{\alpha}\delta_Y(y)=\frac{1}{\alpha}\sum_{y\in Y}R\varphi(y)=\frac{1}{\alpha}\mu\left(R\varphi\right).$$

By substituting the result of the previous equation for $\sum_{x\in X}\varphi(x)$ in equation (*), we get

$\varphi=\frac{1}{\alpha-\beta}SR(\varphi)-\frac{\beta\mu\left(R\varphi\right)}{\alpha\left(\alpha-\beta\right)}$, which is a left inverse formula for *R*. Because we have determined a left inverse for *R*, *R* is one to one. ∎

Block Conditions and Group Actions:

We will show that the main examples satisfy the block conditions (and are hence injective by Theorem 4) by looking at group actions. We begin with a review of concepts.

Definition 6: Let *G* be a group and *X* be a finite set. *G* acts on *X* if there is a function $G\times X\mapsto X$ (denoted $g\cdot x$) such that:
(i) $e\cdot x=x\ \forall x\in X$ (*e* is called the identity element)
(ii) $g_1\cdot(g_2\cdot x)=(g_1\cdot g_2)x\ \forall g_1,g_2\in G,\ \forall x\in X$

For example, let $X=\{x_1,\ldots,x_n\}$ and $G=S_n$ (the symmetric group on *n* objects, i.e. the set of all bijections of the set {1, …, n} onto {1, …, n} under the operation of composition). If $g \in S_n$ and $x_j \in X$, then $g \cdot x_j = x_{g(j)}$. In order to show that this is a group action, we must:

(i) Show that $e \cdot x = x \;\; \forall x \in X$:

$e \cdot x_j = x_{e(j)} = x_j$ because e is the identity function.

(ii) Show that $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) x \;\; \forall g_1, g_2 \in G, \; \forall x \in X$:

$$g_1 \cdot \left(g_2 \cdot x_j\right) = g_1 \cdot x_{g_2(j)} = x_{g_1\left(g_2(j)\right)} = x_{(g_1 \cdot g_2)(j)} = (g_1 \cdot g_2) \cdot x_j .$$

Definition 7: Let G be a group acting on X. A group action is:

1.) Transitive if given $x_1, x_2 \in X$, $\exists g \in G$ such that $g \cdot x_1 = x_2$.
2.) Doubly transitive if given $x, y, w, z \in X$, $\exists g \in G$ such that $g \cdot x = w$ and $g \cdot y = z$ where $x \neq y$ and $w \neq z$.

For example, let $X = \{x_1, \ldots, x_n\}$, Y = 2-element subsets of X, and $G = S_n$. If $g \in G$ and $\{x_j, x_l\} \in Y$, then $g \cdot \{x_j, x_l\} = \{x_{g(j)}, x_{g(l)}\}$. Therefore, an action on X induces an action on Y. To show that G acts doubly transitively on X, given $x_1, x_2, x_3, x_4 \in X$ we need to find $g \in S_n$ such that $g \cdot x_1 = x_2$ and $g \cdot x_3 = x_4$. We can define g to fit the needed conditions. Set $g(1) = 2$ and $g(3) = 4$, and anything else to get sent to itself. Therefore, G acts doubly transitively on X.

Proposition 8: If G acts doubly transitively on X, and the action of G extends to Y, then (X, Y) satisfy the block conditions.

Proof: If G acts transitively on X, given $x, x' \in X$, $\exists g \in G$ such that $g \cdot x = x'$. Then $G_x \to G_{x'}$, as defined by $y \mapsto g \cdot y$, is a bijection. Then we can conclude that $|G_x| = |G_{x'}|$ for $\forall x, x' \in X$, which is the first part of the block conditions.

Given that G acts doubly transitively on X, then given $x, x', w, w' \in X$, with $x \neq x'$ and $w \neq w'$, $\exists g \in G$ such that $g \cdot x = w$ and $g \cdot x' = w'$. Then $G_x \cap G_{x'} \to G_w \cap G_{w'}$ as defined by $y \mapsto g \cdot y$, is a bijection. Then we can conclude that $|G_x \cap G_{x'}| = |G_w \cap G_{w'}|$ for $\forall x, x', w, w' \in X$, which is the second part of the block conditions. ■

Corollary 9: The k-set transform as well as the affine and projective k-plane transforms are injective.

Proof: We appeal to Proposition 8. The k-set transform admits a doubly transitive group action from the example above (just replace 2 element subsets by k element subsets). For the affine k-plane transform, let $X = F^d$ (F = finite field), Y = Affine k-dimensional spaces in X, $GL(F, d)$ = group of bijective, F-linear transformations from F^d to F^d, and $A(F, d)$ = mappings of the form $\vec{x} \mapsto a\vec{x} + \vec{b}$ where $a \in GL(F, d)$, $b \in F^d$.

We will show that $A(F, d)$ acts doubly transitively on X. Observe $A(F, d)$ acts on X with the associated action on Y. Let $x, x', z, z' \in X$ with $x \neq x'$ and $z \neq z'$. Because $GL(F, d)$ is transitive on $F^d - \{0\}$, $\exists g \in GL(F, d)$ such that $g(\vec{x} - \vec{x'}) = \vec{z} - \vec{z'}$. (We can do this because $\vec{x} - \vec{x'}$ and $\vec{z} - \vec{z'}$ are both nonzero.) Define $h \in A(F, d)$ by $h(w) = g \cdot (\vec{w} - \vec{x}) + \vec{z}$. Then $h(x) = z$ and $h(x') = z'$. Therefore, $A(F, d)$ is doubly transitive on $X = F^d$, and the block conditions hold. Proof for the projective k-plane transform is similar. ■

<u>A Radon Transform that does not satisfy the block conditions: The Diaconis-Graham Transform</u>

Let G be an Abelian group. Let $S \subseteq G$, $Y = G$-translated of S (sets of the form $x + S$, $x \in G$). Define $R: C(G) \mapsto C(Y)$ as: $Rf(x + S) = \sum_{u \in x+S} f(u)$. (This transform was introduced by Diaconis and Graham in [4].) We will prove that this transform does not satisfy the block conditions by choosing a specific example. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, $S = \{(1, 0), (0, 0)\}$, and $Y = \{\{(1, 0), (0, 0)\}, \{(1, 1), (0, 1)\}\}$. In this case, $|G_{(1,0)} \cap G_{(0,0)}| = 1$ and $|G_{(1,1)} \cap G_{(1,0)}| = 0$. Therefore, the block conditions are not satisfied.

4. Admissibility

According to Corollary 9, R: $C(X) \rightarrow C(Y)$ is injective in our three main examples. Now, we must ask: When is R onto? In order for R: $C(X) \rightarrow C(Y)$ to be bijective, we need the vector spaces $C(X)$ and $C(Y)$ to be the same dimension. Therefore, if $|X| = n$, then $|Y|$ must be n as well. To meet this condition we throw out some elements of Y to obtain $Y' \subseteq Y$ with $|Y'| = X$.

<u>Definition 10</u>: Let $Y' \subseteq Y$. Y' is <u>admissible</u> if $R: C(X) \mapsto C(Y')$ is bijective.

When Y' is admissible, every element of $C(Y')$ is Rf for some f in $C(X)$, and f can be reconstructed from Rf.

<u>Case 1: The Affine *k*-plane transform</u>

In the Affine k-plane transform, what should Y' be so that $R: C(X) \mapsto C(Y')$ is 1-1? In order to determine what subsets of Y are admissible, we need to first define a spread.

<u>Definition 11</u>: A <u>spread</u> is a collection of subsets of Y that determines a partition of X. For example, if $X = \mathbb{Z}_2^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, and Y = affine lines

$l_1 = \{(0, 0), (1, 0)\}$

$l_2 = \{(0, 0), (0, 1)\}$

$l_3 = \{(0, 0), (1, 1)\}$
$l_4 = \{(0, 1), (1, 1)\}$
$l_5 = \{(1, 0), (1, 1)\}$
$l_6 = \{(0, 1), (1, 0)\}$

then $\{l_1, l_4\}$, $\{l_2, l_5\}$, and $\{l_3, l_6\}$ are all spreads. All of these spreads are formed by translates of a 1-dimensional subspace.

How many spreads are there, given $|F| = q$? The number of spreads is equal to the number of (d-1)-dimensional subspaces in F^d, namely $\frac{q^d - 1}{q - 1}$. Note that when $d = 2$, there are $q + 1$ spreads.

<u>Theorem 12</u>: From Grinberg [5], all admissible subsets of Y are formed by retaining a complete spread in Y and casting out exactly one hyperplane from the other spreads.

Proof in the case $d = 2$:
We first show that this strategy produces the right number of lines. Let $X = F^2$ and $Y =$ lines in X. Note $|X| = q^2$, and that there are $q + 1$ spreads. Each spread has q lines. Therefore, there are $q(q + 1)$ lines. Removing exactly one line from each spread except one means that we remove q lines. So, we have $q(q + 1) - q$ lines left, or $q^2 + q - q = q^2$ lines remaining, to form $Y' \subseteq Y$. Since $|X| = q^2 = |Y'|$, this method produces the right number of lines. It remains to show $R: C(X) \mapsto C(Y')$ is invertible.

Next, we show that the sum of all rows in the submatrix $A' \subseteq A$ corresponding to the lines in the complete spread is a row consisting entirely of 1's, where A' is the point-line incidence matrix that represents $R: C(X) \rightarrow C(Y')$. Because a spread is a partition of X, every $x \in X$ is in the entire spread exactly once. Therefore, all rows of A' representing the lines in the spread contain only one 1 entry per column and the rest are 0's. Then adding the rows together results in a row consisting entirely of 1's.

Now, we will show that the sum of all rows corresponding to a "deleted" spread is all 1's except 0's corresponding to the points on the missing line. In this case, all of the elements of X in the "deleted" spread are represented exactly once, except for the points on the missing line, which are not represented at all. Therefore, the rows corresponding to the deleted spread have one 1 in each column except for the column representing the points on the deleted line, which consists of all zeros. Then the sum of all rows in the deleted spread is all 1's except 0's corresponding to the points on the missing line.

Suppose Y' is obtained by this strategy mentioned in the theorem. Y' is admissible if and only if A' is invertible. We will prove this by using A' to obtain A in a way that shows that their ranks are the same. To obtain A from A', we add in those deleted lines. To obtain the deleted lines, we take the sum of the rows from the deleted spread and subtract it from the sum of the rows from the complete spread. This works because the sum of all rows in the complete spread is all 1's, and the sum of the rows in the deleted spread is all 1's, except for 0's corresponding to the points on the missing line.

Subtracting one from the other results in the row corresponding to the missing line. Since we are obtaining new rows by taking linear combinations of existing rows, we do not change rank of A'. From Theorem 4, rank(A) = $|X|$. So, after filling out A' to obtain A, rank(A') = rank(A) = $|X|$ (full rank). Therefore, this method produces an admissible subset $Y' \subseteq Y$.

For example, let $X = \mathbb{Z}_2^2$ and $Y = \{\text{lines in } \mathbb{Z}_2^2\}$. In this case, the list of lines is:

$l_1 = \{(0, 0), (1, 0)\}$
$l_2 = \{(0, 1), (1, 1)\}$
$l_3 = \{(0, 0), (0, 1)\}$
$l_4 = \{(1, 0), (1, 1)\}$
$l_5 = \{(0, 0), (1, 1)\}$
$l_6 = \{(1, 0), (0, 1)\}$

The spreads are:

$u_1 = \{l_1, l_2\}$
$u_2 = \{l_3, l_4\}$
$u_3 = \{l_5, l_6\}$

By keeping one complete spread, u_1, and throwing out one line from each of the other spreads (l_4 and l_6) we obtain the matrix

$$A' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

A' is invertible, implying that $R: C(X) \mapsto C(Y')$ is bijective. ■

Case 2: The k-set Transform when k=2

In the k-set transform when k=2, X is a set of n elements and Y is a collection of 2-element subsets of X. Therefore, (X, Y) can be identified with the complete graph K_n on n vertices, where X is the set of vertices and Y is the set of edges. In this case, what must occur in order for Y' to be admissible? In order to characterize admissible subsets of Y, we must first introduce terminology that will be used when describing complete graphs.

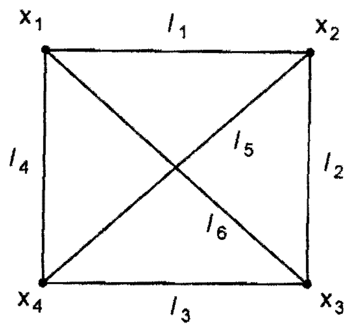
A Y' – path is a sequence of points in X such that consecutive points are joined by an edge in Y', and are distinct. A Y' path is a Y' – cycle if it begins and ends at the same point. A point $x \in X$ lies on an odd Y' cycle if there is a Y' cycle α such that

(i) α begins and ends with x, and
(ii) α has an odd number of edges and therefore an even number of points.

Two Y' cycles are equivalent if their edges form the same set.

Let $X = \{x_1, \ldots, x_4\}$ and $Y = \{l_1, \ldots, l_6\}$.

Figure 3



An example of a Y' path is (x_1, x_2, x_3, x_2), an even Y' cycle is $(x_1, x_2, x_3, x_2, x_1)$, and an odd Y' cycle is $(x_3, x_2, x_4, x_1, x_2, x_3)$.

We will show, via point-line matrix, that our example of an even Y' cycle does not constitute an admissible subset:

Let $Y' = \{l_1, l_2, l_3, l_4\}$. The point-line incidence matrix $A' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. $\det A' = 0$, so A' is not invertible. Therefore, Y' is not an admissible subset.

Now we will show, via point-line matrix, that our example of an odd Y' cycle constitutes an admissible subset:

Let $Y' = \{l_1, l_2, l_5, l_6\}$. The point-line incidence matrix $A' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. $\det A' = 2$, so A' is invertible. Therefore, Y' is admissible.

These examples seem to indicate the following theorem.

Theorem 13: Let $(X, Y) = K_n$ and $Y' \subseteq Y$ with $|Y'| = n$. Y' is admissible if and only if each $x \in X$ lies on an odd Y' cycle.

Proof: Suppose that each vertex lies on an odd cycle. We will call the subgraph (X, Y') and define the Radon transform $\widetilde{R} : C(X) \to C(Y')$. Given $f \in C(X)$, we must show that $\widetilde{R}f \equiv 0$ implies that $f \equiv 0$. Let $x \in X$ and let $\alpha = (x, x_1, \ldots, x_{2n+1})$ be an odd cycle starting at x (so $x_{2n+1} = x$). Now, since $\widetilde{R}f \equiv 0$,

$$0 = \widetilde{R}f(y_1) = f(x) + f(x_1)$$
$$0 = \widetilde{R}f(y_2) = f(x_1) + f(x_2)$$
$$\vdots$$
$$0 = \widetilde{R}f(y_{2n+1}) = f(x_{2n}) + f(x)$$

By manipulating these equations, we get $f(x) = -f(x_1) = f(x_2) = \ldots = f(x_{2n}) = -f(x)$. Therefore, $f \equiv 0$.

Now suppose that α is an even Y'-cycle, so $\alpha = (x_1, \ldots, x_{2n})$. We will show that in this case there are two distinct functions that are sent to 0 under $\widetilde{R}$. Define $f(x_{2k}) = -1$ and $f(x_{2k-1}) = 1$ for $1 \leq k \leq n$, and let f be identically zero away from α. Then we have:

$$\widetilde{R}f(y_1) = f(x_1) + f(x_2) = 1 + (-1) = 0$$
$$\widetilde{R}f(y_2) = f(x_2) + f(x_3) = -1 + 1 = 0$$
$$\vdots$$
$$\widetilde{R}f(y_m) = f(x_{2n-1}) + f(x_{2n}) = 1 + (-1) = 0$$

So this nonzero function maps to the zero function under $\widetilde{R}$, and hence $\widetilde{R}$ is not one-to-one, so Y' is not admissible. Therefore, each $x \in X$ must lie on an odd Y' cycle in order for Y' to constitute an admissible subset. ■

5. Counting Admissible Subsets for the 2-set Transform

Now, knowing how to form admissible subsets in the *2*-set transform, we ask how many such admissible subsets exist.

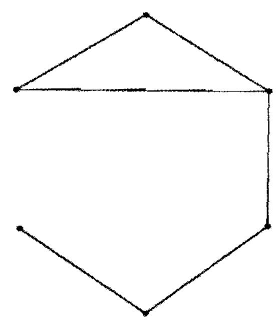
Definition 14: A proper odd cycle is a cycle in which the only repeated vertex is the first and last vertex (i.e. there are no repeated edges).

Admissible subsets can be thought of as subsets of complete graphs on *k* vertices where all vertices lie on a proper odd cycle plus appendages to that cycle. These appendages will be in the form of trees.

Definition 15: An *n*-tree is a connected graph with *n* vertices that does not contain a cycle.

Figure 4 shows an example when $k = 6$, with an admissible subset containing a proper 3-cycle together with a 3-tree.

Figure 4



To count admissible subsets of Y, we need to account for all subsets Y' of Y for which

1) $|Y'| = n = |X|$, and
2) each connected component contains at least one proper odd cycle.

Lemma 16: Conditions 1 and 2 above imply that each component of an admissible Y' contains exactly one proper odd cycle.

Proof.
First we must show that one proper cycle plus attached trees gives the right number of edges. Let k be the number of vertices in the connected component and j the number of vertices in the odd cycle, implying that the odd cycle consists of j edges. Attaching trees with the remaining $k-j$ points will add $k-j$ edges, giving a total of k edges. Therefore, the number of edges equals the number of vertices, so one proper cycle plus attached trees gives the right number of edges. Next, we show that if there are two proper odd cycles in a connected component, this will result in too many edges. If we begin with one proper odd cycle plus attached trees, we must add an edge to create another proper odd cycle. However, we have already shown that one proper odd cycle plus attached trees contains the correct number of edges, so adding an edge to create another proper odd cycle will result in too many edges.■

Let Y' be an admissible subset of Y, and consider k vertices in Y' that form a connected component. Let j = the number of points in the unique proper odd cycle (Lemma 16) in this component (so $j \leq k$, and $j \geq 3$). Then $k-j$ = the number of vertices in this component not lying on the proper odd cycle. These vertices will be attached to the odd cycle via appendages. Also, let λ = a partition of $k-j$ into $k-j$ parts. For example, let $k = 5, j = 3, k-j = 2$. Possible λs in this case are (2, 0) and (1, 1).

Then, to count the number of admissible subsets of Y, we need to compute

$$(1) \qquad \sum_{|\mu|=n} C_\mu \, t_{\mu_1} t_{\mu_2} \ldots t_{\mu_n}$$

where μ is a partition of n, so $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$, C_μ is the number of ways to separate n points into subsets of sizes $\mu_1, \mu_2, \ldots, \mu_n$, and t_{μ_j} is the number of connected admissible sets of μ_j edges on μ_j vertices with $t_{\mu_j}=1$ if $\mu_j = 0$, 1, or 2. In order to evaluate this sum, we need to know how to evaluate C_μ and t_{μ_j}.

Evaluation of C_μ:

If $\lambda = (\lambda_1, \ldots, \lambda_{k-j})$, then we can define the associated *datum* $r_\lambda = (r_1, \ldots, r_{k-j})$ where r_i is the number of times i appears in λ. For example, when $k-j = 11$ and $\lambda = (5, 3, 1, 1, 1, 0, \ldots, 0)$, we have $r_\lambda = (3, 0, 1, 0, 1, 0, \ldots, 0)$.

We will use a multinomial coefficient to determine how to split n into subsets of size $\lambda_1, \ldots, \lambda_{k-j}$, where $\lambda!$ is defined as $\lambda_1! \ldots \lambda_{k-j}!$ At first glance, it seems that we have $\frac{n!}{\lambda_1! \lambda_2! \ldots \lambda_{k-j}!} = \frac{n!}{\lambda!}$ subsets. However, if $|\lambda_g| = |\lambda_h|$, then the same elements could be placed into the subsets of the same size, thus repeating our count. Because the datum counts the number of subsets of each size, we need to divide by $r_\lambda!$, where $r_\lambda!$ is defined by $r_1! \ldots r_{k-j}!$. Thus we have $\frac{n!}{\lambda! r_\lambda!}$ ways to break n vertices into subsets determined by λ.

For example, let $\lambda = (5, 3, 1, 1, 1, 0, \ldots, 0)$ and $r_\lambda = (3, 0, 1, 0, 1, 0, \ldots, 0)$. The multinomial coefficient for λ is $\frac{11!}{5!3!1!1!1!}$. However, the same three elements could be placed into the three 1-element subsets, resulting in the same arrangement of subsets. Dividing by r_λ will account for the repeated subsets, giving us $\frac{11!}{(5!3!1!1!1!)(3!1!1!)}$, which is the correct count. The above discussion constitutes a proof of the following lemma:

Lemma 17: If $|\mu| = k$, then $C_\mu = \frac{k!}{\mu! r_\mu!}$.

Evaluation of t_k:

To find the number of connected admissible sets of k vertices on k edges, we will need to compute the following sum:

$$\sum_{\substack{j \le k \\ j\,odd \\ j \ge 3}} \sum_{\substack{\lambda \\ |\lambda| = k-j}} \left[\begin{pmatrix} \textit{the number} \\ \textit{of ways to} \\ \textit{choose the} \\ j\,\textit{vertices} \\ \textit{to be in the} \\ \textit{odd cycle} \end{pmatrix} * \begin{pmatrix} \textit{the number} \\ \textit{of ways to} \\ \textit{construct the} \\ \textit{cycle with} \\ \textit{the chosen} \\ \textit{vertices} \end{pmatrix} * \begin{pmatrix} \textit{The number} \\ \textit{of ways to} \\ \textit{break the} \\ k-j \\ \textit{remaining} \\ \textit{points} \\ \textit{into subsets} \\ \textit{determined by } \lambda \end{pmatrix} * \begin{pmatrix} \textit{the number} \\ \textit{of ways to} \\ \textit{choose the} \\ \textit{vertex of} \\ \textit{the cycle on} \\ \textit{which to} \\ \textit{attach} \\ \textit{the trees} \end{pmatrix} * \begin{pmatrix} \textit{the number of} \\ \textit{ways to form} \\ \textit{and attach the} \\ \textit{trees to the} \\ \textit{chosen vertex} \end{pmatrix} \right]$$

Now, we will examine each of these necessary components to determine the formula for counting connected admissible subsets.

Number of ways to choose the vertices to be in the odd cycle:

Given k points, we want to choose j points to be in the cycle. In other words, we have $\binom{k}{j}$ ways to choose the points for the odd cycle.

Number of ways to construct the cycle with the chosen vertices:

Pick a vertex from the j points to be the starting point of our cycle. There are $j - 1$ ways to choose the next point in the cycle, $j - 2$ ways to choose the following points, etc. This gives us $(j - 1)!$ cycles. Each cycle is equivalent to its reversal (e.g. $(x_1, x_2, x_3, x_4, x_1)$ is equivalent to $(x_1, x_4, x_3, x_2, x_1)$), so we must divide by 2. Therefore, there are $\frac{(j-1)!}{2}$ ways to construct the cycle with the chosen points.

Number of ways to break the $k - j$ remaining points into subsets determined by λ:

This calculation uses the same methods used in calculating C_μ. In this example, we are calculating how to split $k - j$ into subsets of size $\lambda_1, \ldots, \lambda_{k-j}$, where λ represents a partition of the $k - j$ vertices. For similar reasons shown when calculating C_μ, there are $\frac{(k-j)!}{\lambda! r_\lambda !}$ ways to break the remaining points into subsets determined by λ.

Number of ways to choose the vertex on the cycle to attach the $r_1 + \ldots + r_{k-j}$ trees:

Each tree can be attached to any vertex on the cycle, so there are j ways to do so. r_λ counts the number of nonzero entries in λ, so $r_1 + \ldots + r_{k-j}$ counts the number of subsets of vertices corresponding to tree, and thus is equal to the total number of trees. Since there are $r_1 + \ldots + r_{k-j}$ trees that could be attached to each vertex, we have $j^{r_1 + \ldots + r_{k-j}}$ ways to choose the vertex on the cycle to attach the $r_1 + \ldots + r_{k-j}$ trees.

Number of ways to form and attach trees to the chosen vertex:

Given $m \leq k - j$, we want to find how many (connected) trees can be formed from m vertices.

Definition 18: A spanning tree on m points is a connected graph on m points containing no proper cycles.

Given m points in the tree, there are m^{m-2} spanning trees on m points [3]. We can attach the tree to the cycle from any of the m points, so there are m ways to choose the vertex to use to attach the tree to the proper odd cycle. Therefore, we have $m*m^{m-2}$ or m^{m-1} ways. Since we have r_m subsets of size m, then there are $m^{r_m(m-1)}$ ways to form and attach the trees to the chosen vertex. We must take the product of this evaluation for each $m \le k - j$ to account for all trees being formed from the leftover vertices.

These examinations allow us to conclude:

Lemma 19: The number t_k of connected admissible subsets on k vertices is

$$t_k = \sum_{\substack{j \le k \\ j\, odd \\ j \ge 3}} \sum_{\substack{\lambda \\ |\lambda| = k-j}} \left[\binom{k}{j} * \left(\frac{(j-1)!}{2} \right) * \left(\frac{(k-j)!}{\lambda! r_\lambda !} \right) * \left(j^{r_1 + \ldots + r_{k-1}} \right) * \prod_{m \le k-j} m^{r_m(m-1)} \right].$$

Putting (1) together with Lemmas 17 and 19 allow us to compute a table of numbers of admissible subsets given $|X| = n$:

n	Number of admissable subsets
3	1
4	12
5	162
6	2530
7	45,615
8	937,440
9	$2.1685135 * 10^7$
10	$5.58360144 * 10^8$
11	$1.5850436805 * 10^{10}$
12	$4.9202966692 * 10^{11}$
13	$1.658522958675 * 10^{13}$
14	$6.03402126941952 * 10^{14}$

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