

A NEW PERSPECTIVE ON DAVID LEWIN'S INTERVAL FUNCTION:

THE SYMMETRICAL IFUNC ARRAY

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## A New Perspective on David Lewin's Interval Function:

### The Symmetrical IFUNC array

The twentieth century has seen the emergence and development of a subset of music theory known as set theory, which is derived from the mathematical discipline of group theory. Using mathematical principles set theory generalizes musical features, like the interval, that allow for coherent and consistent analysis. The systems developed by set theorists are especially suited to music that lies outside of functional tonality. Of the many important theorists working in this area, one who has had far-reaching influence is David Lewin. Lewin's seminal work, *Generalized Musical Intervals and Transformations (GMIT)*, proposes generalized systems relating to set theory; among these systems is Lewin's interval function (IFUNC). In his discussion of the IFUNC in *GMIT*, Lewin uses an excerpt from Anton Webern's *Four Pieces for Violin and Piano*, op. 7, no. 3 to demonstrate various applications of the IFUNC.<sup>1</sup> One of these applications investigates how many times each pitch-class interval occurs as a directed interval between notes in two sets in order to make broader observations about temporal and registral placement.

I was intrigued by the IFUNC examples from op. 7 no. 3 because of the orderly disposition of numbers in the resulting IFUNC values. These values display symmetrical

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1. David Lewin, *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1987), 90-99.

features analogous to those of inversionally symmetrical pitch-class sets. Although the symmetrical quality of the ordered IFUNC values, or what I will refer to as the IFUNC array, is not discussed in *GMIT*, another article from 2001 by David Lewin further investigates principles that lead to various distinct IFUNC arrays.<sup>2</sup> Among these special cases is a situation that presents a symmetrical IFUNC array. To Lewin the result of this situation is “perhaps mathematically suggestive, but it is hard to catch a pertinent musical intuition that corresponds to the suggestion.”<sup>3</sup>

It is the goal of this paper to explore the mathematical suggestion observed by Lewin as well as to propose a possible “pertinent musical intuition.” To accomplish this goal I will explore general features of the symmetrical IFUNC array drawing on concepts from the works by David Lewin mentioned above. The following contextual applications will use excerpts from Anton Webern’s op. 5 no. 4, op. 6 no. 1, and op. 7 no. 3— respectively, the *Five Movements for String Quartet*, *Six Pieces of Orchestra*, and *Four Pieces for Violin and Piano*. The focus of these analyses is to uncover a new type of symmetrical relationship within the IFUNC. In preparation for the discussion of the symmetrical IFUNC array, the next section of this paper will review relevant topics in the literature that are important to that discussion.

## REVIEW OF LITERATURE

The IFUNC was first proposed in 1959 by David Lewin and has been further explored and cemented in his 1987 book *Generalized Musical Intervals and*

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2. David Lewin, “Special Cases of the Interval Function between Pitch-Class Sets X and Y,” *Journal of Music Theory* 45, no. 1 (Spring 2001): 1-29.

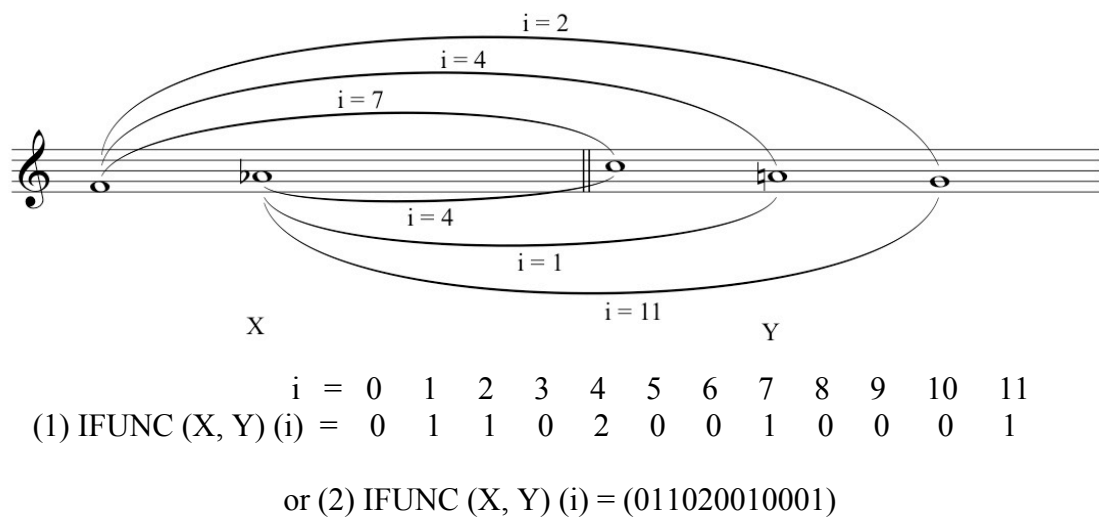
3. *Ibid.*, 11.

*Transformations (GMIT)*. Lewin's IFUNC is similar to Allen Forte's interval vector, but instead of exploring the intervallic relationships within a pitch-class set the interval function catalogs the type and number of intervals between two different pitch-class sets. While Forte's interval vector counts interval classes (1–6), Lewin's IFUNC counts directed pitch-class intervals (0–11) from one set to another set. Chapter 5 of *GMIT* contains Lewin's definition and notation of his IFUNC. Lewin's general definition is a function  $IFUNC(X, Y)(i)$  that shows "how many different ways the interval  $i$  can be spanned between [members of]  $X$  and [members of]  $Y$ ";  $X$  and  $Y$  each being a different pitch-class set and  $i$  being an interval from one member of  $X$  to one member of  $Y$ . Figure 1 gives a visual representation of the IFUNC process and displays two formats in which the IFUNC can be summarized—a table listing each value of  $I$  and its IFUNC value, and an array in which each value is listed in order of intervals 0–11. Throughout this paper I will use the second format most often. The ordered sequence of twelve digits showing the directed pc interval from one set to another I will call the IFUNC array.<sup>4</sup>

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4. David Lewin, "Intervallic Relations Between Two Collections of Notes," *Journal of Music Theory* 3, no. 2 (November 1959): 298-301; Lewin, *GMIT*, 90-99.

FIGURE 1: IFUNC (X, Y) (i)



Lewin proposes in *GMIT* that the IFUNC can be used to explore structural relationships that are not readily apparent on the surface of the music and that might be altogether overlooked without such a tool. To help demonstrate the usefulness of the IFUNC, Lewin analyzes an excerpt from Webern's op. 7 no. 3. In his discussion Lewin calculates several interval functions from pitch-class sets found within the Webern excerpt.

FIGURE 2: IFUNC (X, Z) (i) from op. 7, no. 3

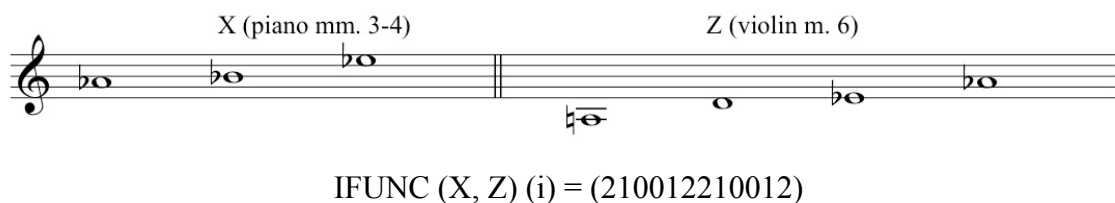


Figure 2 shows an example of an IFUNC Lewin explored in *GMIT*. Lewin discusses how the intervals (i) that appear most frequently can be structural elements. In this particular example (0) is among the most-represented intervals: it happens to be that the last two pitches of Z frame the set X. The same relationship is also found in another important IFUNC within the movement.

In the article from 2001, Lewin continues to explore the possibilities of the IFUNC. Lewin discusses the factors needed to create what he calls “constant-IFUNC pairs,” “alternating-IFUNC pairs,” and what I have labeled “ternary-IFUNC pairs.”

#### FIGURE 3: IFUNC Pairings from “Special Cases” Article

constant-IFUNC pairs: IFUNC (X,Y) = (nnnnnnnnnnnn)  
 alternating-IFUNC pairs: IFUNC (X,Y) = (mnmnmnmnmnm)  
 ternary-IFUNC pairs: IFUNC (X,Y) = (kmnkmnkmnkmn)

where X and Y are each a pitch-class set and k, m, and n are the values for each interval class 0 to 11 spanned between the two sets.

Lewin also defines five “Fourier properties,” first introduced in his 1959 article. When the Fourier properties are found in specific combinations between sets with two to six members they can create one of the above pairs. Using Lewin’s terminology the properties are as follows:

FOURPROP (6): the whole-tone set—the set includes the same number of notes from the WT-1 as from the WT-0 collection.

FOURPROP (4): the diminished seventh chord set—the set includes the same number of notes from each of the three diminished seventh chords.

FOURPROP (3): the augmented triad set—the set includes the same number of notes from one augmented triad as from the augmented triad a tritone away.

FOURPROP (2): set  $K = (0167)$ —the set includes the same number of notes from  $K$  and from  $T_3(K)$ .

FOURPROP (1): the set can be expressed as a disjoint union of tritone sets and/or augmented triad sets. (i.e. set  $X = \{1, 2, 5, 8, 9\}$  and can be expressed as the disjoint union of  $\{1, 5, 9\}$  and  $\{2, 8\}$ .)

The three types of IFUNC pairs listed above result from various combinations of the Fourier properties. Within the discussion on the combination of properties, Lewin presents two IFUNC pairs in which one of the sets exhibits *all* of the properties excluding FOURPROP (1):

Sets -  $\{0, 1, 2\}$  and  $\{0, 2, 3, 5\}$  - IFUNC = (222211000011)  
 Sets -  $\{0, 2, 4\}$  and  $\{0, 1, 3, 6\}$  - IFUNC = (121110101112)

These are the IFUNC arrays that Lewin admits are “perhaps mathematically suggestive.”<sup>5</sup>

Another suggestive idea, found in section 5.5, deals with using the IFUNC to display how inversional relationships may be perceived. In this discussion Lewin touches upon an IFUNC outcome that is “interval-complement symmetrical,” meaning that the values are the same for interval  $i$  as it is for interval  $12-\text{minus-}i$  (e.g. interval 1 and 11, 2 and 10, etc.). The interval-complement symmetrical arrays and the IFUNC arrays mentioned above display the same symmetrical property.<sup>6</sup>

Besides Lewin’s book and two articles mentioned above, there has not been much else written that deals directly with Lewin’s interval function. Lewin’s 1977 article “Forte’s ‘Interval Vector,’ My ‘Interval Function,’ and Regener’s ‘Common-Note Function,’” ten years before *GMIT*, explains Lewin’s IFUNC, or INTF as he calls it in

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5. Lewin, “Special Cases,” 11.

6. *Ibid.*, 17.

this article, in comparison to Forte's interval vector and Regener's common-note function. Lewin is able to express the other two ideas using only his interval function. It is in this article that Lewin explains that Forte's interval vector is a "rigorously harmonic concept" while his interval function is "rigorously contrapuntal in conception."<sup>7</sup>

Articles by Richard Teitelbaum and by Eric Isaacson deal with the similarity of intervallic content between two or more pitch-class sets. Teitelbaum uses the interval vectors of two sets to calculate how similar or dissimilar they are to each other; his "similarity index" is a number between 0 and about 15 (the actual number is irrational). Teitelbaum also calculates the "mean interval vector" or the average occurrence of each interval class within several sets used in a composition. Teitelbaum analyzes atonal works by Schoenberg and Webern, using a computer program, to discover certain generalities for each composer. He shows that Webern prefers sets with specific interval classes, while Schoenberg's sets tend to have a more diverse intervallic content. Isaacson presents his own equation for similarity in comparison with others who have developed systems of similarity, including Forte, Charles Lord and Robert Morris, John Rahn, Lewin, and Teitelbaum.<sup>8</sup>

A 1968 article by Lewin and a later article by Tom Demske both focus on the distribution of pitch classes around a given tonal center. Using a few examples from Schoenberg, Lewin discusses the use of inversional balance and demonstrates how an identification of inversional centers can shed light on how composers distribute pitch

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7. David Lewin, "Forte's Interval Vector, My Interval Function, and Regener's Common-Note Function," *Journal of Music Theory* 21, no. 2 (Fall 1977): 201.

8. Richard Teitelbaum, "Intervallic Relations in Atonal Music," *Journal of Music Theory* 9, no. 1 (Spring 1965): 72-127; Eric J. Isaacson, "Similarity of Interval-Class Content between Pitch-Class Sets: The IcVSIM Relation," *Journal of Music Theory* 34, no. 1 (Spring 1990): 1-28.

classes around such centers. Demske is also interested in pitch centers in what he labels a “registral center of balance” or rcb. These rcbs do not deal with inversional relationships; instead they are the central point of any given pitch-class set. Demske does point out a special case of pitch inversional centers, one which is found in registrally symmetrical sets. Neither one of these articles mentions the interval function, but both make important points about inversional centers that inform to a later discussion in this paper. One last article by Philip Lambert from 2000 uses Lewin’s transformational theory in a practical approach to Webern’s op. 11, *Three Little Pieces for Cello and Piano*. While Lambert does not specifically use the interval function either, he does make reference to pitch symmetry and inversional pairings that have been suggested by Lewin in some of his transformational analyses.<sup>9</sup>

Aside from articles dealing with transformational theory, I reviewed symmetrical pc sets and other structures in the music of Anton Webern to help determine which pieces would be most beneficial for this study. There are many articles written about various types of symmetry in Webern’s music, both atonal and serial. For the purposes of this paper, I will focus only on the freely atonal music because of its use of pitch-class symmetries, which will be important for later discussion. Many of the writings on symmetry in the atonal works, spanning op. 5 through op. 11, centers on the use of pitch-class symmetry around a specific axis to create a stabilizing pseudo-tonic. These include: Bruce Archibald (op. 5 no. 2), Peter Johnson (op. 10 no. 4), Charles Burkhart (op. 5 no.

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9. David Lewin, “Inversional Balance as an Organizing Force in Schoenberg’s Music and Thought,” *Perspectives of New Music* 6, no. 2 (Spring – Summer 1968): 1-21; Tom Demske, “Registral Centers of Balance in Atonal Works by Schoenberg and Webern,” *In Theory Only* 9, nos. 2-3 (1986): 60-76; Philip Lambert, “On Contextual Transformations,” *Perspectives of New Music* 38, no. 1 (Winter 2000): 45-76.

4), as well as Felix Meyer and Anne Shreffler (op. 7).<sup>10</sup> Some authors, like Harold Oliver (op. 6) and John Dean Vander Weg (op. 5), expand their focus to pitch-class symmetries as structural elements in the segmentation of a piece.<sup>11</sup> Other writings on Webern's atonal music explore large-scale symmetries, such as Paul Kabbash's discussion of aggregate symmetry and Robert Clifford's study of contour symmetry in op. 11 no. 1.<sup>12</sup>

## GENERALIZED SYMMETRICAL IFUNC ARRAYS

The best way to illustrate symmetrical relationships is with the visual help of a wheel or clockface with numbers 0–11 representing either pitch-classes ( $C = 0$ ) or pitch-class intervals (1 = a half step “up”). When referring to pitch-classes and pitch-class sets I will call the circular diagram a “pc wheel,” and when referring to interval functions or IFUNC arrays I will call it an “interval wheel.” Figure 3 shows an IFUNC array transformed into a circular diagram or interval wheel. Figure 3a shows a table of two rows and twelve columns, in which the top row lists each  $i$  (pitch-class intervals 0–11) and the second row shows each IFUNC  $(X,Y) (i)$ —that is, the number of occurrences of each  $i$  as a directed pc interval from a note in set  $X$  to a note in set  $Y$ .<sup>13</sup> Figure 3b shows

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10. Bruce Archibald, “Some Thoughts on Symmetry in Early Webern: Op. 5, No. 2,” *Perspectives of New Music* 10, no. 2 (Spring – Summer 1972): 159-163; Peter Johnson, “Symmetrical Sets in Webern’s Op. 10, No. 4,” *Perspectives of New Music* 17, no. 1 (Autumn – Winter 1978): 219-229; Charles Burkhart, “The Symmetrical Source of Webern’s Opus 5, No. 4,” *The Music Forum* 5 (1980): 317-334; Felix Meyer and Anne Shreffler, “Performance and Revision: The Early History of Webern’s Four Pieces for Violin and Piano, Op. 7,” in *Webern Studies*, ed. Kathryn Bailey (Cambridge: Cambridge University Press, 1996), 135-169.

11. Harold Oliver, “Structural Functions of Musical Material in Webern’s Op. 6, No. 1,” *Perspectives of New Music* 6, no. 1 (Autumn – Winter 1967): 67-73; John Dean Vander Weg, “Symmetrical Pitch- and Equivalence-class Set Structure in Anton Webern’s Opus 5,” (PhD diss., University of Michigan, 1983).

12. Paul Kabbash, “Aggregate-Derived Symmetry in Webern’s ‘Early Works,’” *Journal of Music Theory* 28, no. 2 (Autumn 1984): 225-250; Robert Clifford, “Multi-Level Symmetries in Webern’s Op. 11, No. 1,” *Perspectives of New Music* 40, no. 1 (Winter 2002): 198-215.

13. Here and elsewhere I use “note” as a synonym for pitch class.

the same table, only curved around into a circle, with  $i = 11$  abutting  $i = 0$ . The interval wheel in Figure 3 consists of two concentric rings around a central circle; the outer ring corresponds with the top row in Figure 3a while the second ring corresponds with the bottom row. If the wheel can be bisected into two halves that are symmetrical around the bisecting line—where each IFUNC value on one side of the axis reflects the same number on the other side—then the IFUNC can be considered symmetrical. The dividing line between the two halves is the “axis of symmetry.”

FIGURE 3: IFUNC (X, Y) (i)

$$X = \{0, 2, 4, 8\} \quad Y = \{0, 2, 4, 6\}$$

$$\text{IFUNC (X, Y) (i) = (303030202030)}$$

a)

$i =$	0	1	2	3	4	5	6	7	8	9	10	11
IFUNC (X, Y) =	3	0	3	0	3	0	2	0	2	0	3	0

b)

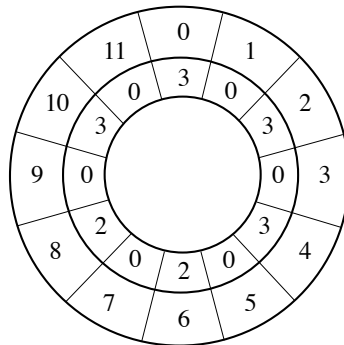
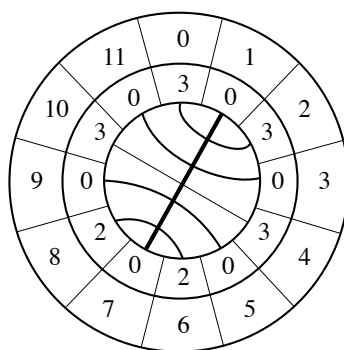


Figure 4 uses the same two sets from the previous figure. The center of the interval wheel shows how the IFUNC array can be divided in half by an axis of symmetry (indicated by the bold line); the thin lines connect identical IFUNC values across the axis and help to make visible the symmetry of the IFUNC array. The location of the axis of symmetry is not a random occurrence, but determined by factors that I will explore next.

FIGURE 4: Symmetry of IFUNC Array



What my research has shown is that the IFUNC array for two inversionally symmetrical sets will be symmetrical.<sup>14</sup> The definition of a symmetrical pitch-class set is similar to that of the symmetrical IFUNC array. When the notes of the set are plotted on the pc wheel, if the wheel can be bisected into two halves where the notes on one side reflect those pitches on the other side, then the pitch-class set is symmetrical. The dividing “axis of symmetry” can be a line between an actual pair of notes (e.g. 1–7 of set X in Figure 5). The axis may also fall between two pitch-class pairs as in set Y of Figure

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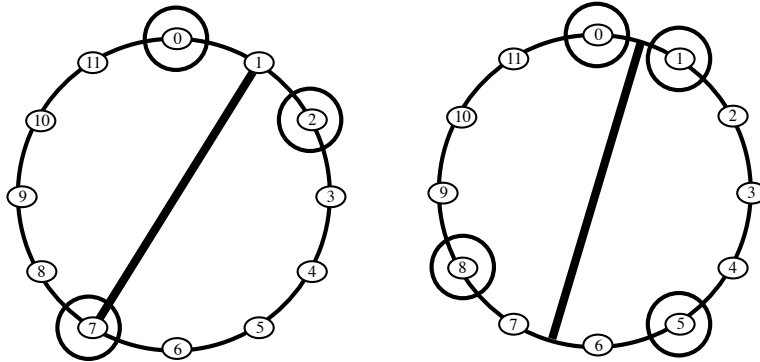
14. While testing to see if symmetrical arrays can occur by other circumstances, I discovered five instances of IFUNC arrays that were symmetrical with just one symmetrical set. Such outcomes seem to be an exception to the rule and later in this paper I will suggest why three of those particular combinations of symmetrical and asymmetrical sets may result in a symmetrical IFUNC array.

5 (0/1–6/7). In Figure 5, the smaller number on the axis of symmetry for set X is 1; I will refer to this number as the “axis value.” The axis value for set Y is 0.5. Axis values will always be in increments of 0.5 between 0 and 5.5 (inclusive).

FIGURE 5: Symmetry of Sets X and Y

$$X = \{0, 2, 7\}$$

$$Y = \{0, 1, 5, 8\}$$



The two sets’ axes of symmetry determine the axis of symmetry for the IFUNC array. If two symmetrical sets, X and Y, are given then the axis of symmetry is computed through the following equation:

$$\text{Axis value for IFUNC (X,Y)} = \text{Axis value Y} - \text{Axis value X}.$$

As seen in Figure 6 using the same sets from Figure 5, if the axis value of X = 1 and the axis value of Y = 0.5, then the axis value for the IFUNC (X, Y) equals 0.5 – 1, or –0.5.

The negative axis value can be adjusted to a positive result by adding 6. Thus the axis value in Figure 6 is –0.5 + 6, or 5.5. Given the same two symmetrical sets, X and Y, the axis of symmetry for the IFUNC (Y, X) is the axis for the IFUNC (X,Y) inverted around 0, but can also be calculated in the same manner as the first.

FIGURE 6: Axis Values

$$X = \{0, 2, 7\} \quad Y = \{0, 1, 5, 8\}$$

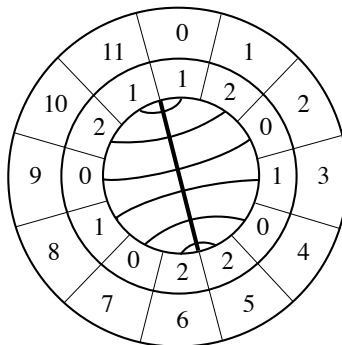
$$\text{Axis value of } X = 1 \quad \text{Axis value of } Y = 0.5$$

a)

$$\text{Axis value of IFUNC array: } 0.5 - 1 = -0.5$$

$$\text{Adjusted axis value for positive result: } -0.5 + 6 = 5.5$$

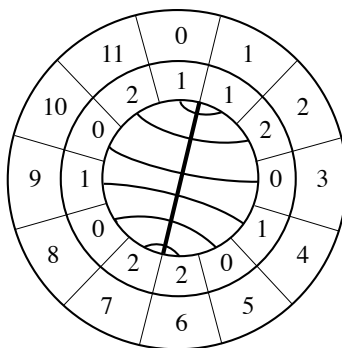
$$\text{IFUNC } (X, Y) (i) = (120102201021)$$



b)

$$\text{IFUNC } (Y, X) (i) = (112010220102)$$

$$\text{Axis value of IFUNC array: } 1 - 0.5 = 0.5$$



In the “Special Cases” article, Lewin demonstrates that sets resulting in one of his IFUNC pairs (see above) will produce the same IFUNC outcome whether one or both sets are transposed. In a similar fashion, transpositions of symmetrical sets will produce the same intervallic relationships across an axis of symmetry, which will have shifted

according to the number of half steps of each set's transposition. The total shift in axis can be calculated with a formula related to the calculation of the IFUNC array axis of symmetry. If the IFUNC (X,Y) axis of symmetry is known between two symmetrical sets, and then one or both sets are transposed, the shift in the axis of symmetry can be calculated by taking the magnitude (1–11 half steps) of the transposition of Y and subtracting the magnitude of the transpositions of X. A positive number n indicates a shift by n clockwise; a negative number –n indicates a shift counterclockwise by n. Figure 7 shows two examples using transpositions of the sets from Figure 6; one results in a positive outcome and the other in a negative. The interval wheel has also been provided again for visual confirmation.

FIGURE 7: Axis shift by Transposition

a)

$X = \{1, 3, 8\}$ : Transposition of  $\{0, 2, 7\}$  by 1 half step  
 $Y = \{4, 5, 9, 0\}$ : Transposition of  $(0, 1, 5, 8)$  by 4 half steps

Shift in axis of symmetry:  $4 - 1 = 3$  half steps clockwise

IFUNC (X, Y) (i) = (021120102201)

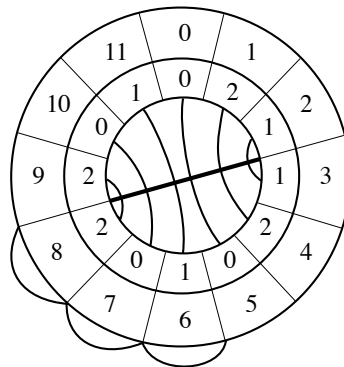


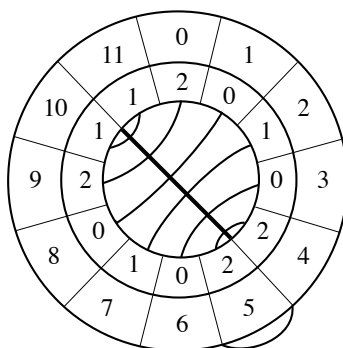
FIGURE 7 cont.

b)

$X = \{3, 5, T\}$ : Transposition of  $\{0, 2, 7\}$  by 3 half steps  
 $Y = \{2, 3, 7, T\}$ : Transposition of  $\{0, 1, 5, 8\}$  by 2 half steps

Shift in axis of symmetry:  $2 - 3 = -1 = 1$  half step counter clockwise

IFUNC (X, Y) (i) = (201022010211)

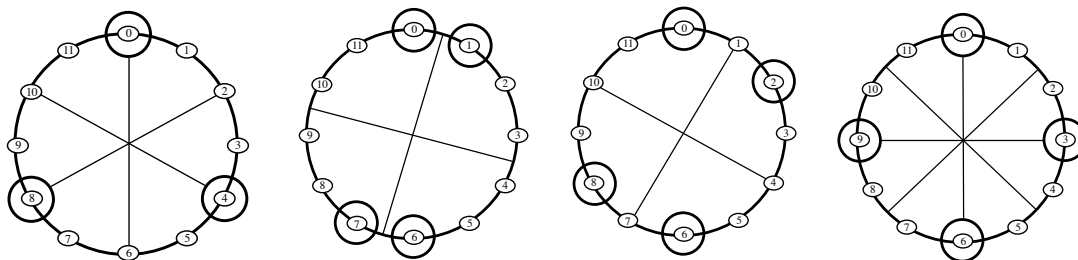


The IFUNC arrays for pitch-class sets with multiple axes of symmetry are particularly interesting. A set with more than one axis of symmetry contains more than one line that bisects its pc wheel. These multi-axis sets are small in number. Out of the five symmetrical trichords, only one contains more than one axis  $\{0, 4, 8\}$ . Of the fourteen symmetrical tetrachords, only three are multi-axis ( $\{0, 1, 6, 7\}$ ,  $\{0, 2, 6, 8\}$ , and  $\{0, 3, 6, 9\}$ ). The IFUNC arrays created by these multi-axis sets and other symmetrical sets are also multi-axis. Figure 8 displays the axes of the four sets mentioned above. The figure also shows three IFUNC combinations between the sets given. All three examples result in multiple axes as well as what I call “tritone pairs,” which result when corresponding intervals across the wheel share the same number of occurrences in the IFUNC array. Also interesting are IFUNC arrays created between multi-axis and

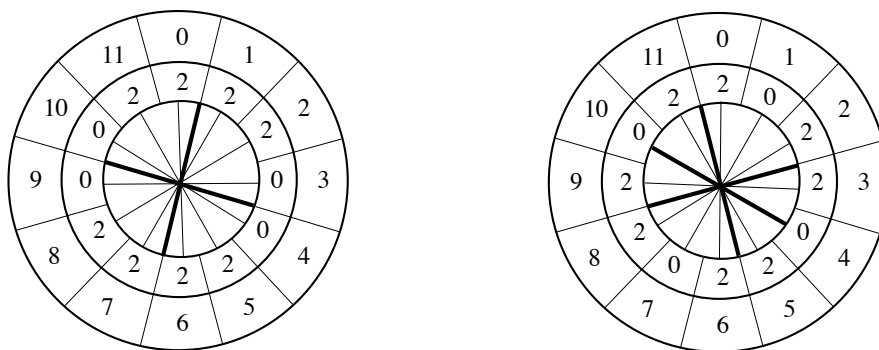
asymmetrical sets. The IFUNC arrays resulting from these pairings do not result in symmetrical outcomes, but usually in tritone pairs.<sup>15</sup>

FIGURE 8: Multi-axis Sets

$$W = \{0, 4, 8\} \quad X = \{0, 1, 6, 7\} \quad Y = \{0, 2, 6, 8\} \quad Z = \{0, 3, 6, 9\}$$



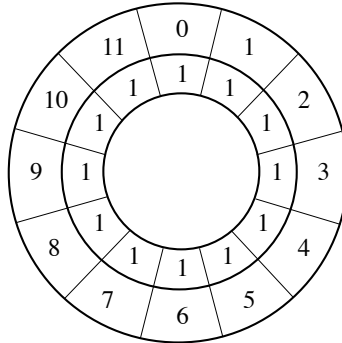
a) IFUNC (X, Y) (i) = (222002222002)    b) IFUNC (Y, Z) (i) = (202202202202)



15. Examples of tritone pairs created by multi-axis and asymmetrical sets will be given in the context of musical excerpts later in this paper. See page 29 for discussion and Figure 13 on pages 31-33.

FIGURE 8 cont.

c) IFUNC (W, Z) (i) = (111111111111)



As for the use of the Fourier properties mentioned earlier, Lewin gives two examples of interval functions that display all properties except FOURPROP(1), resulting in symmetrical outcomes. When I investigated these sets further, I noticed the first example contains two symmetrical pitch-class sets,  $\{0, 1, 2\}$  and  $\{0, 2, 3, 5\}$ , which would seem to foreshadow a symmetrical outcome based on my observations above. To test whether the Fourier properties had any impact on the symmetry (or lack thereof) of the IFUNC of two symmetrical sets, I tested sets with various combinations of properties, including sets having FOURPROP(1), and they always produced a symmetrical outcome. Table 1 shows all of the sets I have used in Figures 3-8 as well as their Fourier properties. The only combination of sets to produce an outcome suggested by Lewin is  $\{0, 4, 8\}$  and  $\{0, 3, 6, 9\}$ , which together manifest all five properties and result in a constant-IFUNC pair. Table 1 shows that a symmetrical IFUNC array can be achieved without the conditions stated by Lewin and thus proves that the properties have little to no impact on the IFUNC arrays of symmetrical sets. The second example, sets  $\{0, 2, 4\}$  and

$\{0, 1, 3, 6\}$ , which excludes FOURPROP (1) and was used by Lewin, proves to be a little more intriguing. This example results in a symmetrical IFUNC array even though one of the sets,  $\{0, 1, 3, 6\}$ , is asymmetrical. It is hard to explain mathematically why a symmetrical outcome would occur out of context. It might be assumed the Fourier Properties have some importance when both sets are asymmetrical, but as will be shown in the contextual applications this not always the case.

TABLE 1: Fourier Properties of Sets used in Figures 3, 5, and 8

FIGURE	SET	FOURIER PROPERTIES
Figure 3	$\{0, 2, 4, 8\}$	0
	$\{0, 2, 4, 6\}$	3
Figure 5	$\{0, 2, 7\}$	4
	$\{0, 1, 5, 8\}$	6
Figure 8	$\{0, 1, 6, 7\}$	6, 3, and 1
	$\{0, 2, 6, 8\}$	3 and 1
	$\{0, 3, 6, 9\}$	6, 3, and 2
	$\{0, 4, 8\}$	4, 2, and 1

## MUSICAL EXAMPLES

While the previous section explored some IFUNC arrays' interval relationships that Lewin calls "mathematically suggestive," the following discussion of Webern examples will explore the "pertinent musical intuition" that such relationships might lead to. I will begin with an excerpt from *Six Pieces for Orchestra*, op. 6, no. 1. The middle section of this piece, mm. 8–14, includes a variety of symmetrical sets. Example 1 is a reduction of mm. 8–14 with some instruments excluded leaving only those instruments that state symmetrical sets, which are marked on the reduction. My criterion for determining

which sets should be paired to yield an IFUNC is that the sets must have some proximal relationship to each other: a set from the beginning of a piece would not be used in an IFUNC with a set from the end of the piece because a listener would perceive no intervallic association between the two. Since Lewin claims his IFUNC is a contrapuntal tool,<sup>16</sup> it makes sense to use the IFUNC for sets that are contrapuntally connected. The combination of sets I chose to investigate include: (G, I), (G, H), (C, I), (H, I), and (C, G), all of which are symmetrical and therefore produce symmetrical IFUNC arrays.

When I calculated the IFUNC values for all the pairs of sets, and when I examined the symmetries of the IFUNC arrays, I still did not feel that I had adequately explained the relationships among the sets in an intuitive musical context. To solve this issue, I created another visual tool to help. Figure 9 outlines the process of this new visual tool using sets G {7, 4, 5, 8} and H {6, 5, 2, 1} from op. 6 no. 1. Figure 9a shows the sets H and G on two pc wheels along with their IFUNC-array. Figure 9b deconstructs the IFUNC array: I calculate the IFUNC of each individual note of H to the set G, finding four intervals which I then plot on an interval wheel. Because G is symmetrical, each of the resulting interval wheels are symmetrical, with axis values 0, 1, 4, and 5. Each interval wheel reflects the interval structure of set G (a member of set class {0, 1, 3, 4}). If the axis values themselves are then placed on a pc wheel, this “consolidation of axes” will reflect the interval structure of set H (a member of set class {0, 1, 4, 5}). This reflection also results in its own symmetrical axis, axis value 2.5, since H is also symmetrical. This consolidation of axes corresponds with the axis of symmetry for the IFUNC array as a whole.

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16. Lewin, “Forte’s Interval Vector,” 201.

## EXAMPLE 1: Webern op. 6, no. 1, reduction, mm. 8-14

(Concert Pitch)

Vln. 1

Set G

Tpt.

Cor. Set I

Fag.

Set H

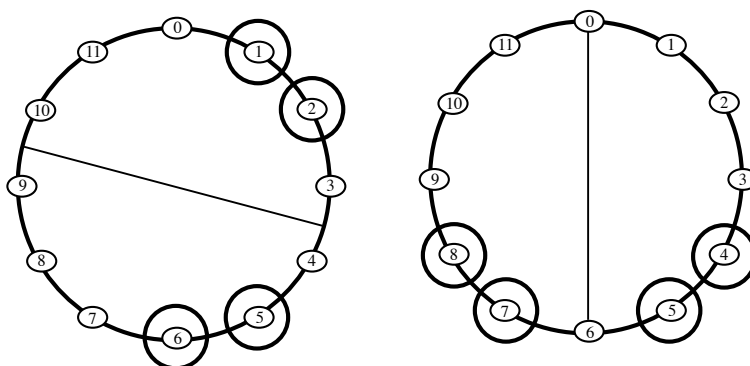
Cl.

Set C

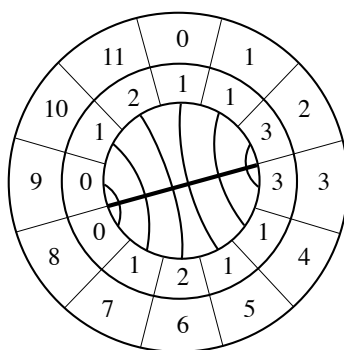
FIGURE 9: IFUNC (H, G) (i)

a)

$$H = \{6, 5, 2, 1\} \quad G = \{7, 4, 5, 8\}$$



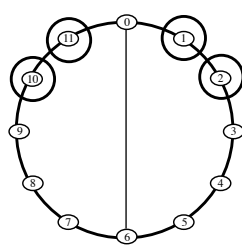
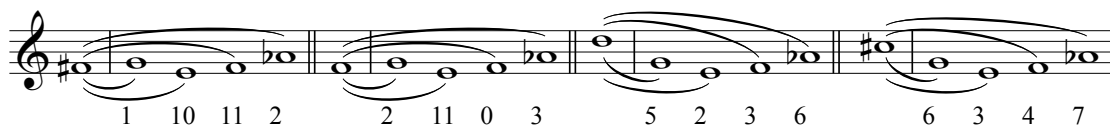
$$\text{IFUNC (H, G) (i) = (113311210012)}$$



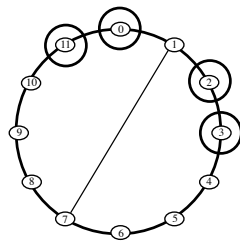
axis value = 2.5

FIGURE 9 cont.

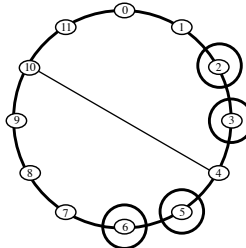
b)



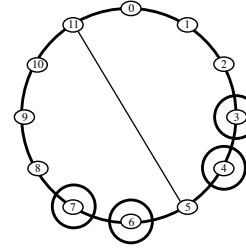
axis value = 0



axis value = 1

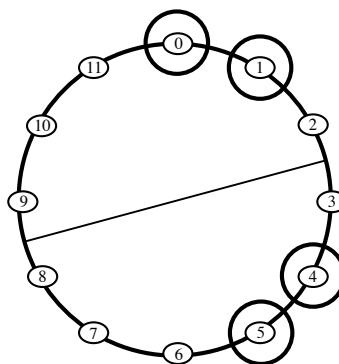


axis value = 4



axis value = 5

Consolidation of axes:



axis value of the axis values = 2.5

While the “deconstruction” process described above, which reveals various symmetries that contribute to an overall symmetrical IFUNC array to create a larger relation helps an intuitive understanding of symmetrical IFUNC arrays, the process is still not completely satisfying. At the beginning of *Four Pieces for Violin and Piano*, op. 7, no. 3 there are a few symmetrical pitch-class sets presented (See Example 2). In this piece I investigated the IFUNC arrays of (W, X), (X, V), (W, V) and (Y, Z). These sets

can be reduced in the same way as the op. 6 sets and their various axes plotted. When exploring the various axes created and their final consolidation, it seems plausible that the intervals found at the axis may be some average of all the intervals present between the two sets.

EXAMPLE 2: Webern op. 7, no. 3, mm. 1-8

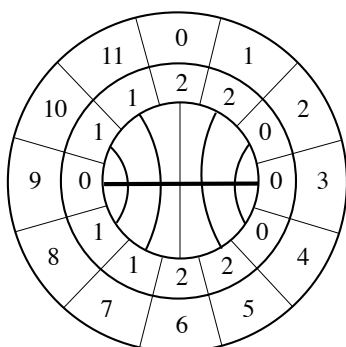
The image displays a musical score for Webern's op. 7, no. 3, measures 1-8. The score is divided into two systems. The first system (measures 1-8) features a piano part with a treble and bass clef. The tempo is marked "Sehr langsam" with a metronome marking of ca. 60. The performance instruction is "mit Dämpfer" (with damper). The dynamic is "ppp ohne cresc." (pianissimo without crescendo). The score includes several boxed sections labeled "Set W am Steg....." (measures 5-6), "Set X" (measures 7-8), "Set V" (measures 1-4), and "Set Y" (measures 7-8). A circled number "5" is placed above measure 5. The instruction "äußerst kurz!" (extremely short!) is written above measure 7. The second system (measures 9-16) features a violin part with a treble clef. The performance instruction is "col legno (weich gezogen)" (col legno, softly drawn). The dynamic is "ppp subito" (pianissimo subito). The score includes boxed sections labeled "Set Z" (measures 9-10), "Set Z3" (measures 11-12), and "Set Y" (measures 13-14). The score is marked with "s" (sforzando) and "p" (piano) dynamics.

I found it instructive to calculate the arithmetic mean (average) of the IFUNC array by multiplying each I by its IFUNC value, then dividing the sum of those products by the number of intervals (i) with a non-zero IFUNC value. The arithmetic mean is usually—

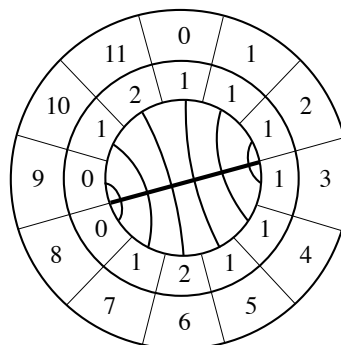
but not always—close to the axis value. Figure 10 shows three IFUNC pairs from op. 7 as well as the mean of each IFUNC array. In Figure 10a and 10b the mean does not correspond with the axis value while in Figure 10c it does. My proposition is that the axis intervals are the *median* intervals of each interval function in a way similar to Demske's registral centers of balance. The axis intervals are those intervals that are directly in the middle of all the intervals on the interval wheel. Such median intervals can be either a single interval or the two intervals that lie in the center. Like Demske's rcbs, the median intervals do not need to be present in the IFUNC array: In Figure 10a, for example, neither  $i = 3$  nor  $i = 9$  is an interval in the IFUNC, yet the median interval has an axis value of 3. Because these IFUNC arrays are symmetrical what occurs is an equal distribution of intervallic half-steps on either side of the median intervals. In other words,  $\text{IFUNC}(X, Y)(i + n)$  is equal to  $\text{IFUNC}(X, Y)(i - n)$  for all intervals  $i \pm n$  around the median intervals. Using the IFUNC axis deconstruction described in the previous musical example provides a more concise way of viewing the median intervals. Figure 11 shows the deconstruction of each of the IFUNC pairs of Figure 10. When the axes are consolidated onto the interval wheel the median intervals are expressed as the axis. In these examples, one of the median intervals can also be expressed as the mean average of the consolidated axes; this is also the case for the op. 6 example, but I have not found the mean average of axes to be the same as the axis for all IFUNC arrays.

FIGURE 10: Mean values

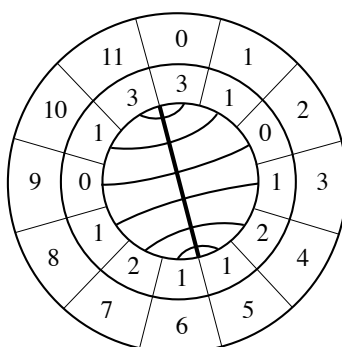
- a) IFUNC (W, X) (i) = (220002211011)    W = {4, 2, T, 9}    X = {8, T, 3}    Mean = 5



- b) IFUNC (X, V) (i) = (11111210012)    X = {8, T, 3}    V = {T, 9, 1, 2}    Mean = 5.5



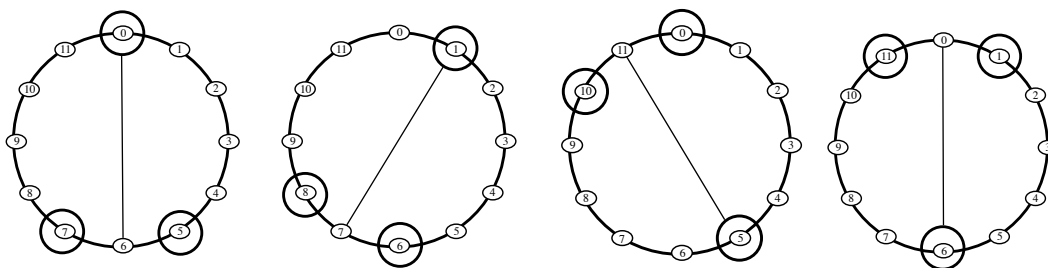
- c) IFUNC (W, V) (i) = (310121121013)    W = {3, 2, T, 9}    V = {T, 9, 1, 2}    Mean = 5.5



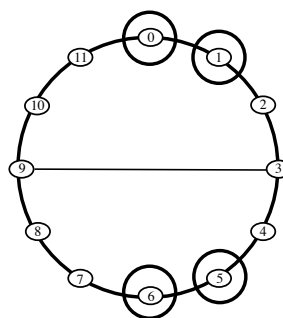
The argument for a median interval can even be applied to those IFUNC arrays that Lewin described as “interval-complement symmetrical” (see Review of Literature). In the case of interval-complement symmetry, the median interval is 0 with an equal distribution of intervallic half-steps on either side of 0, which also happen to be related by inversion. Furthermore, in theory, any two symmetrical sets whose axes of symmetry fall on a pitch-class pair, instead of between, can become interval-complement symmetrical if the right transpositions of each set are used.

FIGURE 11: IFUNC deconstruction

a) IFUNC deconstruction (W, X)  $W = \{4, 2, T, 9\}$   $X = \{8, T, 3\}$



Consolidation of axes:



b) IFUNC deconstruction (X, V)  $X = \{8, T, 3\}$   $V = \{T, 9, 1, 2\}$

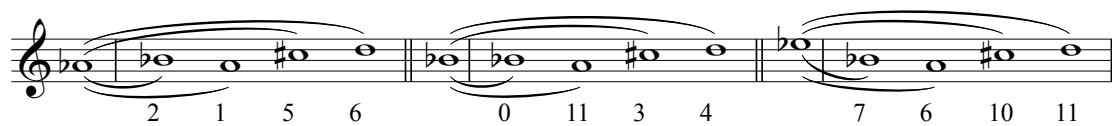
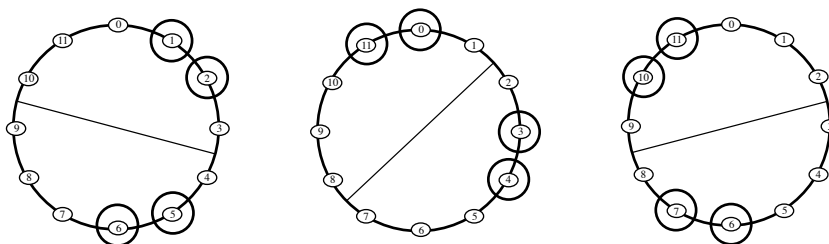
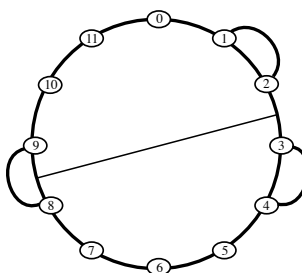
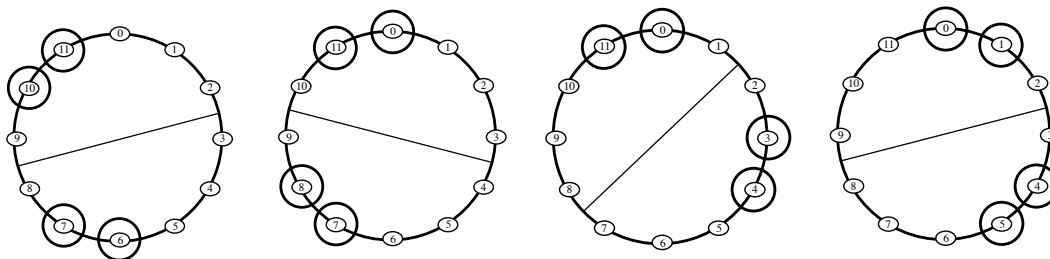


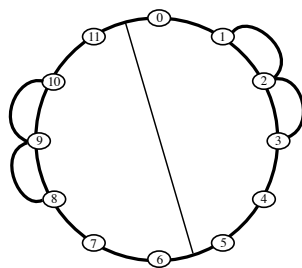
FIGURE 11 cont.



Consolidation of axes:

c) IFUNC deconstruction (W, V)  $W = \{3, 2, T, 9\}$   $V = \{T, 9, 1, 2\}$ 

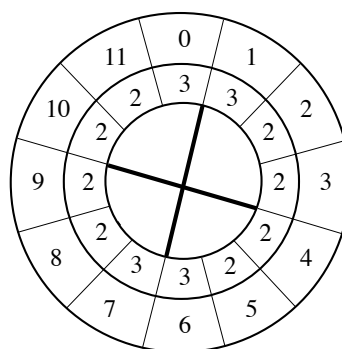
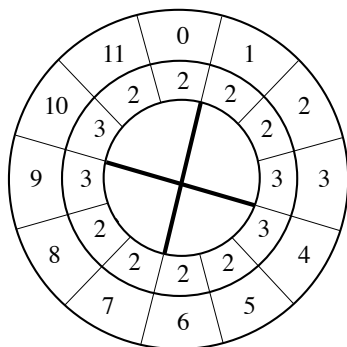
Consolidation of axes:



Op. 7 no. 3 also contains the multi-axis set (Z). Set Z and its transposition  $Z_3$  are combined with Y, an asymmetrical set in the IFUNC (Y, Z) and IFUNC (Y,  $Z_3$ ). Even though Y is asymmetrical, both IFUNC arrays are symmetrical around more than one axis (Figure 12). It is interesting that the IFUNC array for sets Y {E, 1, 2, 3, 4, 5, 6} and Z {2, 3, 8, 9}, a member of set class {0, 1, 6, 7}, is symmetrical even though Y itself is asymmetrical and Z has FOURPROP (1). Another feature present in the IFUNC (Y, Z), which appears frequently in the interval functions of op. 5 no. 4, are the tritone relationships across the interval wheel mentioned earlier. This feature will be explored in more detail in the next piece discussed.

FIGURE 12: Op. 7, multi-axis

- a) IFUNC (Y, Z) (i) = (222332222332)  
 Y = {E, 1, 2, 3, 4, 5, 6}    Z = {2, 3, 8, 9}
- b) IFUNC (Y,  $Z_3$ ) (i) = (332222332222)  
 Y = {E, 1, 2, 3, 4, 5, 6}     $Z_3 = \{E, 0, 5, 6\}$



Webern's *Five Movements for String Quartet*, op. 5, no. 4 also uses the set  $\{0, 1, 6, 7\}$  multiple times and in three different transpositions, labeled as sets Y, V and M. Each time  $\{0, 1, 6, 7\}$  is used in an IFUNC the array displays the tritone pair relationships found in the IFUNC (Y, Z) from op. 7 no. 3 discussed above, except for once in op. 5, the IFUNC (W, V).<sup>17</sup> These complementary relationships are present whether the other set is symmetrical or not and are the result of the built-in tritone relationships of set  $\{0, 1, 6, 7\}$ , which are also characteristic of all sets that contain two axes (refer to Figure 8). The IFUNCs of (Y, X), (Y, W) and (M, L) are presented in Figure 13. In the figure, the deconstruction for each member of Y and M has a pair that is transposed by a tritone. The IFUNC arrays for the sets in the Figure 13 do not contain an axis of symmetry, but do propose another IFUNC relationship that is similar to the constant-, alternating-, and ternary-IFUNCs discussed earlier. Like the equal divisions of the octave, the outcome of these sets divide the IFUNC array into two equal halves while alternating-IFUNCs divided it into six equal parts, and ternary-IFUNCs divide it into four equal parts. It happens that op. 5 no. 4 also contains a set that divides the IFUNC into three equal parts,  $\{0, 4, 8\}$ . The set  $\{0, 4, 8\}$  is shown in context of op. 5 in Figure 14.

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17. The outcome is a curious anomaly; this is the only time I have encountered an IFUNC that is *not* a transposed version of another IFUNC using the same prime forms. The set V is a transposed form of (0167) as is Y. The IFUNC (Y, W) results in the complementary relationships, while the IFUNC (W, V) and IFUNC (V, W) does not.

EXAMPLE 3: Webern op. 5, no. 4

a) mm. 1-9

Sehr langsam (♩ = ca 58) IV

zögernd im tempo

Set Z mit Dämpfer *ppp* am Steg... *ppp* Set Y *pizz.* *ppp* arco am Steg... *pp* *ppp*

mit Dämpfer *ppp* mit Dämpfer *ppp* Set X *ppp* Set W

5 Set U rit. *ppp* am Steg... *äußerst ruhig* *ppp* Set T

tempo so sanft als möglich *ppp* am Steg... rit. *verklingend* Set R

*pizz.* *ppp* *3* Set S *3* *3* *3* *3*

b) mm. 12-13

rit. tempo

am Steg... *ppp* *elächtig* Set L

*pizz.* *ppp* Set M *pizz.* *ppp*

FIGURE 13: Tritone pairs

a) IFUNC (Y, X) (i) = (110121110121)     $Y = \{6, E, 5, 0\}$      $X = \{3, 4, 6\}$

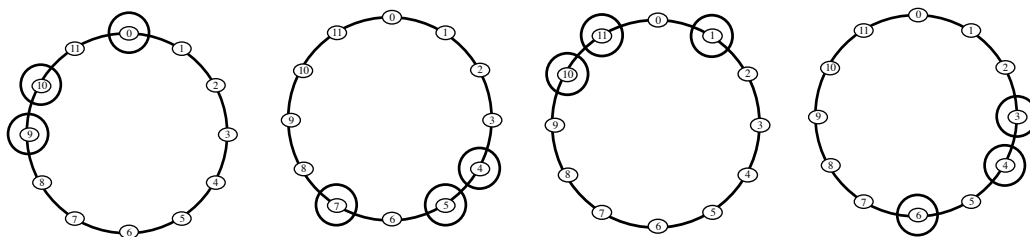
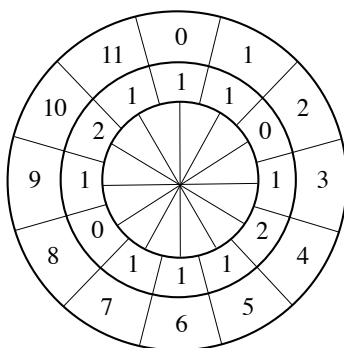


FIGURE 13 cont.

b) IFUNC (Y, W) = (132011132011)     $Y = \{6, 2, 5, 0\}$      $W = \{4, 6, 1, 7\}$

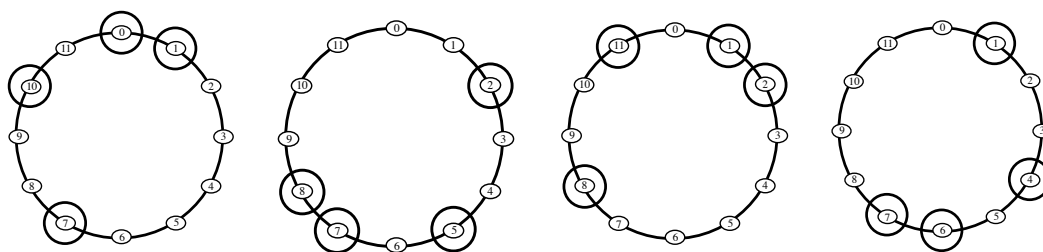
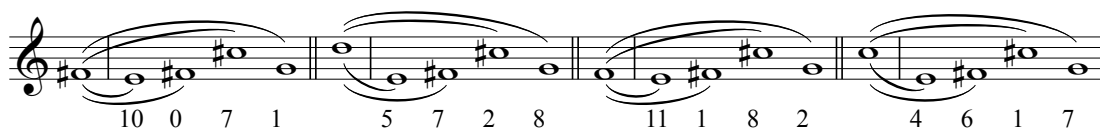
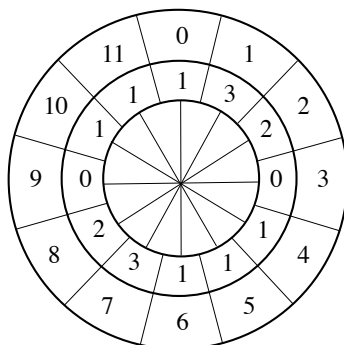
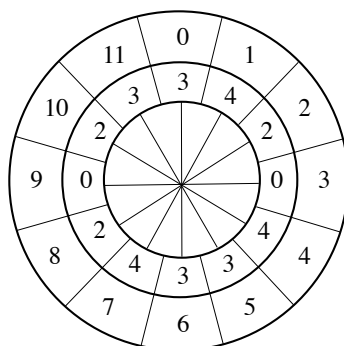
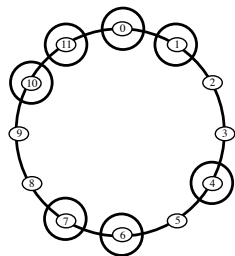


FIGURE 13 cont.

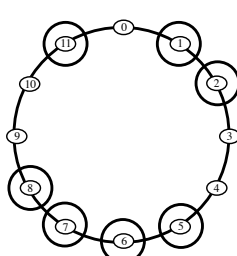
c) IFUNC (M, L) (i) = (342023342023)<sup>18</sup>    M = {8, 1, 7, 2}    L = {8, 0, 2, 7, 9, 3, 6}



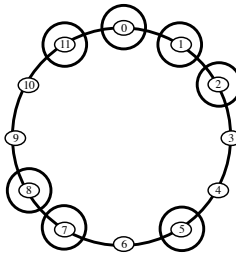
IFUNC (8, L)



IFUNC (1, L)



IFUNC (7, L)



IFUNC (2, L)

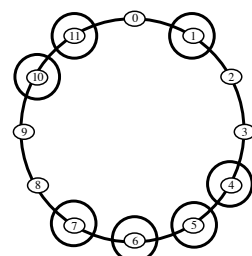
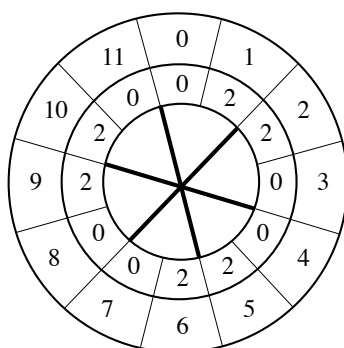


FIGURE 14: IFUNC (S, R) (i)

S = {2, T, 6}    R = {E, 8, 0, 3}

IFUNC (S, R) (i) = (022002200220)



18. To conserve space, the staff has been left out. Instead a short-hand label has been provided for reference.

The set  $\{0, 4, 8\}$  is responsible for such a division because it has three axes of symmetry (see Figure 8). There are only two other sets that have this same property:  $\{0, 1, 4, 5, 8, 9\}$  and  $\{0, 1, 2, 4, 5, 6, 8, 9, T\}$ . When  $\{0, 4, 8\}$  is combined with another symmetrical set, the resulting IFUNC features three axes of symmetry and a division of the IFUNC into equal thirds (Figure 14). Unfortunately, in combining the transposed set S found in the piece with an asymmetric set T, there was no IFUNC array symmetry nor any other feature similar to the tritone pairs found with the set  $\{0, 1, 6, 7\}$ .

Similar to the previous work discussed, op. 5 no. 4 also includes an asymmetrical set, U, that produces a symmetrical IFUNC array when combined with the symmetrical set Y, another member of set class  $\{0, 1, 6, 7\}$ . When confronted with the IFUNC array of (Y, U), the IFUNC array of (Y, Z) from op. 7 no. 3 and the second example by Lewin from earlier,<sup>19</sup> I intuited that there must be some common thread among the three pairs of sets in order for them to produce a symmetrical IFUNC array despite the asymmetry of one of the sets. The IFUNC arrays of op. 5 and op. 7 share an important feature: they both contain the set  $\{0, 1, 6, 7\}$ . As shown above, however, sets with the prime form  $\{0, 1, 6, 7\}$  do not necessarily produce symmetrical IFUNC arrays (see Figure 13), and Lewin's example does not use the set  $\{0, 1, 6, 7\}$ . The next logical step would be to seek a connection with the Fourier properties, as Lewin has suggested. Again, however, while the Lewin example *does* exclude FOURPROP (1), the set  $\{0, 1, 6, 7\}$  has that very property. I propose then that the asymmetrical sets used in these IFUNCs are related because they *imply* their own axis of symmetry. Looking at the examples from Op. 5 and from other works that Lewin investigates, and applying the technique of IFUNC

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19. Sets -  $\{0, 2, 4\}$  and  $\{0, 1, 3, 6\}$  - IFUNC = (121110101112)

deconstruction, we can understand an implied axis of symmetry for their respective IFUNC arrays (Figure 15). Of course this interpretation also poses a problem because the set  $\{0, 1, 3, 6\}$  used in the Lewin example is also used in op. 5 no. 4, set W, and when used in the IFUNC (Y, W) it does not yield a symmetrical result, only the tritone relationship.

FIGURE 15: Implied Axes

a) op. 5, no. 4: IFUNC (Y, U) (i) = (222101222101)     $Y = \{6, E, 5, 0\}$      $U = \{0, 5, 1, 2\}$

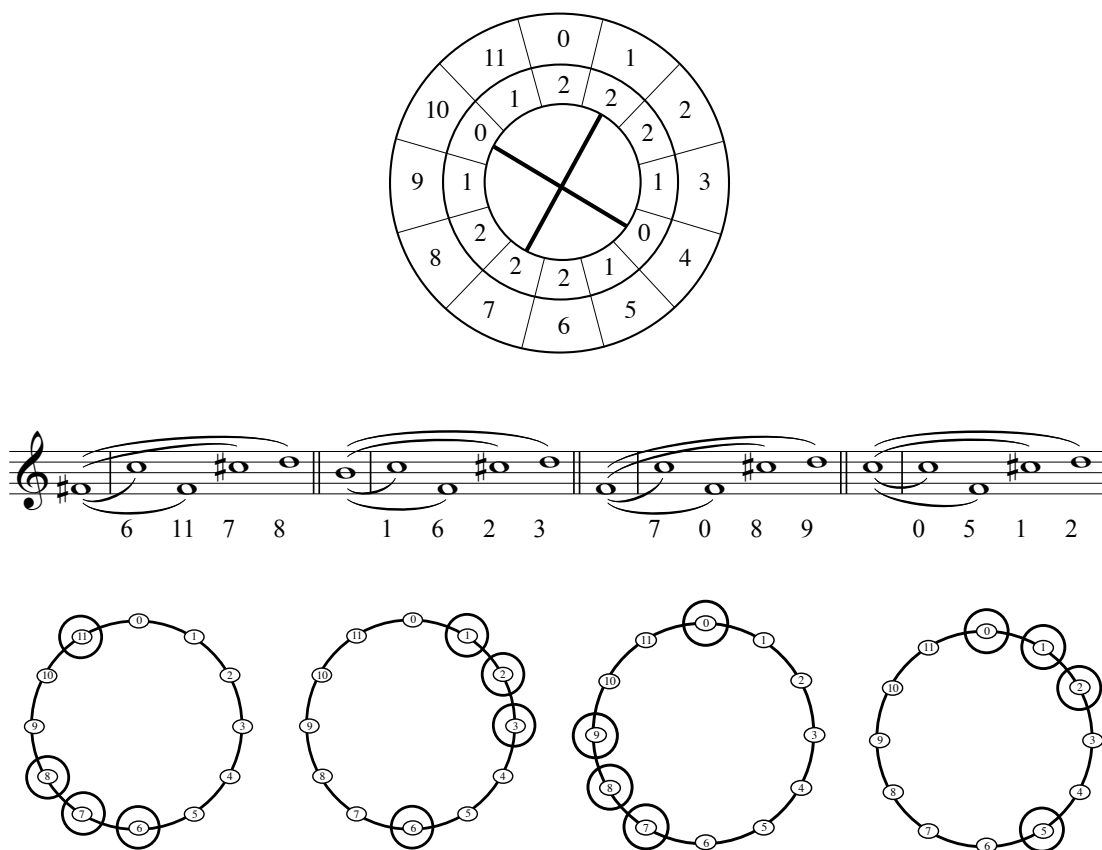
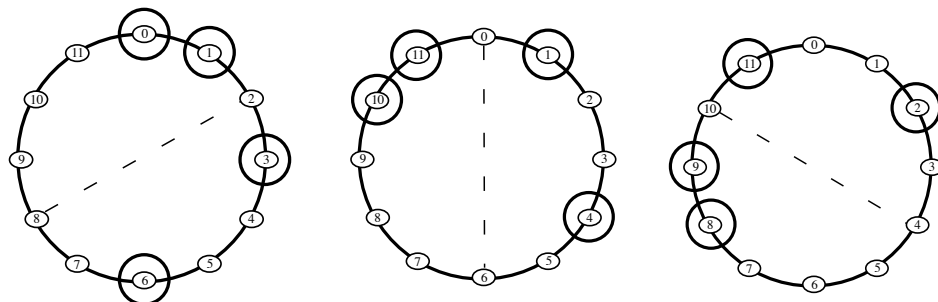
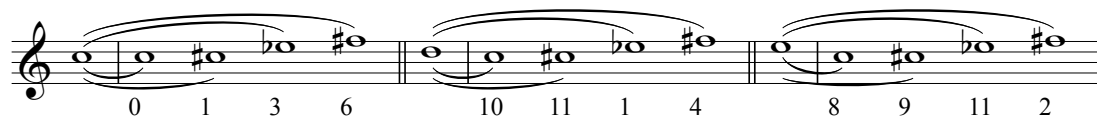
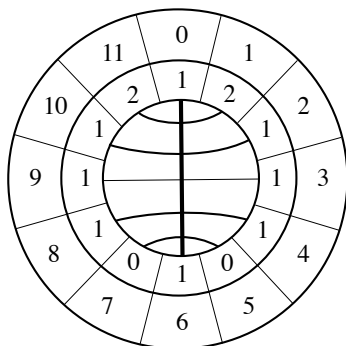
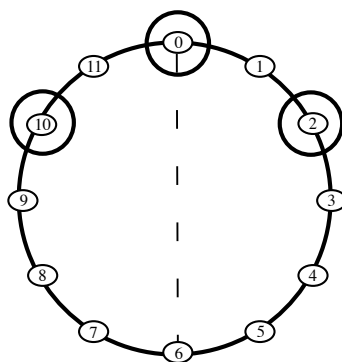


FIGURE 15 cont.

b) Lewin example: IFUNC  $(X, Y)$  (i) = (121110101112)  $X = \{0, 2, 4\}$   $Y = \{0, 1, 3, 6\}$



Consolidation of implied axes:



## CONCLUSION

After reviewing the work of David Lewin and being intrigued by his offhand comment that symmetrical IFUNC arrays are “mathematical suggestion,” I have developed a system for determining how and why a symmetrical IFUNC array results. While there are specific circumstances, such as the use of two symmetrical sets, that always result in a symmetrical array, some anomalies occur that pose their own questions. The main questions are what characteristics of the two pc sets cause the IFUNC array to be symmetrical, and are these characteristics something that can be modeled mathematically? I have tried to propose a solution by suggesting an implied axis, but this solution is not founded in consistent mathematical principles and still leads much to be desired and discovered for a proper solution. The problem of the Fourier properties, and their seeming lack of importance in the symmetrical IFUNC arrays is also an area open for further study. It seems that since similar arrays result from sets with different Fourier properties, there may still be another yet undiscovered property or properties that guide the symmetrical IFUNC array, especially in the case of IFUNCs containing an asymmetric set.

I have also proposed a “pertinent musical intuition” for the symmetrical IFUNC array through the idea of median intervals. Whether a listener truly perceives the median intervals intuitively, analogous to Demske’s registral centers of balance, is up for further debate. My goal was not so much to declare a hard and fast rule, but to propose a possible solution that seems to have some mathematical and musical merit. The next obvious step in validating the idea of median intervals would be to see if particular axes, and therefore certain median intervals, are more prevalent than others throughout a work.

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