An Analysis of the Henstock-Kurzweil Integral

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by

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§1 INTRODUCTION.

Since the inception of integration theory, mathematicians have sought to find an integral which would make integration and differentiation truly inverse processes.

Today, one of the first theories of integration that most mathematicians learn about is the Riemann integral. The Riemann integral, although very powerful is not without its flaws.

One of the first mathematicians to exploit the flaws in Riemann's integral was the French mathematician Johann Peter Gustav Lejeune Dirichlet. Dirichlet constructed the following bounded function which is not Riemann integrable

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function illustrates an inherent flaw in Riemann's integral by showing that it cannot integrate functions with too many discontinuities. Mathematicians began to notice other problems with the Riemann's integral as well. For example, they realized that the Riemann integral cannot integrate every derivative. Hence, as an operator, it is not a true inverse to the derivative. Moreover, in order to integrate functions over infinite intervals or functions with vertical asymptotes, one must introduce an improper Riemann integral. Subsequently, these flaws led mathematicians to see integration in a whole new light. Prior to this time, the mathematical community had rarely considered pathological functions such as Dirichlet's. As a result, the very foundation of analysis seemed to be on unsteady ground.

It was not until nearly one-hundred years later that a young mathematician named Henri Lebesgue developed a theory of integration that could handle most of the problems that plagued the Riemann integral. Lebesgue examined the effect of partitioning the range of a function rather than the domain thus allowing more control over small variations in the graph of the function. However, Lebesgue's theory was much more complicated than Riemann's and forced mathematicians to learn very deep mathematical concepts most of which were relatively unfamiliar and new to mathematics. After some time, as is their nature, mathematicians began finding flaws in Lebesgue's integral. First of all, like the Riemann integral, the Lebesgue integral is not a true inverse to differentiation. In addition, in the most general definition of the Lebesgue integral, the integrability of a function f is contingent upon the integrability of $|f|$. Although Lebesgue's integral would prove to be a valuable tool for the working mathematician, it was not the end of the story. Nearly twenty years after Lebesgue's death, two mathematicians simultaneously developed an integral that integrates any derivative making it a true inverse to differentiation. Furthermore, this new integral generalizes both the Riemann and Lebesgue integral. It is this theory that we discuss today.

Let $I = [a, b] \subseteq \mathbb{R}$ be a compact interval. Let $f : I \to \mathbb{R}$ be a real-valued function. We will investigate the Henstock-Kurzweil or "$\mathcal{HK}$" integral and compare it to the integrals of Riemann and Lebesgue. In the classic version of the Fundamental Theorem of Calculus, given a compact interval $I = [a, b] \subseteq \mathbb{R}$ and functions $f : I \to \mathbb{R}$ and $F : I \to \mathbb{R}$ with $F'(x) = f(x)$ for all $x \in I$, neither the Riemann, or "$\mathcal{R}$" integral nor the Lebesgue, or "$\mathcal{L}$" integrals guarantees that

$$\int_a^b f = F(b) - F(a).$$

However, the $\mathcal{HK}$ integral does guarantee this result. Furthermore, the $\mathcal{HK}$ integral is an non-absolute integral in the sense that a function f may be $\mathcal{HK}$ integrable without $|f|$ being $\mathcal{HK}$ integrable. In addition, the space $\mathcal{HK}(I)$ of Henstock-Kurzweil integrable functions defined on a compact interval $I = [a, b] \subseteq \mathbb{R}$ cannot be extended by adding on improper integrals in the sense that if a function f has an improper integral, then f is already $\mathcal{HK}$ integrable.

In this paper we look at some of these results. We begin by investigating specific properties of the $\mathcal{HK}$ integral which illustrate how a relatively small change in the definition of the $\mathcal{R}$ integral can have far reaching consequences. We follow this by defining the $\mathcal{HK}$ integral and looking at some of its most powerful results. Next, we examine some of the differences between the $\mathcal{HK}$ integral and the integrals of Riemann and Lebesgue. In these sections, we look at various functions which are $\mathcal{HK}$ integrable and yet are neither Riemann nor Lebesgue integrable. One such function resembles the Volterra function, but with a much simpler construction. In particular, we show that

$$\mathcal{R}(I) \subsetneqq \mathcal{L}(I) \subsetneqq \mathcal{HK}(I)$$

where $\mathcal{R}(I)$ and $\mathcal{L}(I)$ denote the space of Riemann and Lebesgue integrable functions defined on a compact interval $I \subseteq \mathbb{R}$ respectively. In addition, we look at the benefit of being a non-absolute integral in the sense that a function f can be Henstock-Kurzweil integrable without $|f|$ being Henstock-Kurzweil integrable. We exploit this property by showing that the $\mathcal{HK}$ integral can integrate any conditionally convergent series while the Lebesgue integral cannot. Finally, we discuss the consequences of using the $\mathcal{HK}$ integral in various areas of applied mathematics.

Contents

§2 BASIC DEFINITIONS.

We begin by introducing some preliminary definitions which serve to illustrate the prowess of the $\mathcal{HK}$ integral. Let $I = [a, b]$ be a compact interval in $\mathbb{R}$ with $a < b$.

A *partition* $\mathcal{P}$ is a finite collection of non-degenerate closed intervals $\{I_i\}_{i=1}^n$ whose union is I. Here, $I_i = [x_{i-1}, x_i]$, where

$$a = x_0 < x_1 < \ldots < x_n = b.$$

We call each closed interval $I_k = [x_{k-1}, x_k]$ a *subinterval of the partition* $\mathcal{P}$.

A *tagged partition* $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ is a finite set of ordered pairs where the closed intervals I_i form a partition of I and the numbers $t_i \in I_i$ are the corresponding *tags.*

The *mesh* $\|\mathcal{P}\|$ of a partition $\mathcal{P}$ is given by $\max\{\ell([x_{i-1}, x_i]) \ : \ i = 1, 2, \ldots, n\}$ where $\ell([x_{i-1}, x_i]) = x_i - x_{i-1}$.

Given a tagged partition $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ of I, and a function $f : I \to \mathbb{R}$, the real-number

$$\mathcal{S}(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)\Delta x_i,$$

where $\Delta x_i = x_i - x_{i-1}$, is the *Riemann Sum of f induced by* $\dot{\mathcal{P}}$.

Remark: As we will see, in our approach to integration we view the integral as a limit of Riemann sums. By using tagged partitions, we allow ourselves some flexibility with the placement of tags in that adding or eliminating points from a partition $\dot{\mathcal{P}}$, often the tags themselves, if done with care, will not effect the value of the Riemann sum.

For example, let $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition of I. Let t_k be an interior point of the subinterval $I_k = [x_{k-1}, x_k]$. Now, let $\dot{\mathcal{Q}}$ be the partition of I obtained from $\dot{\mathcal{P}}$ by adding the new partition point $\xi = t_k$, so that

$$a = x_0 < \ldots < x_{k-1} < \xi < x_k < \ldots < x_n = b.$$

We can now use ξ as the tag for both subintervals $[x_{k-1}, \xi]$ and $[x_k, \xi]$ of $\dot{\mathcal{Q}}$. Since $t_k = \xi$, and

$$f(t_k)(x_k - x_{k-1}) = f(\xi)(\xi - x_{k-1}) + f(\xi)(x_k - \xi),$$

$$\mathcal{S}(f;\dot{\mathcal{P}}) = \mathcal{S}(f;\dot{\mathcal{Q}}).$$

This process can also be reversed by merging two subintervals which share a tag ξ. In this case, the tag of the resulting subinterval will no longer be an endpoint. Thus, when using tagged partitions, we may assume any of the following:

- Every tag t_k is an endpoint of a subinterval I_k.

- No point t_k is a tag for two distinct subintervals I_k and I_{k+1}.

Let $f : I \to \mathbb{R}$ be a real-valued function. We say that f is *Riemann or $\mathcal{R}$ integrable on I* if there is a real number A such that for every $\epsilon > 0$ there is a number $\delta_\epsilon > 0$ such that if $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^{n}$ is any tagged partition of I with $\|\mathcal{P}\| < \delta_\epsilon$, then

$$\left|\mathcal{S}(f;\dot{\mathcal{P}}) - A\right| < \epsilon.$$

It can be shown that the number A is unique, and in this case,

$$A = \mathcal{R}\int_a^b f$$

or if it is clear that we are using a Riemann integral,

$$A = \int_a^b f.$$

Remark: If f is Riemann integrable on I then it is known that f is bounded on I. Using an alternate scheme, we can view the Riemann integral through the lens of the following terminology.

Let $f : A \to \mathbb{R}$ be a bounded real-valued function and A a non-empty subset of $\mathbb{R}$. We define the *oscillation* of f on A to be

$$\omega_f(A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x).$$

Let $f : I \to \mathbb{R}$ be a bounded real-valued function. Let $\mathcal{P}$ be a partition of I. Then define

$$M_j = \sup_{x\in[x_{j-1},x_j]} f(x)\ ,\ m_j = \inf_{x\in[x_{j-1},x_j]} f(x)\ ,$$

$$\mathcal{U}(\mathcal{P}; f) = \sum_{j=1}^{n} M_j \Delta x_j \text{ and } \mathcal{L}(\mathcal{P}; f) = \sum_{j=1}^{n} m_j \Delta x_j$$

where $\Delta x_j = (x_j - x_{j-1})$. The following criteria for Riemann integrability is useful: *f is Riemann integrable if and only if there exists a partition $\mathcal{P}$ of I such that for each $\epsilon > 0$,*

$$\mathcal{U}(\mathcal{P}; f) - \mathcal{L}(\mathcal{P}; f) < \epsilon.$$

We can rephrase the above criteria in terms of the oscillation since

$$\begin{aligned} \mathcal{U}(\mathcal{P}; f) - \mathcal{L}(\mathcal{P}; f) &= \sum_{j=1}^{n} M_j \Delta x_j - \sum_{j=1}^{n} m_j \Delta x_j \\ &= \sum_{j=1}^{n} (M_j - m_j) \Delta x_j \\ &= \sum_{j=1}^{n} \omega_f([x_{j-1}, x_j]) \Delta x_j. \end{aligned}$$

Let $A \subseteq \mathbb{R}$. The *outer measure of A* is defined to be

$$\inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \,\middle|\, A \subseteq \bigcup_{k=1}^{\infty} U_k \right\}$$

where $\{U_k\}_{k=1}^{\infty}$ is a collection of nonempty, open, bounded intervals.

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ a real-valued function.

We call f a *null function* if the set $\{x \in A : f(x) \neq 0\}$ has outer measure zero.

A subset A of $\mathbb{R}$ is said to be a *null set* if its outer measure is zero.

Let $f : I \to \mathbb{R}$. The *variation of f over I* is given by

$$Var(f; I) = \sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \,\middle|\, \mathcal{P} = \{x_0, x_1, \dots, x_n\} \text{ is a partition of } I \right\}.$$

If $Var(f; I) < \infty$, we say that f is of *bounded variation on I*. We denote the collection of functions on I that are of bounded variation by $\mathcal{BV}(I)$.

Let $I = [a, b] \subseteq \mathbb{R}$ be a compact interval. Let $f : I \to \mathbb{R}$ be an $\mathcal{HK}$ integrable function (see section 4.) We say that f is *absolutely integrable on* I if $|f|$ is also integrable on I. A function that is integrable on I but not absolutely integrable on I is said to be *non-absolutely integrable on* I.

§3 GAUGES.

We now discuss different methods of measuring the fineness of partitions. In the traditional approach to the Riemann integral, one measures the fineness of a partition $\dot{\mathcal{P}}$ with regard to the mesh $\|\dot{\mathcal{P}}\|$ of $\dot{\mathcal{P}}$ in the sense that the length of each subinterval I_k of $\dot{\mathcal{P}}$ must be no greater than the length of $\|\dot{\mathcal{P}}\|$. In Lebesgue theory, the concept of fineness is developed from the theory of outer measure. In order to develop the Henstock-Kurzweil integral, we introduce the following terminology. Let $I = [a, b]$ be a compact interval in $\mathbb{R}$.

(3.1) Definition. A function $\delta : I \to (0, \infty)$ is called a *gauge*. For each $t \in I$ the interval around t *controlled by the gauge* δ is the interval

$$B(t; \delta(t)) = (t - \delta(t), t + \delta(t)).$$

(3.2) Definition. Let $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^{n}$ be a tagged partition of I. Given a gauge δ on I, we say that $\dot{\mathcal{P}}$ is *δ-fine* if

$$I_i \subseteq (t_i - \delta(t_i), t_i + \delta(t_i)) \text{ for all } i = 1, 2, \ldots, n.$$

If a tagged partition $\dot{\mathcal{P}}$ is δ-fine we say that $\dot{\mathcal{P}}$ *is subordinate to* δ.

(3.3) Definition. The Riemann integral relies on what is known as a *constant gauge.* Let $I = [a, b]$ be a compact interval, and let δ_0 be any positive real number. The function $\delta : I \to (0, \infty)$ on I given by

$$\delta(t) = \delta_0 \text{ for all } t \in I$$

is a constant gauge on I. Any tagged partition $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^{n}$ on I will be subordinate to δ if and only if

$$I_i \subseteq (t_i - \delta_0, t_i + \delta_0) = B(t_i; \delta_0) \text{ for all } i = 1, 2, \ldots, n.$$

One of the primary benefits of using gauges lies in their ability to force a given point to be a tag for a partition $\dot{\mathcal{P}}$. For example, let $I = [a,b] \subseteq \mathbb{R}$ and let $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition of I. Let $\delta : I \to (0,\infty)$ be given by

$$\delta(t) = \begin{cases} \frac{1}{4} & \text{if } t = a \\ \\ \frac{1}{2}|t-a| & \text{if } a < t \leq b. \end{cases}$$

Then, for any δ-fine partition $\dot{\mathcal{P}} = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$ the tag t_1 for $[a, x_1]$ is forced to be $t_1 = a$ as we now show. To see this, since $\dot{\mathcal{P}}$ is δ-fine, we must have that

$$[a, x_1] \subseteq (t_1 - \delta(t_1), t_1 + \delta(t_1)).$$

Hence,

$$t_1 - \delta(t_1) < a,$$

and this would imply $t_1 - \frac{1}{2}|t_1 - a| < a$ or equivalently $t_1 - a < \frac{1}{2}(t_1 - a)$ if $a < t_1 \leq b$, which is a contradiction. Thus, a must be the tag for $[a, x_1]$.

From this a very natural question arises. Given a compact interval I and a gauge $\delta : I \to (0,\infty)$, is it always possible to find a tagged partition $\dot{\mathcal{P}}$ which is δ-fine? The answer is yes, as given by the following theorem.

(3.4) Cousin's Theorem. *If $I = [a,b] \subseteq \mathbb{R}$ is a nondegenerate compact interval, and δ is a gauge on I, then there exists a tagged partition of $[a,b]$ that is δ-fine.*

For a proof of Cousin's Theorem, we refer the reader to [1]. Thus, given any compact interval $[a,b] \subseteq \mathbb{R}$, and a gauge δ on $[a,b]$, a δ-fine partition of $[a,b]$ always exists. Now that we have some preliminary definitions and results out of the way, we define the "Henstock-Kurzweil" integral.

§4 THE $\mathcal{HK}$ INTEGRAL.

We begin by defining the $\mathcal{HK}$ integral. Let $I = [a,b] \subseteq \mathbb{R}$ be a compact interval.

(4.1) Definition. Let $f : I \to \mathbb{R}$ be a function. We say that f is *Henstock-Kurzweil, $\mathcal{HK}$, Gauge, or Generalized Riemann Integrable on I* if there is some real number A such that for every $\epsilon > 0$ there is a gauge

$\delta_\epsilon : I \to (0, \infty)$ such that if $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ is any tagged partition of I that is δ_ϵ-fine, then

$$\left|\mathcal{S}(f; \dot{\mathcal{P}}) - A\right| < \epsilon.$$

As with the Riemann integral, one can show that A is unique. In this case,

$$A = \mathcal{HK} \int_a^b f$$

or if it is clear that we are using an $\mathcal{HK}$ integral,

$$A = \int_a^b f.$$

In some cases, we can use an appropriately defined constant gauge δ_0 and construct our partitions so that the length of the mesh is always smaller than δ_0. In this way, it is easy to see that the $\mathcal{R}$ integral is a special case of the $\mathcal{HK}$ integral. Indeed, in the definition of the $\mathcal{R}$ integral we defined our limit in terms of the mesh of a partition. However, after some reconsideration, we actually see that the mesh $\|\dot{\mathcal{P}}\|$ of a partition $\dot{\mathcal{P}}$ is just a constant gauge. In fact, by using gauges, we are making only a small change to the classic definition of the Riemann integral. However, this small change will prove to have large consequences. Intuitively, whether using the Riemann approach or the Henstock-Kurzweil approach, we are using the fineness of the subintervals of the partition over which the function is defined to approximate the integral. The Riemann approach measures that fineness by taking the mesh of the partition. So, the lengths of subintervals are all less than or equal to some real number. By using a gauge, we allow more variation in the lengths of those subintervals. In this way, we get a better approximation of the area beneath a curve via Riemann sums. For intervals upon which the curve is changing rapidly, we use subintervals with sufficiently small length and for intervals where the change is slow, we use subintervals of larger length. This turns out to be one of the key advantages of the $\mathcal{HK}$ integral. In addition, a gauge allows us to enclose a countable set of points $\mathbb{A}$ within a collection of subintervals $\{J_k\}_{j=1}^\infty$ so that $\bigcup_{k=1}^\infty J_k$ has very small outer measure thus nullifying the effect that $\mathbb{A}$ has on the Riemann sums. This is a handy tool when dealing with discontinuities. Furthermore, as we have already shown, a gauge can force certain "problem" points to be tags. This allows us to deal with asymptotes. Finally, as we will see, the use of gauges will give an improved version of the "Fundamental Theorem of Calculus." We now look at some of the important results tied to the $\mathcal{HK}$ integral.

§5 PROPERTIES OF THE $\mathcal{HK}$ INTEGRAL.

(5.1) Theorem. Let $I = [a, b]$ be a compact interval in $\mathbb{R}$. Let $\mathbb{A} \subseteq I$ be a set of outer measure zero. Let φ be defined as follows:

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{A} \\ 0 & \text{if } x \in I \setminus \mathbb{A}. \end{cases}$$

Then, $\varphi \in \mathcal{HK}(I)$ and

$$\int_a^b \phi = 0.$$

Proof: Let $\epsilon > 0$ be given. Let $\{J_k\}_{k=1}^{\infty}$ be a countable collection of open intervals such that

$$\mathbb{A} \subseteq \bigcup_{k=1}^{\infty} J_k \text{ and } \sum_{k=1}^{\infty} \ell(J_k) < \epsilon.$$

We define a gauge on I as follows. If $t \in I \setminus \mathbb{A}$, put $\delta_\epsilon(t) = 1$. If $t \in \mathbb{A}$, let $k(t)$ be the smallest index k such that $t \in J_k$ and choose $\delta_\epsilon(t) > 0$ such that $(t - \delta_\epsilon(t), t + \delta_\epsilon(t)) \subseteq J_{k(t)}$.

Now, let $\dot{\mathcal{P}} = \{(I_i, t)\}_{i=1}^{n}$ be a δ_ϵ-fine partition of I. If $t \in I \setminus \mathbb{A}$, then $\varphi(t) = 0$ and the Riemann sum $\sum_{t_i \in I \setminus \mathbb{A}} \varphi(t_i)\Delta x_i = 0$. Since, $\dot{\mathcal{P}}$ is δ_ϵ-fine, if $t \in \mathbb{A}$,

$$J_{k(t_i)} \supseteq (t - \delta_\epsilon(t), t + \delta_\epsilon(t)) \supset I_i.$$

So, for each $k \in \mathbb{N}$, the (nonoverlapping) intervals I_i with tags in $\mathbb{A} \cap J_k$ have total length less than or equal to $\ell(J_k)$ by countable subadditivity. So for each $k \in \mathbb{N}$, the Riemann sum $\sum_{t_i \in I_k} \varphi(t_i)\Delta x_i \leq \ell(J_k)$. Therefore,

$$0 \leq \sum_{t_i \in I \setminus \mathbb{A}} \varphi(t_i)\Delta x_i + \sum_{k=1}^{\infty} \sum_{t_i \in I_k} \varphi(t_i)\Delta x_i \leq \ell(J_k) \leq \sum_{k=1}^{\infty} \ell(J_k) < \epsilon.$$

Thus, $\varphi \in \mathcal{HK}(I)$ and

$$\int_a^b \phi = 0.$$

□

The next result, the proof of which can be found in [1] reveals one of the primary differences between the $\mathcal{HK}$ integral and the Lebesgue integral.

(5.2) Theorem. Let $\sum_{k=1}^{\infty} a_k$ be any convergent series. Say $\sum_{k=1}^{\infty} a_k = A$. Let us define a sequence $\{c_n\}_{n=0}^{\infty}$ as $c_n = 1 - \frac{1}{2^n}$. Now, let $h : [0,1] \to \mathbb{R}$ be defined as follows:

$$h(x) = \begin{cases} 2^k a_k & \text{if } x \in [c_{k-1}, c_k) \ k \in \mathbb{N} \\ 0 & \text{if } x = 1 \end{cases}$$

In this case, $h \in \mathcal{HK}([0,1])$ and

$$\int_0^1 h = A = \sum_{k=1}^{\infty} a_k.$$

Another impressive result which comes from the $\mathcal{HK}$ integral is the so called "Hake's Theorem." One can find the proof of this in [1]

(5.3) Hake's Theorem. A function $f \in \mathcal{HK}([a,b])$ if and only if $f \in \mathcal{HK}([a,c])$ for every $c \in [a,b)$ and $\lim\limits_{c \to b^-} \int_a^c f < \infty$. In this case,

$$\int_a^b f = \lim_{c \to b^-} \int_a^c f.$$

Hake's Theorem tells us that the space $\mathcal{HK}(I)$ cannot be extended by adjoining "improper integrals." If such an integral exists, then the integrand must already belong to $\mathcal{HK}(I)$. This is not so for Riemann and Lebesgue integrals. We now examine one of the most powerful aspects of the $\mathcal{HK}$ integral - the Fundamental Theorem of Calculus.

§6 THE FUNDAMENTAL THEOREM OF CALCULUS.

Unlike the $\mathcal{R}$ and $\mathcal{L}$ integrals, the $\mathcal{HK}$ integral can integrate every derivative. In order to fully formulate the Fundamental Theorem of Calculus for $\mathcal{HK}$ integrals, we first give some preliminary definitions. Let $I = [a,b] \subseteq \mathbb{R}$ be a compact interval.

(6.1) Definition. Let $F, f : I \to \mathbb{R}$. We say that F is a *Primitive of f on I* if $F'(x)$ exists and $F'(x) = f(x)$ for all $x \in I$.

(6.2) Definition. Let $F, f : I \to \mathbb{R}$. We say that F is an *a-primitive, c-primitive or f-primitive of f on I* if F is continuous on I and there is a null, countable or finite set, respectively, $E \subseteq I$ so that $F' = f$ on $I \setminus E$. We call the set E the *exceptional set* for f.

(6.3) Definition. Suppose that $f \in \mathcal{HK}(I)$ and let $u \in I$. The function $F_u : I \to \mathbb{R}$ given by

$$F_u(x) = \int_u^x f$$

is called an *indefinite integral of f with base point u*. If the base point is the left endpoint of I or is well understood, then we may omit the subscript. Any function that differs from F_u by a constant is called an *indefinite integral of f*.

We can now begin our discussion of the Fundamental Theorem of Calculus for $\mathcal{HK}$ integrals. To understand why we get an improved version of the Fundamental Theorem, we must first consider the classic definition of the derivative.

(6.4) Definition. Given a differentiable function $F : [a, b] \to \mathbb{R}$, we say that F is *differentiable at $t \in [a, b]$*, with derivative $f(t)$, if for all $\epsilon > 0$ there is a $\delta_\epsilon(t) > 0$ such that if $0 < |x - t| < \delta_\epsilon(t)$, where $x \in [a, b]$, then

$$\left| \frac{F(x) - F(t)}{x - t} - f(t) \right| < \epsilon.$$

Notice that the number $\delta_\epsilon(t)$ can just as easily be thought of as a function of t and ϵ. Thus, if a function F is differentiable at a point $t \in [a, b]$, there is a built-in gauge. It is this very idea that lies at the heart of the following result.

(6.5) The Fundamental Theorem of Calculus I. Let $I = [a, b] \subseteq \mathbb{R}$ be a compact interval. Let $f : I \to \mathbb{R}$ be a function. If f has a c-primitive F on I, then $f \in \mathcal{HK}(I)$ and

$$\int_a^b f = F(b) - F(a). \tag{1}$$

Notice that in the hypothesis of the Fundamental Theorem of Calculus for the $\mathcal{HK}$ integral we need not assume that $f \in \mathcal{HK}(I)$. This is not the case for the $\mathcal{R}$ integral, and for the Lebesgue integral additional constraints are necessary to make equation (1) valid. Consequently, this version of the Fundamental Theorem of Calculus makes differentiation and integration truly inverse processes. Furthermore, we have the following version of the Fundamental Theorem of Calculus II.

(6.6) The Fundamental Theorem of Calculus II. If $f \in \mathcal{HK}(I)$ then any indefinite integral F is continuous on I and is an a-primitive of f on I. Thus, there exists a null set $\mathbb{A} \subseteq I$ such that

$$F'(x) = f(x) \text{ for all } x \in I \setminus \mathbb{A}.$$

For the proofs of the above theorems, we refer the reader to [1].

Remark: A primary feature of the $\mathcal{HK}$ integral is its relationship to the c-primitive. The following statements are meant to add clarity to the subtle distinction between c-primitives and indefinite integrals.

- An $\mathcal{HK}$ integrable function f always has an indefinite integral and every indefinite integral of a function in $\mathcal{HK}(I)$ is an a-primitive.
- An $\mathcal{HK}$ integrable function does not always have a c-primitive.
- If F is a c-primitive of $f : I \to \mathbb{R}$, then $f \in \mathcal{HK}(I)$ and F is an indefinite integral of f.
- If F is an a-primitive of $f \in \mathcal{HK}(I)$, then F need not be an indefinite integral of f.

Examples or counterexamples for each of these assertions can be found in [1].

Now that we have discussed some preliminary results, we consider the effect of these results. We first show that $\mathcal{R}(I) \subsetneqq \mathcal{HK}(I)$.

§7 FUNCTIONS WHICH ARE $\mathcal{HK}$ INTEGRABLE BUT NOT $\mathcal{R}$ INTEGRABLE.

The very first integral that most mathematicians learn about is Riemann's. While the importance of the Riemann integral to a general theory of integration cannot be overemphasized, it is not without its drawbacks. In this section we examine some of those drawbacks. We first show the power of the $\mathcal{HK}$ version of the Fundamental Theorem of Calculus.

(7.1) Example. Consider the function

$$f(x) = \begin{cases} x^{-\frac{1}{4}} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

Since f is not bounded, it fails to be Riemann integrable. Let $F : [0,1] \to \mathbb{R}$ be given by $F(x) = \frac{4}{3}x^{\frac{3}{4}}$. Then, F is continuous on $[0,1]$ and $F'(x) = f(x)$ for all $x \in (0,1]$ but $F'(0)$ does not exist. Thus, F is an f-primitive for f on $[0,1]$ with exceptional set $E = \{0\}$. Therefore, by the Fundamental Theorem of Calculus I for the $\mathcal{HK}$ integral (6.5),

$$\int_0^1 f(x)\, dx = F(1) - F(0) = \frac{4}{3}.$$

In practice, we write

$$\int_0^1 x^{-\frac{1}{4}}\, dx$$

with the understanding that the integrand is zero at $x = 0$.

In any undergraduate real analysis course, students learn that the Riemann integral can handle any function with a finite number of discontinuities. This idea is later extended by saying that the set of discontinuities of a Riemann integrable function must have measure zero. Using this idea, we can create bounded functions which are not Riemann integrable. To show that such functions are actually $\mathcal{HK}$ integrable we construct a gauge which will enclose the points of discontinuity in a set with small outer measure making the contribution of said discontinuities to the Riemann sums negligible. To illustrate the full power of this technique, we will now construct such a function.

(7.2) Example. Let $\mathbb{A}$ be a countable, dense subset of $[0,1]$. Examples include, but are not limited to, the algebraic numbers and the rational numbers. We claim that the following function is $\mathcal{HK}$ integrable but not $\mathcal{R}$ integrable.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{A} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{A} \end{cases}$$

Proof: We begin by showing that f is not $\mathcal{R}$ integrable. To this end, let $\epsilon = \frac{1}{2}$, and let $\mathcal{P} = \{I_i\}_{i=1}^n$ be a partition of $[0,1]$. Since $\mathbb{A}$ and $[0,1] \setminus \mathbb{A}$ are both dense in $[0,1]$, for each subinterval I_i induced by $\mathcal{P}$, I_i contains both an element of $\mathbb{A}$ and an element of $[0,1] \setminus \mathbb{A}$. So, $M_i = 1$ and $m_i = 0$ for $i = 1, 2, \ldots, n$. Clearly, $\mathcal{L}(\mathcal{P}; f) = 0$, and

$$\begin{aligned}
\mathcal{U}(\mathcal{P}; f) &= \sum_{j=1}^{n} M_i \Delta x_j \\
&= \sum_{j=1}^{n} 1 \Delta x_j \\
&= \sum_{j=1}^{n} \Delta x_j \\
&= (1-0) \\
&= 1.
\end{aligned}$$

Thus,

$$\mathcal{U}(\mathcal{P}; f) - \mathcal{L}(\mathcal{P}; f) = 1 - 0 = 1.$$

Since $\mathcal{P}$ was arbitrary, we see that

$$\mathcal{U}(\mathcal{P}; f) - \mathcal{L}(\mathcal{P}; f) \geq \epsilon = \frac{1}{2}$$

for all partitions $\mathcal{P}$. Therefore, f is not Riemann integrable on $[0,1]$.

It is clear, by Theorem (5.1), that $f \in \mathcal{HK}([0,1])$. We will now examine this assertion in further detail by using an explicit gauge. Since $\mathbb{A}$ is a countable set, it is enumerable. Let $\mathbb{A} = \{r_1, r_2, \ldots\}$ be an enumeration of $\mathbb{A}$. Let $\epsilon > 0$ be given. Consider the following gauge on $[0,1]$.

$$\delta_\epsilon(t) = \begin{cases} 1 & \text{if } t \in [0,1] \setminus \mathbb{A} \\ \frac{\epsilon}{2^{k+1}} & \text{if } t = r_k \in \mathbb{A}. \end{cases}$$

Let $\dot{\mathcal{P}}$ be a δ_ϵ-fine, tagged partition of $[0,1]$. Since $f = 0$ on $[0,1] \setminus \mathbb{A}$, the tags in $[0,1] \setminus \mathbb{A}$ contribute nothing to the Riemann sum of f induced by $\dot{\mathcal{P}}$. So, since $\dot{\mathcal{P}}$ is δ_ϵ-fine, we need only to consider the following sum:

$$\left| \mathcal{S}(f; \dot{\mathcal{P}}) \right| \leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

Therefore, $f \in \mathcal{HK}([0,1])$ and

$$\int_0^1 f = 0.$$

$\square$

In the above example, we constructed a gauge which allowed us to nullify the effect that the points of $\mathbb{A}$ had on the Riemann sums. By doing this, we have effectively made f identically zero on $[0,1]$. However, this is not surprising since, alternatively, we could have shown that $f \in \mathcal{HK}([0,1])$ using Theorem (6.5) and considering the function $F(x) = 0$ for all $x \in [0,1]$. Notice that F is a c-primitive for f on $[0,1]$ with exceptional set $\mathbb{A}$. Hence, by the $\mathcal{HK}$ version of the Fundamental Theorem of Calculus,

$$\int_0^1 f(x)dx = F(0) - F(1) = 0.$$

We have now shown that $\mathcal{R}(I) \subsetneqq \mathcal{HK}(I)$. However, the above functions, or variations therein, are so well known that in some sense they are trivial examples. We now construct one final example which will show the power of the Fundamental Theorem of Calculus for $\mathcal{HK}$ integrals. The following is a bounded function which is the derivative of another function, yet which is not Riemann integrable. The initial construction of this function was suggested by Goffmann.[2]

(7.3) Example.

We now construct an oscillating function on a set of positive outer measure by using a method similar in spirit to the construction of the classic Volterra function.

We begin with the unit interval, which we denote by $K_1 = [0,1]$. Consider $x_1 = \frac{1}{2}$, the midpoint of K_1. First, we delete the interval $G_1 = I_1 = (\frac{3}{8}, \frac{5}{8})$ from K_1. Note that I_1 is symmetric about x_1 and $\ell(I_1) = \frac{1}{4}$. We are now left with two closed intervals, $K_2 = [0, \frac{3}{8}]$ and $K_3 = [\frac{5}{8}, 1]$. Let

$$L_1 = K_2 \cup K_3.$$

Let $x_2 = \frac{3}{16}$ and $x_3 = \frac{13}{16}$, the midpoints of K_2 and K_3 respectively. We now delete the open intervals $I_2 = (\frac{5}{32}, \frac{7}{32})$ and $I_3 = (\frac{25}{32}, \frac{27}{32})$, which are symmetric about x_2 and x_3, from $K_2 = [0, \frac{3}{8}]$ and $K_3 = [\frac{5}{8}, 1]$ respectively. Note that

$$\ell(I_k) = \frac{1}{16} = [\ell(I_1)]^2 \text{ for } k = 2, 3.$$

Let

$$G_2 = I_2 \cup I_3 = \left(\frac{5}{32}, \frac{7}{32}\right) \cup \left(\frac{25}{32}, \frac{27}{32}\right).$$

This leaves us with the four closed intervals

$$K_4 = \left[0, \frac{5}{32}\right], \; K_5 = \left[\frac{7}{32}, \frac{3}{8}\right], \; K_6 = \left[\frac{5}{8}, \frac{25}{32}\right] \text{ and } K_7 = \left[\frac{27}{32}, 1\right].$$

Let

$$L_2 = K_4 \cup K_5 \cup K_6 \cup K_7.$$

Let $x_4 = \frac{5}{64}, \; x_5 = \frac{19}{64}, \; x_6 = \frac{45}{64}$ and $x_7 = \frac{59}{64}$ which are the midpoints of $K_4, \; K_5, \; K_6,$ and K_7 respectively.

Next, we delete the open intervals

$$I_4 = \left(\frac{9}{128}, \frac{11}{128}\right), \; I_5 = \left(\frac{37}{128}, \frac{39}{128}\right), \; I_6 = \left(\frac{89}{128}, \frac{91}{128}\right) \text{ and } I_7 = \left(\frac{117}{128}, \frac{119}{128}\right)$$

from $K_4 = \left[0, \frac{5}{32}\right], \; K_5 = \left[\frac{7}{32}, \frac{3}{8}\right], \; K_6 = \left[\frac{5}{8}, \frac{25}{32}\right]$ and $K_7 = \left[\frac{27}{32}, 1\right]$ respectively. Note that

$$\ell(I_k) = \frac{1}{64} = [\ell(I_0)]^3 \text{ for } k = 4, 5, 6, 7.$$

Let

$$G_3 = I_4 \cup I_5 \cup I_6 \cup I_7 = \left(\frac{9}{128}, \frac{11}{128}\right) \cup \left(\frac{37}{128}, \frac{39}{128}\right) \cup \left(\frac{89}{128}, \frac{91}{128}\right) \cup \left(\frac{117}{128}, \frac{119}{128}\right).$$

If we continue this process inductively, we obtain the following construction:

- $G_1 = I_1$.
- $G_n = I_{2^{n-1}} \cup I_{2^{n-1}+1} \cup \ldots \cup I_{2^{n-1}+(2^{n-1}-1)}$ for $n \geq 2$, where $n \in \mathbb{N}$.
- $\ell(I_k) = [\ell(I_1)]^n = \left(\frac{1}{4}\right)^n$ where $k = 2^{n-1}, 2^{n-1} + 1 \ldots, 2^{n-1} + (2^{n-1} - 1)$ for each $n \in \mathbb{N}$.
- $I_j \cap I_k = \emptyset$ for all $j \neq k$
- $G_j \cap G_k = \emptyset$ for all $j \neq k$.

For each $n \in \mathbb{N}$, we define $\lambda^*(G_n)$ to be the outer measure of G_n. Since G_n is the union of 2^{n-1} disjoint open intervals, each of length $[\ell(I_1)]^n = \left(\frac{1}{4}\right)^n$, by countable additivity,

$$\begin{aligned}\lambda^*(G_n) &= 2^{n-1}[\ell(I_1)]^n \\ &= 2^{n-1}\left(\frac{1}{4}\right)^n \\ &= \left(\frac{1}{2}\right)^{n+1}.\end{aligned}$$

Now, let us define the set

$$G = \bigcup_{k=1}^{\infty} G_k.$$

By countable additivity,

$$\begin{aligned}\lambda^*(G) &= \lambda^*\left(\bigcup_{k=1}^{\infty} G_k\right) \\ &= \sum_{k=1}^{\infty} \lambda^*(G_k) \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k+1} \\ &= \frac{1}{2}.\end{aligned}$$

Notice that after the nth iteration in the construction of G we are left with a set L_n, a union of 2^n closed intervals each of which we denoted by K_k for some $k \in \mathbb{N}$. Then,

$$K = \bigcap_{n=1}^{\infty} L_n = [0,1] \setminus G$$

is a fat Cantor set with outer measure $\lambda^*(K) = \frac{1}{2}$. Since the Cantor set is known to be totally disconnected, its only connected sets are singletons. Thus, given any point $x \in [0,1] \setminus G$, and any open interval $U = (x-\epsilon, x+\epsilon)$ containing x, $U \cap G \neq \emptyset$. Hence, $x \in \text{cl}(G)$ and $[0,1] \subseteq \text{cl}(G)$. Clearly, $[0,1] \supseteq \text{cl}(G)$. So, $[0,1] = \text{cl}(G)$ and G is dense in $[0,1]$. Thus, G has the following properties:

- G is a union of pairwise disjoint open intervals.

- G is dense in $[0,1]$.

- G has outer measure $\lambda^*(G) = \frac{1}{2}$.

We can now begin constructing a bounded function f which is the derivative of another function, but which is not Riemann integrable. We begin with the open interval $I_1 = (\frac{3}{8}, \frac{5}{8})$ of length $\frac{1}{4}$, and the point $x_1 = \frac{1}{2}$ which is the midpoint of I_1. Let $J_1 = [\frac{15}{32}, \frac{17}{32}]$ be the closed interval of length $\frac{1}{16} = \left(\frac{1}{4}\right)^2 = [\ell(I_1)]^2$ which is symmetric about $x_1 = \frac{1}{2}$. Define $f(\frac{1}{2}) = 1$, $f(\frac{15}{32}) = f(\frac{17}{32}) = 0$. Let f be linear and continuous on the open intervals $(\frac{15}{32}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{17}{32})$ connecting $f(\frac{15}{32})$ to $f(\frac{1}{2})$ and $f(\frac{1}{2})$ to $f(\frac{17}{32})$ respectively. Define $f = 0$ on $I_1 \setminus J_1$. Next, consider the open intervals $I_2 = (\frac{5}{32}, \frac{7}{32})$ and $I_3 = (\frac{25}{32}, \frac{27}{32})$, each of length $\frac{1}{16}$, which are symmetric about $x_2 = \frac{3}{16}$ and $x_3 = \frac{13}{16}$ respectively. From I_2 and I_3, we construct the closed intervals $J_2 = [\frac{23}{128}, \frac{25}{128}]$ and $J_3 = [\frac{103}{128}, \frac{105}{128}]$ of length $\frac{1}{64} = \left(\frac{1}{16}\right)^2 = [\ell(I_k)]^2$, for k = 2, 3, symmetrically about x_2 and x_3, respectively. Define $f(\frac{3}{16}) = f(\frac{13}{16}) = 1$, $f(\frac{23}{128}) = f(\frac{25}{128}) = f(\frac{103}{128}) = f(\frac{105}{128}) = 0$. Let f be linear and continuous on the open intervals $(\frac{23}{128}, \frac{3}{16})$, $(\frac{3}{16}, \frac{25}{128})$, $(\frac{103}{128}, \frac{13}{16})$, and $(\frac{13}{16}, \frac{105}{128})$, connecting $f(\frac{23}{128})$ to $f(\frac{3}{16})$, $f(\frac{3}{16})$ to $f(\frac{25}{128})$, $f(\frac{103}{128})$ to $f(\frac{13}{16})$, and $f(\frac{13}{16})$ to $f(\frac{105}{128})$. Finally, define f to be identically zero on $I_k \setminus J_k$ where $k = 2, 3$. Continuing this process inductively, we obtain the following construction:

Let I_n be the nth open interval, of length $\left(\frac{1}{4}\right)^n$ in the construction of G. Let $J_n = [a_n, b_n] \subseteq I_n$ be a closed interval which is symmetric about the midpoint x_n of I_n so that $\ell(J_n) = [\ell(I_n)]^2 = \left[\left(\frac{1}{4}\right)^n\right]^2$ for each $n \in \mathbb{N}$. Define the function $f : [0,1] \to [0,1]$ follows:

Let $f(x_n) = 1$ and $f(a_n) = f(b_n) = 0$ for each n. Let f be linear and continuous on the open intervals (a_n, x_n) and (x_n, b_n) connecting $f(a_n) = 0$ to $f(x_n) = 1$ and $f(x_n) = 1$ to $f(b_n) = 0$. Define $f(x) = 0$ for all $x \in I_n \setminus J_n$. Finally, define $f = 0$ on $[0,1] \setminus G$. Since $0 \le f(x) \le 1$ for all $x \in [0,1]$, f is bounded on $[0,1]$. Our function can be seen in the following diagram:

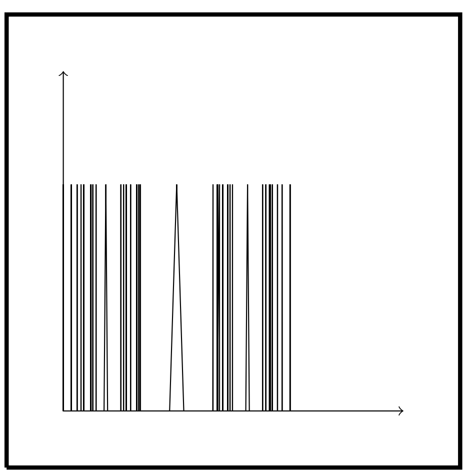

Figure 1: $\mathcal{HK}$ integrable but not $\mathcal{R}$ integrable

We claim that f is not Riemann integrable. To see this, first let $\mathcal{O}$ be an open subset of $[0,1]$ such that $\mathcal{O} \cap K \neq \emptyset$. Since $\mathcal{O}$ is open, we can find distinct points x and y in $\mathcal{O} \cap K$. Without loss of generality, suppose that $x < y$. Then, since K is a "Cantor-like" set, it contains no isolated points and is totally disconnected. So, by construction of G, $J_n \subseteq I_n \subseteq (x, y) \subseteq \mathcal{O}$ for some positive integer n.

Let $\mathcal{P} = \{0 = x_0 < x_1 < \ldots < x_n = b\}$ be any partition of $[0,1]$. Now, let us consider

$$\begin{aligned} K \setminus \{1\} &= K \setminus \{1\} \cap \bigcup_{j=1}^{n} [x_{j-1}, x_j) \\ &= \bigcup_{j=1}^{n} \left[(K \setminus \{1\}) \cap [x_{j-1}, x_j) \right]. \end{aligned}$$

Recall, $\lambda^*(G) = \frac{1}{2}$. So,

$$\lambda^*(K) = \lambda^*([0,1] \setminus G) = \frac{1}{2}.$$

Thus,

$$\begin{aligned} \frac{1}{2} &= \lambda^*(K) \\ &= \lambda^*(K \setminus \{1\}) \\ &= \sum_{j=1}^{n} \lambda^*(K \setminus \{1\}) \cap [x_{j-1}, x_j)) \\ &\leq \sum_{K \cap (x_{j-1}, x_j) \neq \emptyset} \ell((x_{j-1}, x_j)). \end{aligned}$$

When $K \cap (x_{j-1}, x_j) \neq \emptyset$, we can find a positive integer $n = n(j)$ such that

$$J_n \subseteq I_n \subseteq (x_{j-1}, x_j).$$

Therefore,

$$M_j = 1 \quad \text{and} \quad m_j = 0, \text{ when } K \cap (x_{j-1}, x_j) \neq \emptyset.$$

Hence,

$$\begin{aligned}\mathcal{U}(\mathcal{P};f)-\mathcal{L}(\mathcal{P};f) &= \sum_{j=1}^{n}(M_j-m_j)\Delta x_j \\ &\geq \sum_{K\cap(x_{j-1},x_j)\neq\emptyset}(M_j-m_j)\Delta x_j \\ &= \sum_{K\cap(x_{j-1},x_j)\neq\emptyset}\Delta x_j \\ &\geq \frac{1}{2}.\end{aligned}$$

Thus, $f \notin \mathcal{R}([0,1])$.

□

We now show that f is the derivative of the following improper integral. Consider the function

$$F(x)=\sum_{k=1}^{\infty}\int_{J_k\cap[0,x]} f(t)dt.$$

We claim that

$$F'(x)=f(x) \text{ for all } x\in[0,1].$$

Choose an open interval $I\subseteq[0,1]$ so that $I\cap([0,1]\setminus G)\neq\emptyset$. Then, $I\supseteq I_n\supseteq J_n$ for some positive integer n. Choose $n\in\mathbb{N}$ so that $I\cap J_n\neq\emptyset$ and let $S_n=\ell(I_n)$. Then, by construction, $0\leq S_n\leq\frac{1}{2}$. Hence, from the fact that $0\leq x\leq\frac{1}{2}$ if and only if $2(x-x^2)\geq x$, we conclude that

$$\frac{1}{2}[S_n-[S_n]^2]\geq\frac{1}{4}S_n.$$

Now, $I\cap([0,1]\setminus G)\neq\emptyset$, $I\cap J_n\neq\emptyset$, and $\ell(J_n)=[\ell(I_n)]^2=[S_n]^2$. So,

$$\ell(I\cap I_n)\geq\frac{1}{2}\ell(I_n)-\frac{1}{2}[\ell(J_n)]=\frac{1}{2}[S_n-[S_n]^2]\geq\frac{1}{4}S_n$$

since I_n and J_n are both symmetric about the midpoint of I_n. Therefore,

$$16[\ell(I\cap I_n)]^2\geq S_n^2, \tag{2}$$

and by monotonicity,

$$\ell(I\cap J_n)\leq\ell(J_n)=[S_n]^2.$$

Hence, with this last inequality together with (2), we conclude

$$\ell(I \cap J_n) \leq \ell(J_n) = [S_n]^2 \leq 16[\ell(I \cap I_n)]^2.$$

Now, consider the set

$$\mathcal{J} = \{n \in \mathbb{N} : I \cap J_n \neq \emptyset\}.$$

Then,

$$\begin{aligned}
\sum_{n\in\mathcal{J}} \ell(I \cap J_n) &\leq 16 \sum_{n\in\mathcal{J}} [\ell(I \cap I_n)]^2 \\
&\leq 16 \left[\sum_{n\in\mathcal{J}} \ell(I \cap I_n)\right]^2 \\
&\leq \left[\lambda^*\left(\bigcup_{n\in\mathcal{J}} I \cap I_n\right)\right]^2 \\
&= 16\left[\lambda^*\left(I \cap \bigcup_{n\in\mathcal{J}} I_n\right)\right]^2 \\
&\leq 16[\ell(I)]^2. \qquad (3)
\end{aligned}$$

We now show that $F'(x) = f(x)$ for all $x \in [0,1]$. Let $x \in [0,1] \setminus G$, and let $y \in [0,1]$ with $y \neq x$. Without loss of generality assume $x < y$. Then,

$$\begin{aligned}
F(y) - F(x) &= \sum_{n\in\mathcal{J}} \int_{J_n\cap[0,y]} f(t)dt - \sum_{n\in\mathcal{J}} \int_{J_n\cap[0,x]} f(t)dt \\
&= \sum_{n\in\mathcal{J}} \left(\int_{J_n\cap[0,y]} f(t)dt - \int_{J_n\cap[0,x]} f(t)dt \right) \\
&= \sum_{n\in\mathcal{J}} \left(\int_{J_n\cap[x,y]} f(t)dt \right) \\
&\leq \sum_{n\in\mathcal{J}} \ell(J_n \cap [x,y]).
\end{aligned}$$

By assumption, $x < y$. So $F(x) \leq F(y)$ and $F(y) - F(x) \geq 0$. Therefore, since $0 \leq f(t) \leq 1$ for all t,

$$0 \leq \frac{F(y) - F(x)}{y - x} = \frac{1}{y - x} \sum_{n\in\mathcal{J}} \int_{J_n\cap[x,y]} f(t)dt \leq \frac{1}{y - x} \sum_{n\in\mathcal{J}} \ell\left([x,y] \cap J_n\right) \leq \frac{1}{y - x} 16(y-x)^2 \leq 16(y-x), \text{ by (3).}$$

Note: By using the closed interval $[x, y]$, rather than the open interval (x, y), we have not changed the outer measure of (x, y), so the result still holds by (3). Thus,

$$0 \leq \frac{F(y) - F(x)}{y - x} \leq 16(y - x).$$

Then,

$$0 \leq F'(x) = \lim_{y \to x} \frac{F(y) - F(x)}{y - x} \leq \lim_{y \to x} 16(y - x) = 0.$$

Therefore, since $f(x) = 0$ for all $x \in [0, 1] \setminus G$,

$$F'(x) = f(x) \text{ for all } x \in [0, 1] \setminus G.$$

Now, let $x_0 \in G$. Consider

$$F(x) = \sum_{n \in \mathcal{J}} \int_{J_n \cap [0,x]} f(t)dt.$$

Since $x_0 \in G,\ x_0 \in I_n$ for some $n \in \mathbb{N}$. Note that if $x \in I_n \setminus J_n$ for some $n \in \mathbb{N}$, then

$$F'(x) = f(x).$$

So, let $x \in J_n$, and suppose $x > x_0$. Then

$$F(x) - F(x_0) = \sum_{n \in \mathcal{J}} \int_{J_n \cap [x_0,x]} f(t)dt = \int_{x_0}^{x} f(t)dt,$$

since $f \equiv 0$ on $I_n \setminus J_n$. So,

$$F(x) = F(x_0) + \int_{x_0}^{x} f(t)dt.$$

Since f is continuous on each I_n, we may apply the Fundamental Theorem of Calculus II, giving us

$$\begin{aligned} F'(x) = \frac{d}{dx}(F(x)) &= \frac{d}{dx}\left(F(x_0) + \int_{x_0}^{x} f(t)dt\right) \\ &= 0 + f(x). \end{aligned}$$

Thus, $F'(x) = f(x)$. If $x \leq x_0$ we may use the above proof with the identity

$$F(x_0) - F(x) = -\int_{x_0}^{x} f(t)dt.$$

Therefore, $F'(x) = f(x)$ for all $x \in G$, so $F'(x) = f(x)$ for all $x \in [0,1]$. Hence, f is the derivative of F on $[0,1]$. In other words, F is the primitive of f on $[0,1]$. Hence, by The Fundamental Theorem of Calculus for $\mathcal{HK}$ integrals, and Hake's Theorem, $f \in \mathcal{HK}([0,1])$.

$\square$

The above examples not only show some of the most powerful applications of gauges, but also the deficiencies of the Riemann integral. The previous function f definitely seems like it should be Riemann integrable, we are after all simply adding up the areas of triangles, but we have shown that it is not. The function oscillates much too wildly on a set with positive outer measure. However, the function does belong to $\mathcal{HK}([0,1])$. Furthermore, in Example (7.2) we showed that the Henstock-Kurzweil integral is able to handle certain pathological functions that the $\mathcal{R}$ integral simply cannot. In that example, we constructed a gauge which encloses each point of $\mathbb{A}$ in an open interval with arbitrarily small length. By doing that, we obtained a collection of sets whose union has an arbitrarily small outer measure, ultimately rendering the effect that the set $\mathbb{A}$ has on the Riemann sums of a partition $\dot{\mathcal{P}}$ negligible. However, these examples have revealed something deeper. The use of gauges allows us to bridge the very small gap between $\mathcal{HK}(I)$ and $\mathcal{L}(I)$. In fact, as we will now show, it is only a class of highly oscillatory functions which lie in the space between.

§8 FUNCTIONS WHICH ARE $\mathcal{HK}$ INTEGRABLE BUT NOT $\mathcal{L}$ INTEGRABLE.

In any basic measure theory class, one learns about the Lebesgue integral in steps. For brevity, we characterize the class of Lebesgue integrable functions in terms of the $\mathcal{HK}$ integral. The proof of this assertion can be found in [5].

(8.1) Theorem. Let $I = [a,b] \subseteq \mathbb{R}$ be a compact interval. Let $f : I \to \mathbb{R}$ be a function defined on I. Then, f is Lebesgue integrable if and only if $|f|$ is $\mathcal{HK}$ integrable. In either case, the integrals agree.

An application of this result will help us to show that $\mathcal{HK}(I)$ is a space of non-absolutely integrable functions. Consequently, $\mathcal{L}(I)$ is a space of absolutely integrable functions. Hence, a function f can be Henstock-Kurzweil integrable without $|f|$ being Henstock-Kurzweil integrable, but this is not true for Lebesgue integrable functions. In this way, we can think of $\mathcal{HK}(I)$ as being analogous to the class of conditionally convergent series and $\mathcal{L}(I)$ as being analogous to the class of absolutely convergent series. To classify those functions which are absolutely integrable, we employ the following result.

(8.2) Theorem. Let $I = [a, b] \subseteq \mathbb{R}$ be a compact interval. Let $f : I \to \mathbb{R}$ be a function defined on I. Let $f \in \mathcal{HK}(I)$. Then, $|f|$ is $\mathcal{HK}$ integrable if and only if the indefinite integral

$$F(x) = \int_a^x f$$

has bounded variation on I. In this case,

$$\int_a^b |f| = Var(F; I).$$

The proof of this assertion can be found in [1].

Thus, Theorem 8.1 and Theorem 8.2 together show that if F is of bounded variation on I, then F' is Lebesgue integrable on I. Using these characterizations, we look at some functions which are $\mathcal{HK}$ integrable but not Lebesgue integrable.

(8.3) Example. Consider the function

$$F : [0, 1] \to \mathbb{R}$$

given by

$$F(x) = \begin{cases} x^2 \cos\left(\frac{\pi}{x^2}\right) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

This function has a derivative given by:

$$f(x) = \begin{cases} 2x \cos\left(\frac{\pi}{x^2}\right) + \frac{2\pi}{x} \sin\left(\frac{\pi}{x^2}\right) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that F is not of bounded variation on $[0, 1]$, which will in turn implies that $F' = f \notin \mathcal{L}([0, 1])$. To see this, consider the sequence $x_k = \frac{1}{\sqrt{k}}$ where $k = 1, 2, \ldots, N$. Then,

$$\begin{aligned}
\sum_{k=1}^{N} |F(x_k) - F(x_{k+1})| &= \sum_{k=1}^{N} \left| \frac{1}{k} \cos k\pi - \frac{1}{k+1} \cos(k+1)\pi \right| \\
&= \sum_{k=1}^{N} \left(\frac{1}{k} + \frac{1}{k+1} \right) \\
&= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{N} + \frac{1}{N+1}\right) \\
&= 2(1) + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{N}\right) - 1 + \frac{1}{N+1} \\
&= 2\sum_{k=1}^{N} \frac{1}{k} - 1 + \frac{1}{N+1}.
\end{aligned}$$

Since the harmonic series diverges, we see that this series goes to $+\infty$ as $N \to +\infty$. Thus, F is not of bounded variation on $[0, 1]$. So, $f = F' \notin \mathcal{L}([0, 1])$. However, according to the Fundamental Theorem of Calculus I for $\mathcal{HK}$ integrals, since F is a primitive for f on $[0, 1]$, we see that $f = F'$ does belong to $\mathcal{HK}([0, 1])$, and

$$\mathcal{HK} \int_0^1 f(x)dx = F(1) - F(0) = -1.$$

In the spirit of Theorem (5.2), we will now construct another example of a function which is not Lebesgue integrable but is $\mathcal{HK}$ integrable. This example illustrates the strength of non-absolute integrability.

(8.4) Example. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \log n}{n}.$$

We claim that this series is not absolutely convergent. Consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \log n}{n} \right| = \sum_{n=1}^{\infty} \frac{\log n}{n}.$$

For all $n \geq 3$, $\frac{\log n}{n} \geq \frac{1}{n}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is known to diverge, by the comparison test,

$$\sum_{n=1}^{\infty} \frac{\log n}{n}$$

diverges as well. Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \log n}{n}$$

is not absolutely convergent. However, letting $a_n = \frac{\log n}{n}$, we see that

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\log n}{n} = 0.$

- Letting $f(x) = \frac{\log x}{x}$, $f'(x) = \frac{1-\log x}{x^2}$, and

- since $f'(x) < 0$ for all $x > e$, a_n is a decreasing sequence.

So, by the alternating series test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\log n}{n}$$

is a convergent series. Thus, by Theorem 5.2, the function $\kappa : [0,1] \to \mathbb{R}$ given by:

$$\kappa(x) = \begin{cases} 2^k \frac{(-1)^{k+1}\log k}{k} & \text{if } x \in [c_{k-1}, c_k) \text{ for } k \in \mathbb{N} \\ 0 & \text{if } x = 1 \end{cases}$$

where $c_k = 1 - \frac{1}{2^k}$ is $\mathcal{HK}$ integrable. However, we claim that κ is not absolutely integrable, hence not Lebesgue integrable. To see this, let

$$K(x) = \int_0^x \kappa(t)dt$$

be an indefinite integral for κ with base point 0. Note that $K(0) = 0$ and

$$K(c_n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}\log k}{k}.$$

Now, consider

$$\begin{aligned} |K(c_n) - K(c_{n-1})| &= \left| \sum_{k=1}^{n} \frac{(-1)^{k+1}\log k}{k} - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}\log k}{k} \right| \\ &= \frac{\log n}{n} \end{aligned}$$

Using parts of our sequence c_k as a partition of $[0,1]$, we set

$$\mathcal{P} = \{0 = y_0, y_1, \ldots, y_n = 1\}$$

where

$$y_0 = 0,\ y_1 = c_1,\ y_2 = c_2,\ \ldots,\ y_{n-1} = c_{n-1},\ y_n = 1.$$

Then,

$$\sum_{i=1}^{n} |K(y_i) - K(y_{i-1})| \geq 0 + \frac{\log 2}{2} + \frac{\log 3}{3} + \ldots + \frac{\log(n-2)}{n-2} + \frac{\log(n-1)}{n-1}.$$

Taking $n \to \infty$, we see that $K \notin \mathcal{BV}([0,1])$, since $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges. Therefore, $|\kappa|$ is not $\mathcal{HK}$ integrable on $[0,1]$. Hence, κ is not Lebesgue integrable on $[0,1]$. However, by Theorem (5.2), since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \log n}{n}$ does converge, $\kappa \in \mathcal{HK}([0,1])$ and

$$\mathcal{HK} \int_0^1 \kappa = \sum_{k=1}^{\infty} \frac{(-1)^{n+1} \log n}{n} = \gamma \log 2 - \frac{1}{2}[\log 2]^2$$

where γ is the Euler-Mascheroni constant given by

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right] \approx 0.5772 \ \ (see[8].)$$

Here, κ fails to be Lebesgue integrable because $|\kappa|$ is not $\mathcal{HK}$ integrable. However, upon deeper inspection, we realize that $|\kappa|$ fails to be $\mathcal{HK}$ integrable because $\int_0^x \kappa \notin \mathcal{BV}([0,1])$. Indeed, by simply looking at the first few partial sums we can see that the function oscillates wildly. The following diagram shows the graph of $K(c_{20})$:

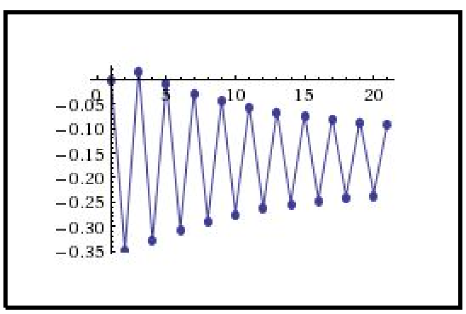

Figure 2: $K(c_{20})$

We will now look at one final function which belongs to $\mathcal{HK}(I)$ but not $\mathcal{L}(I)$. In the process we show the power of Hake's Theorem.

(8.5) Example. Consider the following function.

$$f(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^3}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We cannot express the integral

$$\int_0^x f$$

with elementary functions. However, consider the function

$$g(x) = \int_0^x \frac{\sin t}{t} dt.$$

It is a commonly known fact that $\lim_{x\to\infty} g(x) = \frac{\pi}{2}$ [9]. Armed with this fact, we can now evaluate the integral of f as an improper Riemann integral in the following manner. Let $t = \frac{1}{x^3}$, then

$$\begin{aligned} \int_0^1 \frac{1}{x}\sin\left(\frac{1}{x^3}\right) dx &= \frac{1}{3}\int_1^\infty \frac{\sin t}{t} dt \\ &= \frac{1}{3}\left(\lim_{x\to\infty} g(x) - g(1)\right) \\ &= \frac{1}{6}\left(\pi - 2g(1)\right) < \infty. \end{aligned}$$

Now, let $u = \frac{1}{x}$ and $dv = \sin x\, dx$. Then, $du = -\frac{1}{x^2}$ and $v = -\cos x$, and we have

$$\begin{aligned} \int_1^A \frac{\sin x}{x} dx &= -\frac{\cos x}{x}\Big|_1^A - \int_1^A \frac{\cos x}{x^2} dx \\ &= \frac{\cos 1}{1} - \frac{\cos A}{A} - \int_1^A \frac{\cos x}{x^2} dx. \end{aligned}$$

Now,

$$\lim_{A\to\infty} \int_1^A \frac{\cos x}{x^2}\, dx = \int_1^\infty \frac{\cos x}{x^2}\, dx$$

exists since

$$\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2} \quad \text{on } [1, \infty),$$

and

$$\int_1^\infty \frac{1}{x^2}\, dx < \infty.$$

So,

$$\lim_{A\to\infty}\int_1^A \frac{\cos x}{x^2}dx$$

converges. Hence,

$$\int_0^1 \frac{1}{x}\sin\left(\frac{1}{x^3}\right)dx = \frac{1}{3}\int_1^\infty \frac{\sin t}{t}dt = \frac{1}{3}\lim_{A\to\infty}\left(\frac{\cos 1}{1} - \frac{\cos A}{A} - \int_1^A \frac{\cos x}{x^2}dx\right)$$

$$= \frac{1}{3}\left(\cos 1 - \int_1^\infty \frac{\cos x}{x^2}dx\right).$$

So, f is improper Riemann integrable, and Hake's Theorem guarantees that it is a "proper" $\mathcal{HK}$ integral. However, f is not Lebesgue integrable. To see this, we use the fact that a Lebesgue integrable function must be absolutely integrable. For $k \in \mathbb{N}$ consider

$$\int_\pi^{(k+1)\pi}\left|\frac{\sin t}{t}\right|dt = \sum_{j=1}^k \int_{j\pi}^{(j+1)\pi}\left|\frac{\sin t}{t}\right|dt. \text{ Then, letting } u = t - j\pi,\ du = dt. \text{ So}$$

$$= \sum_{j=1}^k \int_0^\pi \frac{|\sin(u+j\pi)|}{u+j\pi}du$$

$$= \sum_{j=1}^k \int_0^\pi \frac{\sin u}{u+j\pi}du, \quad \text{since } 0 \le u \le \pi.$$

So, since $\frac{1}{u+j\pi} \ge \frac{1}{(j+1)\pi}$ for $0 \le u \le \pi$, we have

$$\sum_{j=1}^k \int_0^\pi \frac{\sin u}{u+j\pi}du \ge \sum_{j=1}^k \frac{1}{(j+1)\pi}\int_0^\pi \sin u\,du$$

$$\ge \frac{2}{\pi}\sum_{j=1}^k \frac{1}{j+1}.$$

So,

$$\sum_{j=1}^k \int_{j\pi}^{(j+1)\pi}\left|\frac{\sin t}{t}\right|dt = \frac{2}{\pi}\sum_{j=1}^k \frac{1}{j+1}.$$

Therefore,

$$\begin{aligned}\int_0^1 \left|\frac{1}{x}\sin\left(\frac{1}{x^3}\right)\right| dx &= \frac{1}{3}\int_1^\infty \left|\frac{\sin t}{t}\right| dt \\ &= \frac{1}{3}\int_1^\pi \left|\frac{\sin t}{t}\right| dt + \frac{1}{3}\sum_{j=1}^{n}\int_{j\pi}^{(j+1)\pi} \left|\frac{\sin t}{t}\right| dt \\ &= \frac{1}{3}\int_1^\pi \left|\frac{\sin t}{t}\right| dt + \frac{2}{3\pi}\sum_{j=1}^{n}\frac{1}{j+1} \\ &\geq \frac{2}{3\pi}\sum_{j=1}^{n}\frac{1}{j+1}.\end{aligned}$$

Taking $n \to \infty$, we see that

$$\int_0^1 \left|\frac{1}{x}\sin\left(\frac{1}{x^3}\right)\right| dx$$

diverges since the harmonic series is divergent. So, f is not absolutely integrable hence not Lebesgue integrable. Again the function is not $\mathcal{L}$ integrable because $\int_0^x f \notin \mathcal{BV}([0,1])$.

Notice, in the above example, we have inadvertently found a function which does not belong to $\mathcal{HK}([0,1])$. Namely,

$$h(x) = \left|\frac{1}{x}\sin\left(\frac{1}{x^3}\right)\right|.$$

For if $h \in \mathcal{HK}([0,1])$, then both

$$h(x) = \left|\frac{1}{x}\sin\left(\frac{1}{x^3}\right)\right|$$

and

$$f(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^3}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

belong to $\mathcal{HK}([0,1])$, which would imply that $f \in \mathcal{L}([0,1])$. However, as we just showed, this is not the case. Therefore, $h \notin \mathcal{HK}([0,1])$. This illustrates an important point. Although the $\mathcal{HK}$ integral is a very powerful tool, it cannot integrate every function.

Since it is a well known fact that $\mathcal{R}(I) \subsetneq \mathcal{L}(I)$, we have shown that

$$\mathcal{R}(I) \subsetneq \mathcal{L}(I) \subsetneq \mathcal{HK}(I).$$

We conclude the paper by discussing various strengths of the $\mathcal{HK}$ integral over the R and $\mathcal{L}$ integrals in applied mathematics.

§9 APPLICATIONS OF THE $\mathcal{HK}$ INTEGRAL.

Until now, we have mainly examined the differences between the $\mathcal{HK}$ integral and the classic Riemann and Lebesgue integrals by using pathological functions. However, Kurzweil originally developed the $\mathcal{HK}$ integral as a way to solve complicated differential equations. Indeed, due to the improved Fundamental Theorem of Calculus, the $\mathcal{HK}$ integral is a powerful tool for solving differential equations especially those involving highly oscillatory functions.

(9.1) Example. Recall from Example (8.3), the function $F : [0, 1] \to \mathbb{R}$ be given by:

$$F(x) = \begin{cases} x^2 \cos\left(\frac{\pi}{x^2}\right) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$$

with derivative

$$F'(x) = \begin{cases} 2x \cos\left(\frac{\pi}{x^2}\right) - \frac{2\pi}{x} \sin(\frac{\pi}{x^2}) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Now, consider the initial value ODE:

$$y'(t) = t^2 y(t) + F'(t), \;\; y(0) = 0, \;\text{ and }\; 0 \leq t \leq 1.$$

Recall that F' is not Lebesgue integrable. So, $\tilde{F}(x, t) = t^2 y(t) + F'(t)$ cannot be solved by Lebesgue or Riemann integration. However, consider the following function:

$$y(t) = e^{\frac{t^3}{3}} \left[\mathcal{HK} \int_0^t e^{-\frac{s^3}{3}} F'(s) ds \right].$$

Then,

$$y(0) = 0$$

and

$$y'(t) = t^2 e^{\frac{t^3}{3}} \left[\mathcal{HK} \int_0^t e^{-\frac{s^3}{3}} F'(s) ds \right] + e^{\frac{t^3}{3}} \frac{d}{dt} \left[\mathcal{HK} \int_0^t e^{-\frac{s^3}{3}} F'(s) ds \right].$$

Now, by the Fundamental Theorem of Calculus II for $\mathcal{HK}$ integrals,

$$\frac{d}{dt} \left[\mathcal{HK} \int_0^t e^{-\frac{s^3}{3}} F'(s) ds \right] = e^{-\frac{t^3}{3}} F'(t),$$

and

$$y'(t) = t^2 e^{\frac{t^3}{3}} \left[\mathcal{HK} \int_0^t e^{-\frac{s^3}{3}} F'(s) ds\right] + e^{\frac{t^3}{3}} e^{-\frac{t^3}{3}} F'(t) = t^2 \left[e^{\frac{t^3}{3}} \left(\mathcal{HK} \int_0^t e^{-\frac{s^3}{3}} F'(s) ds\right)\right] + F'(t),$$

or equivalently,

$$y'(t) = t^2 y(t) + F'(t), \quad 0 \le t \le 1,$$

and $y(0) = 0$. Thus,

$$y(t) = e^{\frac{t^3}{3}} \left[\mathcal{HK} \int_0^t e^{-\frac{s^3}{3}} F'(s) ds\right]$$

is indeed a solution to the aforementioned ODE.

Although differential equations play an important part in the applicability of the $\mathcal{HK}$ integral, working mathematicians who regularly study the $\mathcal{HK}$ integral and its applications are also making headway in areas such as probability, statistics, physics and finance. For example, a Brownian motion $X = (X_t)(0 < t, X_0 = 0)$ with drift rate (rate at which the average changes) μ_t and variance σ_t can be constructed from a standard Brownian motion $W = (W_t)$ by the equation:

$$X_t = \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds.$$

Although the first integral is $\mathcal{L}$ integrable, the second, which is called a "Stochastic integral," is not. Almost all of the sample paths given by $x(s)(0 < s \le t, x(0) = 0)$ - which are continuous functions between topological spaces that produce a set of values according to the "rules" of the process - are of unbounded variation. So, for any path x if

$$\mu_x((u, v])$$

is even as simple as

$$x(v) - x(u),$$

then the integral

$$\int_0^t d\mu_x$$

does not exist. This is due to the fact that the Lebesgue integral is an absolutely convergent integral. When considered seperately, the sums of all paths x so that

$$x(v) - x(u) \ge 0$$

and

$$x(v) - x(u) < 0$$

diverge to ∞ and $-\infty$ respectively. However, when considered as a $\mathcal{HK}$ integral, the integral exists. Indeed, given a partition

$$\mathcal{P} = \{0 = u_0, u_1, \ldots, u_n = t\}$$

of $[0, t]$,

$$\sum_{j=1}^{n} \mu_x((u_j, u_{j-1}]) = x(u_1) - x(u_0) + x(u_2) - x(u_1) + \ldots + x(u_n) - x(u_{n-1})$$

$$= x(u_n) - x(u_0) = x(t).$$

So, it turns out that

$$\int_0^t d\mu_x$$

is finite and is equal to $x(t)$. In terms of Brownian motion, the stochastic integral on $(0, t]$ is given by

$$\int_0^t dW_s = W_t.$$

Furthermore, much research has been done on the uses of the $\mathcal{HK}$ integral in quantum mechanics. Particularly when considering transforms and Feynman paths, which are often highly oscillatory by nature.

§10 FINAL COMMENTS.

The $\mathcal{HK}$ integral is one of the most powerful methods of integration currently being researched by mathematicians. By using gauges, one can evaluate functions on more of a local level than one can with the traditional Riemann integral. This seemingly small change to the traditional definition of the Riemann integral has proven to have far reaching consequences. For example, the $\mathcal{HK}$ integral makes integration and differentiation truly inverse processes. The fact that the $\mathcal{HK}$ integral is a non-absolutely convergent integral makes it ideal for integrating functions which oscillate wildly, a feature not always available with the $\mathcal{L}$ integral. This allows one to look at the integration process as a whole, rather than being forced to consider the negative and nonnegative cases separately as is often the case in Lebesgue integration theory. However, this advantage does have its drawbacks. To date no one has developed a suitable norm for the space $\mathcal{HK}(I)$. However, all of the above results can be generalized to $\mathbb{R}^n$ along with $\mathcal{HK}$ versions of "Fubini's Theorem" for iterated integrals and the "Divergence Theorem" similar to the ones learned in second or third year calculus courses. In addition, Hake's theorem tells us that there are no improper $\mathcal{HK}$ integrals, another result unavailable in R and $\mathcal{L}$ integration theory. Furthermore, the definition of the Henstock-Kurzweil integral is relatively simple and requires no knowledge of measure theory making it a viable alternative to the traditional Riemann integral. In fact, there is a petition to replace the $\mathcal{R}$ integral in undergraduate analysis courses with the $\mathcal{HK}$ integral. The use of gauges may help students develop a deeper understanding of epsilon-delta style proofs, and give them some rudimentary intuition about outer measure.

The $\mathcal{HK}$ integral's ability to integrate functions of a highly oscillatory nature has proven to be a valuable tool in many areas of applied mathematics including differential equations, stochastic probability and quantum mechanics. Although, no one has found a suitable norm for the space of Henstock-Kurzweil integrable functions, there are semi-norms available. Even so, one cannot deny the Henstock integral's value as an important supplement to the theory of integration.

Although the $\mathcal{HK}$ integral may not be the apotheosis of integration theory, it is an important mathematical tool with far reaching consequences. Its sheer simplicity and ability to generalize all previous integrals make it an important addition to integration theory, and although the $\mathcal{HK}$ integral cannot totally replace the Lebesgue integral, it is a tool that should be considered by any serious student of integration theory.

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