Chaos Theory and Probability

An Honors Thesis (HONRS 499)

by

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Purpose of Thesis

Books and other literature written on chaos theory, fractals and the uncertainty principle are sometimes difficult for the average reader to understand. As a result, the reader may walk away feeling more frustrated or confused than at the start of their research. Dr. Emert's colloquium classes on fractals were extremely enjoyable and thought provoking, but not beyond comprehension. This presentation was written to help people understand the basic concepts behind chaos theory and how it relates to the world around us. Examples involving random walks, Brownian motion and Heisenberg's uncertainty principle are used to illustrate the intimate relationship between chaos theory and probability.

Acknowledgements

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For as long as humans have roamed the earth, there has been a drive to understand the world in which they live. The universe is a far cry from being simplistic and many people from all walks of life have dedicated their lives to this objective. Many would readily agree that life has more than its share of chaotic and unpredictable moments, but through this search for knowledge, some theories have surfaced to explain this chaos and mayhem.

At first glance, there may seem to be very little in common with chaos theory and probability. Chaos theory focuses on how erratic and unpredictable an outcome is while probability focuses on the certainty of a particular outcome. It is like the positive and negative side of the same coin. Chaos theory is a reminder that the future is uncertain and unpredictable while probability provides a certain amount of security in predictability.

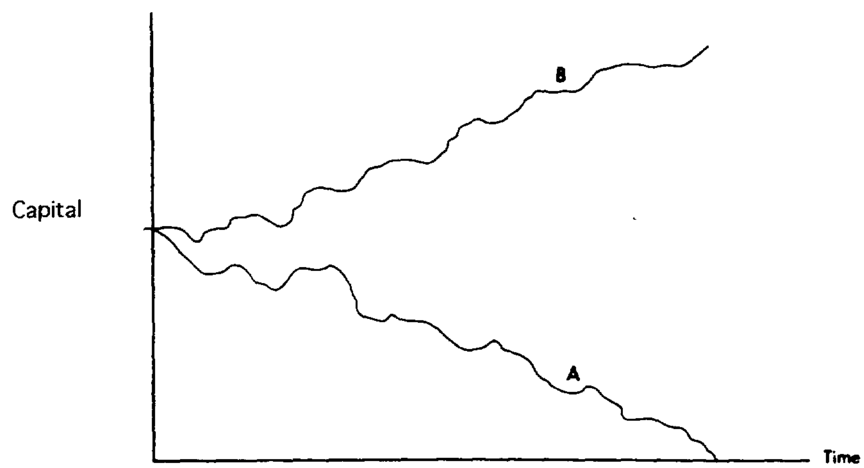
Chaos theory is one of the main properties relating to fractals. Simply stated, fractals are objects or pictures that have a fraction of a dimension. The chaos properties pertain to the fact that it cannot be predicted with complete certainty how a typical fractal will look. A fractal also has self-similar properties which means that a smaller version looks very similar to the larger version. This can be seen in a fern leaf. The whole shape of the leaf is similar to the shape of the leaves that branch off the middle vein. These leaves are also made up of smaller leaves that have the same shape. (See figure 1) While a fern may be a good illustration to describe self-similarity, there are better ways to explain chaos such as through a random walk.

A random walk traces the path of an object like a road on a map. It is called 'random' because unlike the road on a map, the path the object is going to take is uncertain. Random walks can be simulated using lines and points on a plane. Starting at an initial point on a plane, there is an equal probability of going in any direction a distance of one unit. There is no way to accurately predict which direction will be taken. The second step, like the first step, has the same freedom to go in any direction. In this way, previous decisions do not affect future results. The only difference between each advancement is that the starting points will not necessarily be the same. A random walk is created when this process is indefinitely repeated.

What makes such an arbitrary stroll so interesting is that given an infinite amount of time, the probability of reaching any particular point on the plane, even the initial starting position, is one. Although the steps are random as to which direction is chosen and the resulting path is totally chaotic, there are predictable outcomes in determining the destination of the traveling points.

Figure 1

Figure 2



One of the uses of a random walk is found in forming a model for the success or failure of an organization. The plane is a two dimensional graph where the x axis is time and the y axis is the organization's capital. (See figure 2.) The two factors that alter the level of capital are the initial capital of the starting company and the dynamic process governing the changes over time. Fluctuations in the organizational capital can be represented by a totally arbitrary random walk where the chance of having a successful company, (a 'high' position on the y axis), is just as probable as falling victim of bankruptcy, (when y is less than or equal to zero) (Levinthal, 403). Since the objective is to become successful, most organizations try to put the odds in their favor by bringing in management to make the kinds of decisions that influence the random walk of capital in an upward direction.

Even when specific decisions are made in hopes of increasing the organizational capital, there is an uncertainty as to whether this will be the resulting outcome. At each point in time, the change in the organization's capital can be represented by a normal distribution with a corresponding mean and variance. If the expected value (mean) of this distribution is zero, then the fluctuations of the organizational capital are considered a pure random walk. An expected value of zero means there is equal probability for the capital to either increase or decrease. The mean can be influenced by the decisions the firm makes. A positive expected value means the capital is more likely to increase than decrease. The variance is a measurement of how much the actual move may differ from the expected move (Levinthal, 402). It is the expected range in which the organizational capital may fluctuate after a certain amount of time.

The random walk model generates some familiar patterns of organizational capital. There is an initial 'honeymoon' period, the 'liability of adolescence' stage and the general stability of an established organization that can withstand financial challenges which would bankrupt a new firm (Levinthal, 401). However, specific increases and decreases of the organization's capital cannot be exactly predicted.

The random walk model is also an excellent representation for the fluctuations in the stock market. There are two theories for trying to predict what the stock market is going to do: firm foundations and castles in the air (Malkiel, 23). The firm foundations theory is based on the fact that there is a calculatable value of stock in a certain company based on that company's capital and success. If the stock is being bought or sold at a different value, then one can rest assured that the stock will eventually come back to its actual worth. The idea is to never pay more than the stock's current worth and to sell when the going rate is higher than its true value.

The castles in the air theory is based on what one expects the stock to do; or more accurately, what one can convince another that the stock is going to do. If a buyer can be convinced that a certain stock is selling for less than its future value, then it would be wise to buy the stock. Even if it does not increase in value, the buyer can then sell it by convincing someone else that the stock will increase in value.

Even though these theories seem to be contradictory, there is truth in both perspectives. The name of the game is trying to figure out what the stock will do in the future so that one can buy low and sell high. Short term changes are difficult to impossible to predict and fluctuations in value do not necessarily reflect the true value of a product. It all depends on what price shares in a company can be bought or sold. If the stock is in high demand, it will push the price up and likewise, if there is low demand for certain stock, the price will drop.

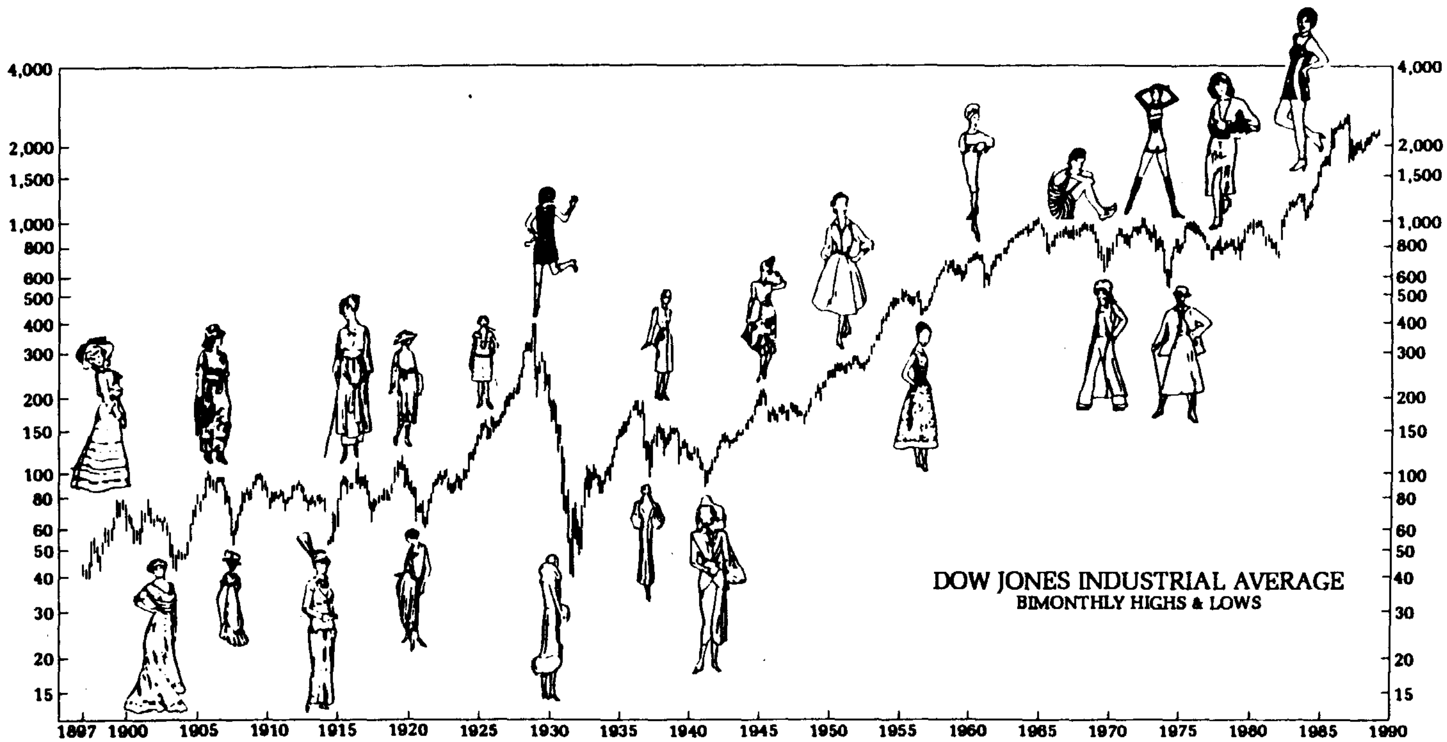
Frequently, charts are drawn to record the history of the movement of certain stock prices. Then an effort is made to predict what the stock will do next based on previous data. There are more than enough systems, theories, patterns and indicators that try to interpret the history of stock prices to predict the future values (such as the super bowl and the hemline indicators-- See figure 3), but random walks can best represent the actual fluctuations in stock market prices.

To illustrate this point, Dr. Burton Malkiel, a professor of economics at Princeton University, had his students construct a normal stock chart showing the movements of a hypothetical stock initially selling at $50 per share. At the end of every day, they would flip a coin and if it was a head, it would be assumed that the stock increased 1/2 a point and if was a tail, it was assumed that the stock decreased 1/2 a point. The resulting charts illustrated several different patterns and formations. One in particular was chosen and Dr. Malkiel showed it to a chartist friend of his who immediately demanded to know what company it was because it was obvious that the stock would be up 15 points by next week. He apparently did not appreciate the fact that the chart had been determined by the toss of a coin (Malkiel, 135). There are many truths and explanations of the world around us that are not always appreciated or even believed.

Like Dr. Malkiel, Robert Brown made a discovery that was not readily accepted. What Mr. Brown saw is now known as Brownian

Figure 3

The Hemline Indicator



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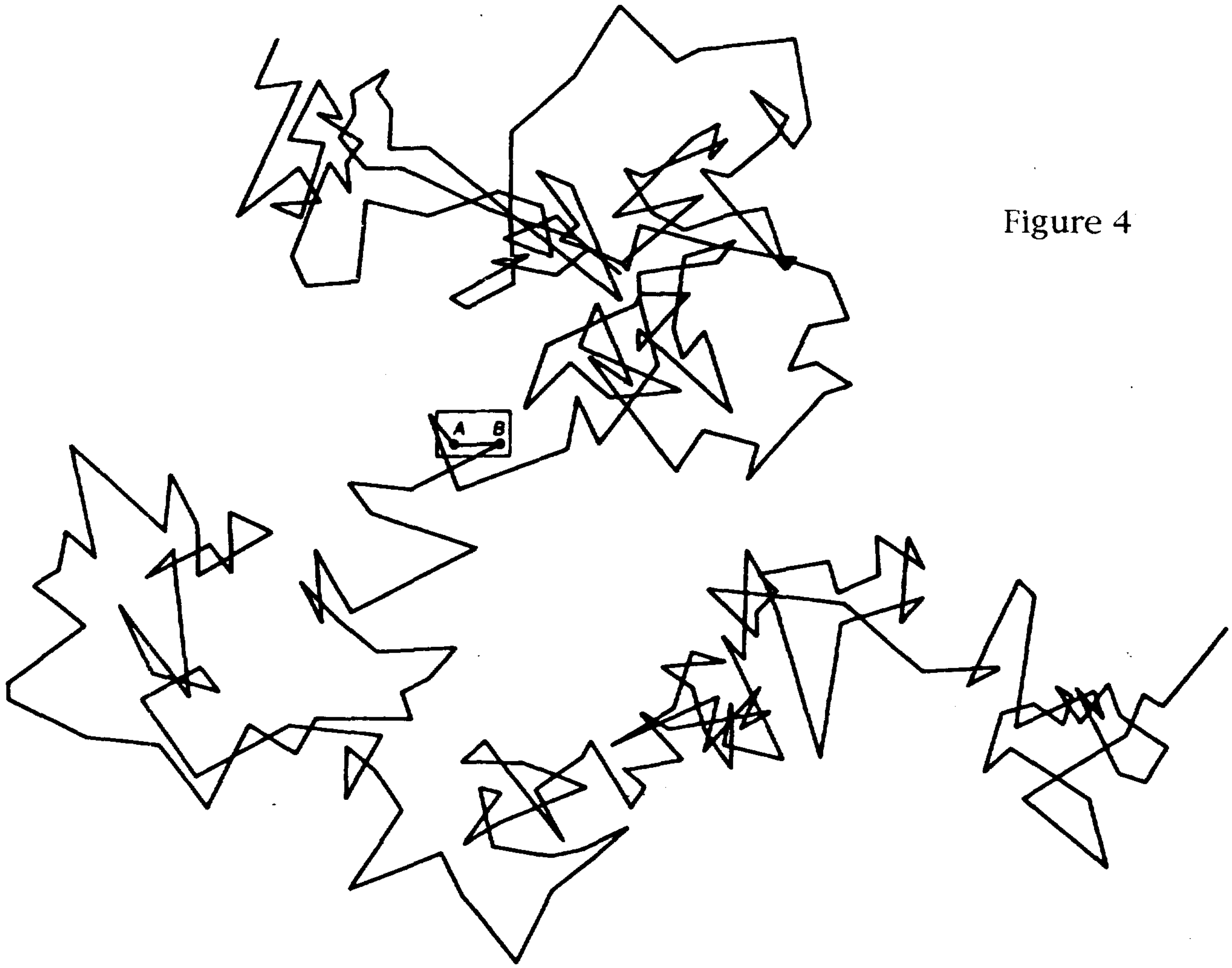
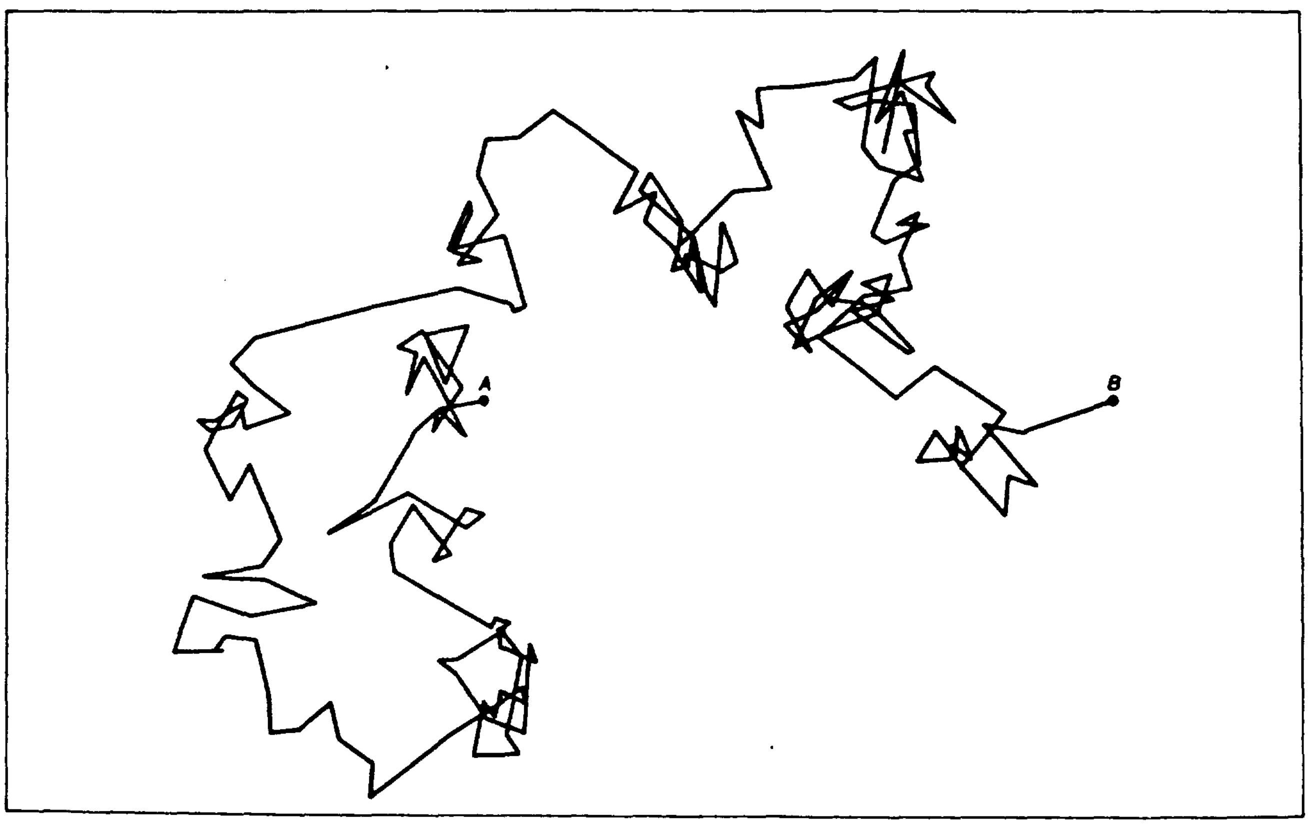
motion and can be described as a special type of random walk. It all began in the early 19th century when a Scottish botanist, Robert Brown, was watching pollen suspended in water. Brownian motion describes the erratic movements of small particles suspended in liquid due to the collisions with surrounding molecules (Peitgen, 400). To get a better idea of the impact of the surrounding molecules, try to picture what it would be like to stand in the midst of a mob of people each one having a different destination in mind and direction in which they wish to go. The standing figure no longer possesses freedom of choice. The mob's destiny becomes his own. It is estimated that a Brownian particle undergoes about 10^{21} collisions per second (Lavenda, 77). When this motion is graphed by periodically plotting the position of a particle at equal time intervals, the resulting picture looks like a random walk. (See figure 4.)

What makes Brownian motion special is that if any segment is taken and enlarged, a more detailed picture is produced from the movements inside that segment and one would find another self-similar random walk. Likewise, if part of that segment is enlarged again, one would see that each line segment would also be a random walk. Even more fascinating is considering what would happen if one would graph Brownian motion at infinitely small time intervals--

Figure 4





the result would be a plane! The Hausdorff dimension can be determined to be 2 (Schroeder, 141).

Most people have seen Brownian motion in action without realizing it. A simple illustration can be seen when watching what happens to a drop of food coloring in a glass of water. The dye swirls and dances until the color finally stabilizes and reaches a state of equilibrium throughout the liquid. The drop of dye has an initial velocity from gravity which causes its downward movement and the swirling is caused by the resistance it receives from the water. Then the dispersement of the dye is caused by the water molecules mixing with the dye molecules which are all in constant motion.

In a similar fashion, if a permeable membrane of Brownian particles is inserted in the center of a box, the concentration of the diffusing particles at different moments in time can be estimated using normal bell curves. Instead of the medium being a liquid, in this case it is air, but the principle remains the same. The Brownian particles mix and swirl with the air in the box until equilibrium is achieved between the pull of gravity and diffusion. There is a normal curve that corresponds to the concentration of particles for each moment in time. At time t=0, the particles are all gathered in the permeable membrane. Then the particles immediately begin to disperse away from this highly concentrated region. The normal curve has its high point directly in line with the position of the membrane. The higher the position of the curve, the more concentrated the particles are. As time elapses, the particles venture further and further away from their starting point and the curve becomes lower and broader until all the particles have reached an equilibrium throughout the box. (See figure 5.)

The probability that a Brownian particle can be found in a certain region can also be determined through the use of normal bell curves. The curves are probability density functions. This means that the probability a particle can be found in a certain region is the area under the curve of that region. The area underneath a standard normal curve is directly related to the probability that a certain Brownian particle can be located within the chosen region. For example, if the area under the curve between -1 and 1 is calculated to be 0.5, then there would be a 50 percent chance that the particle could be found within that region.

The displacement of a Brownian particle at a particular time t can also be estimated. The distance that a particle travels away from its origin can be calculated via root-mean-square displacement. Root-mean-square displacement can be found by first squaring the displacement of each particle at some time t, then finding the

Figure 5

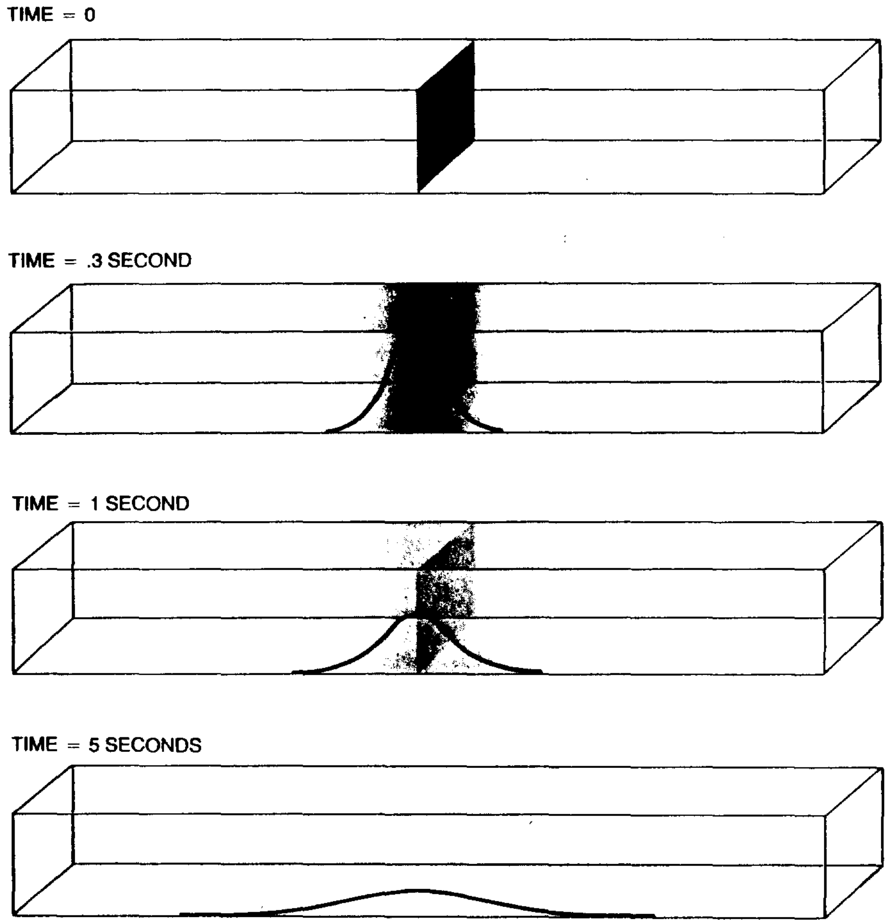
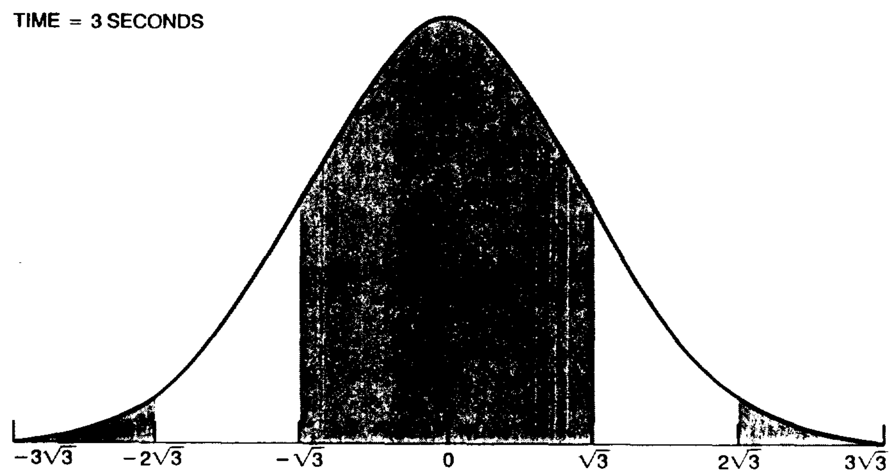


Figure 6



average of these results and finally taking the square root of the average. The answer establishes two things for this particular time t: 1) the probability that the Brownian particles have stayed within the region identified by the root-mean-square displacement is approximately 0.68 and 2) the probability that the Brownian particles have strayed further than twice the root-mean-square displacement is less than 0.05 (Lavenda, 77). (See figure 6.)

The path of a Brownian particle is a random walk influenced by surrounding particles and gravity in the same way business decisions influence an organization's capital. Even though the future can be influenced, there is a distinct and undeniable element of uncertainty. No one can know for sure if a particle is going to go in a particular direction just like no one can know with absolute certainty what an organization's capital is going to do from day to day. Heisenberg realized this and the concept of being unsure of the future is now known as Heisenberg's uncertainty principle. However straight forward the uncertainty principle may appear, it seems to be one of the most misunderstood concepts in the field of physics and mathematics.

There are many different ways to describe a new concept. One is to start with an example. This seems to be the most popular choice when introducing the uncertainty principle, but in this particular case the examples prove to be highly misleading. John Gribbin goes to great lengths to describe what Werner Heisenberg was trying to explain by the uncertainty principle and in a sentence, here is what he says, "...(what) the uncertainty principle tells is that, according to the fundamental equation of quantum mechanics, there is no such thing as an electron that possesses both a precise momentum and a precise position." If one can determine to a high degree of accuracy what the momentum of an electron is, the consequence is an inaccurate knowledge of the position. Likewise, if one can accurately determine the position of an electron, then the momentum cannot be accurate. There is a trade off in the knowledge of either position or momentum. This lack of knowledge, known as the uncertainty principle, is not due to poor equipment or underdeveloped technology. Rather, Heisenberg realized that this uncertainty is an unavoidable part of physics just as gravity affects nearly all forms of physical mathematics.

Why would anyone need or want to know the momentum or position of an electron? This question stems from a theory proposed by the deterministic model. The determinists concluded that the future could be predicted as long as the right data was available. For example, if one were interested in knowing the future position of a particle, they would have to know the precise position and momentum of that particle at some particular time. If the future position of a small particle such as an electron could be predicted, then theoretically the same concept could be extended to larger things. We could know the destiny of the world!

Heisenberg began with the fundamental equation of quantum mechanics and derived the formula $(\Delta q)^*(\Delta p) > h/(2\pi)$ where h is Planck's constant, Δq is the change in the position of the particle, and Δp is the change in the momentum of the particle. The interpretation of this equation is that the more accurate the position (or momentum) is known, the smaller Δq (or Δp) becomes. Given a small delta value, the momentum (or position) must be large enough to compensate so the equation can still be satisfied. As Heisenberg stated, one cannot know an electron's position and momentum simultaneous. Since the future cannot be predicted, the deterministic viewpoint cannot be utilized (Gribbin, 157).

Probability plays an important roll in the uncertainty principle. Although one cannot know the exact position and momentum of an electron, there is no reason why they cannot be estimated with relatively high certainty. However, there is a trade-off in this guessing: the more accurate the estimate is for one, the probability for an accurate estimate in the other will continue to drop.

The most common example illustrates what happens when one tries to determine either position or momentum of an electron. If one could view an electron, some sort of light would be needed to see it. The light photons must reflect off of the electron in order for it to be seen. Unfortunately, the position and momentum of the electron is affected by the photons that are hitting it. The simple act of observing the electron affects the outcome of the experiment.

To illustrate another point, assume the electron will not move under the influence of the photons. How accurate then would the observer be able to measure the electron's position? The fact that light is made of up waves adds a certain limitation to the accuracy of the position of the electron. The accuracy cannot be any better than the distance between the wave crests of light. One could use light with shorter wave lengths, but the greater frequency will have a greater affect on the electron that is being observed. So there is a trade-off for accuracy versus the observation method affecting the outcome of the experiment.

From these examples, it is often concluded that the uncertainty principle means the act of observing a situation alters the outcome. While this may be true in certain scenarios, the uncertainty principle says only that the future is uncertain. Such a statement is not readily accepted in the logical world of mathematics and physics without numbers to support the theory. In this case, Heisenberg was not trying to prove a certain philosophy concerning predestination, but the formula he derived from the fundamental equation of quantum mechanics has proved that future outcomes cannot be predicted with absolute certainty.

Scientifically speaking, there is no way to predict the future with complete certainty. Random walks and Brownian motion reflect the Heisenberg uncertainty principle since movement of a particle or an organization's capital can be influenced by outside factors but not specifically guided. There is still an element of uncertainty in any situation.

Fortunately, results can be estimated using probability. Probability offers a method for measuring how certain a specific outcome or series of events is. Chaos theory is a much more powerful tool because the models incorporate probability and uncertainty. These models are developed through an iterative process based on probability and uncertainty properties which produce the self-similarity that is seen in fractals. Chaos theory provides a more robust model of the world and surrounding events but as explained by the uncertainty principle, no event is completely predictable.

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