

**A Study of the Regular Polytopes:
Hypercubes, Simplexes, and Crosspolytopes**

An Honors Thesis (HONRS 499)

by

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ABSTRACT:

The main purpose of the project was to study the regular polytopes in high dimensions and discuss possible applications for the study of high dimensions. Another driving purpose of the project was to build three dimensional representations of various polytopes. Along with building models, computer aided drawings were completed.

Imagine human beings living in a cave, where they have been raised from birth. Their legs are chained so that they cannot move, and they can only see what is on the wall before them. Behind them a fire is blazing. Between the fire and the prisoners a low wall is built, like a screen. People pass along the wall carrying all sorts of materials, whose shadows appear over the wall. The chained human beings only see the shadows which the fire throws on the wall of the cave. Truth to these chained prisoners is the shadows on the wall. This is the beginning of Plato's *Allegory of the Cave* from the **Republic**, 370 B.C. The study of hypercubes, simplexes, and crosspolytopes involves a clear understanding of the concept of space and dimensions. These people lived in a world of limited freedom and dimension. Hypercubes, simplexes and crosspolytopes push these limits beyond even our three-dimensional world.

To most people, space means the moon and the stars; to the mathematician, space means dimensions. People have studied the world and its three dimensions for ages; the Egyptians and the Greeks began this study of dimensions several thousands of years ago. Students learn two-dimensional Euclidean Geometry in high school and often earlier. While mere three dimensional applications tantalize physicists and mathematicians, higher dimensions provide more to be explored, albeit difficult to fully comprehend. The fourth dimension and higher dimensions baffle many, but to the well-trained eye these are beautiful and inviting. Mathematicians try not to limit themselves; they always look to delve into

new material. The concept of dimensions above two or three is limited only by the reader's imagination and time to contemplate these *new* dimensions. A polytope is a geometric figure bounded by portions of lines, planes, or hyperplanes. One should recognize the three and two dimensional polytopes as polyhedra, and polygons, respectively. The word itself was coined by Hoppe around 1882. Other people influential to the subject include Mrs. Alicia Boole Stott (a daughter of G. Boole) around 1902. L. Schläfli began work with this concept of regular figures in high dimensions under the name of polyscheme in 1852. (Coxeter, preface)

Before describing the regular polytopes, first consider the concept of dimension. A straight jacket's design keeps a person from injuring himself or others. This garment effectively refrains one's hands and arms from any movement. Tricks to escape have been conceived, but if an extra dimension was available, escape would be simple. The jacket would then become useless with respect to its purpose. Develop a mental picture of a dog in a fenced yard. The dog cannot escape from his trap unless he uses the third dimension to jump up and over the fence. The fourth dimension rises from the third in a manner similar to the way third dimension rises from the second dimension. The third dimension contains packed slices of two space. A three dimensional sphere consists of infinitely many two-dimensional circles. Likewise, a two-dimensional is the sphere is sliced into circles by planes. From the third dimension, the fourth dimension looks like these slices except now the slices are three dimensional. This fourth dimension is in a direction mutually orthogonal to the three universal directions of length, width, and height. This is what allows the straight jacket to fall off in the fourth dimension.

An equilateral triangle has three edges, sides, of equal length and three vertices, points. Polygons include such figures as triangles, squares, and pentagons; three-sided, four-sided, and five-sided two-dimensional figures, respectively. The endpoints of edges are known as vertices. A regular polygon is defined as being both equilateral and equiangular (Jacobs, 498). Polyhedra can be defined as solids bounded by parts of intersecting planes (Jacobs, 541). The polyhedra are three dimensional polytopes. Solids have faces which consist of edges and vertices, and the collection of faces, intersecting planes, creates the solid.

The following is a proof that only five regular polyhedra exist in three space (Firby and Gardiner, 86-87). V represents the number of vertices; E the number of edges in the solid; and F the number of faces. Additionally, a is the number of edges producing a face, and b is the number of edges connecting at a vertex. For example a square has four edges to a face, $a = 4$, and two edges meet at a vertex, $b = 2$. Begin with the following two equations:

$$2E = aF \text{ ①}$$

$$2E = bV \text{ ②}$$

Two faces come together at an edge and two edges come together at a vertex, hence the number of edges have been counted twice. Transform the equations by dividing by 2 and ① becomes:

$$\frac{2}{a} E = F \text{ ③};$$

and ② becomes:

$$\frac{2}{b} E = V \text{ ④}.$$

Using the Euler characteristic formula:

$$\chi = V - E + F \text{ ⑤.}$$

Substitute ③ and ④ into equation ⑤ which yields the following equation:

$$\chi = \frac{2}{b} E - E + \frac{2}{a} E \text{ ⑥}$$

reducing to

$$\chi/2E = 1/a + 1/b - 1/2 \text{ ⑦.}$$

Topologically the surface of these figures is a sphere; therefore, substitute the Euler characteristic of a sphere, which is 2, into equation ⑦. The equation then becomes

$$1/a + 1/b = 1/2 + 1/E \text{ ⑧}$$

since the number of Edges, $E > 0$ then

$$1/a + 1/b > 1/2 \text{ ⑨.}$$

If a and b are chosen such that the equation is satisfied, the combination defines a three dimensional regular polyhedron.

Examine all possibilities for a and b of equation ⑨, where $a, b \geq 3$:

If $a = 3$ and $b = 3$, then $1/3 + 1/3 = 2/3 > 1/2$.

This produces the Tetrahedron.

If $a = 3$ and $b = 4$, then $1/3 + 1/4 = 7/12 > 1/2$.

This produces the Octahedron.

If $a = 3$ and $b = 5$, then $1/3 + 1/5 = 8/15 > 1/2$.

This produces the Dodecahedron.

If $a = 3$ and $b \geq 6$, then $1/3 + 1/b \leq 1/2$.

This combination is not possible.

If $a = 4$ and $b = 3$, then $1/4 + 1/3 = 7/12 > 1/2$.

This produces the Cube.

If $a = 4$ and $b \geq 4$, then $1/4 + 1/b \leq 1/2$.

This combination is not possible.

If $a = 5$ and $b = 3$, then $1/5 + 1/3 = 8/15$ since $> 1/2$.

This produces the Icosahedron.

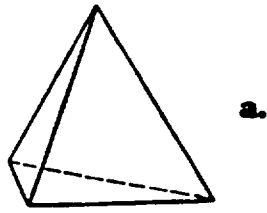
If $a \geq 5$ and $b \geq 4$, then $1/a + 1/b \leq 9/20$.

This combination is not possible.

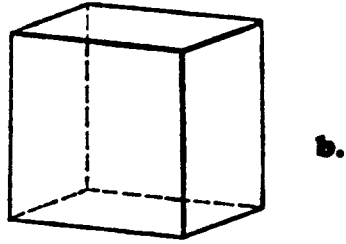
This exhausts the possible combinations, thus the cases yield the five platonic solids: tetrahedron, octahedron, dodecahedron, cube, and icosahedron {Fig. 1 a-e}.

The previous proof shows the number of regular solids in 3-space. It has been shown by H. S. M. Coxeter (Coxeter: Regular Polytopes, 134) that in 4-dimensions there are exactly 6 regular polytopes and then in dimensions higher than 4 there are exactly 3. The Tesseract, which is now known as the 4-Cube or the Hypercube, 4-Simplex, 4-Crosspolytope, 24-cell, 120-cell, and 600-cell form a complete list of the regular polytopes in the fourth dimension. The Hypercube is analogous to the cube, while the

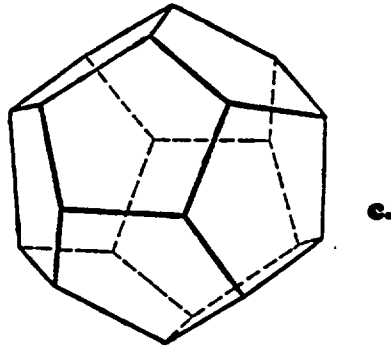
Tetrahedron



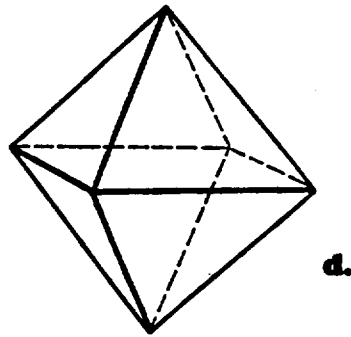
Cube



Dodecahedron



Octahedron



Icosahedron

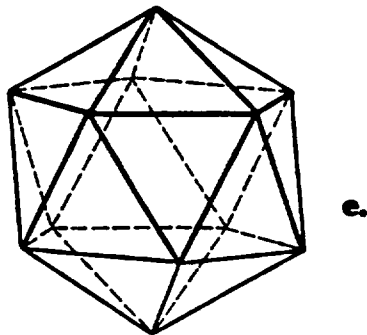


Figure 1

Simplex is from the family of the tetrahedron, and the Crosspolytope derives from the Octahedron. The 24-, 120-, and 600-cell polytopes have no three dimensional relatives. The 5-Hypercube, 5-Simplex, 5-Crosspolytope are the only regular polytopes in the fifth dimension. In the n th dimension the regular polytopes are the n -cube, n -simplex, and the n -crosspolytope.

The cube family is built through what is referred to as *motions*. If one examines the process for drawing a square, regular polygon, generalizations can be made for all hypercubes. Begin with two parallel line segments one unit long and one unit apart. Line A has endpoints a and b , and line B has endpoints c and d . The motion connects corresponding vertices through adding an edge from a to c and an edge from b to d . These two new edges are perpendicular to the original edges. Take two parallel squares and connect corresponding vertices, as in the previous example. This motion produces the cube. The next step takes a leap of faith since it involves the understanding of higher dimensions. Continue the motion method by taking two parallel cubes and connect corresponding vertices with edges {Fig. 2}. This pattern could continue until the artist's mind became too cluttered to decipher the number of dimensions. Mathematically the motion process continues indefinitely.

The coning process describes the method of building the simplex family. As the name implies, the process closely resembles building a cone. Begin with a vertex. Place another vertex one unit away and connect them with an edge. Cone the line with another vertex, by adding edges from the original vertices to the new vertex, one unit away from the original vertices. The resulting figure will be an equilateral triangle. As

before choose a new vertex one unit away from each existing vertex and cone to the new vertex. This pushes up another dimension and goes into the third dimension, creating a regular tetrahedron. Another coning produces a regular 4-simplex. Use the existing 4 vertices and connect with edges to another vertex so that all the edges are of equal length {Fig. 3}. The coning method can also continue indefinitely.

The process of suspension is used to build the crosspolytope family from the octahedron. This uses the concept of suspending an n dimensional crosspolytope in $n + 1$ dimension and adding connecting edges from the n dimensional crosspolytope to the two new vertices in the $n + 1$ dimension. For example, suspend a square between a pair of vertices, with a vertex above and one below. This suspension process produces an octahedron. Now suspend the octahedron between two new points, remembering that the distance between each pair of vertices is equal, otherwise they would not be regular crosspolytopes {Fig. 4}. This new suspension represents the 4-crosspolytope. As before, the process could continue indefinitely.

As a nature of the models, there is a distinct loss of regularity in the representations of the higher dimensional regular figures in two and three dimensions. "Of course, from a theoretical point of view, illustrations are superfluous and no kind of visual aids can alter the fact that a hypercube is nothing but an abstract notion. But illustrations help us to get a more concrete idea of these notions and thus they have great heuristic value." (Toth, 129) To the three dimensional world the fourth dimension is unattainable except through the mind, abstraction. The two and three dimensional models are merely shadows of what would be there in the

Figure 2

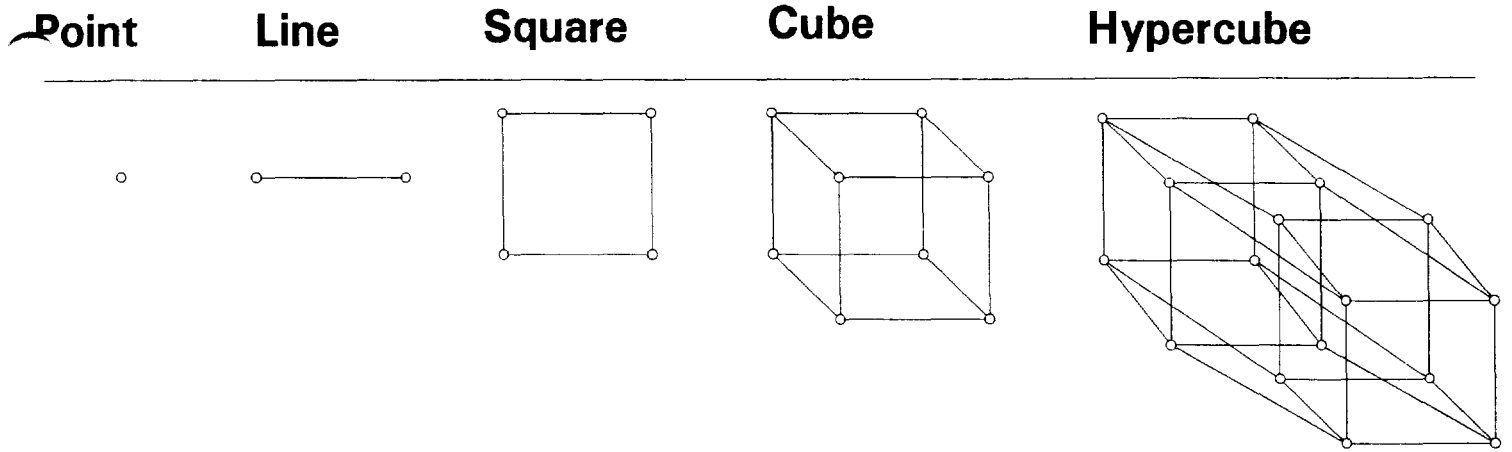


Figure 3

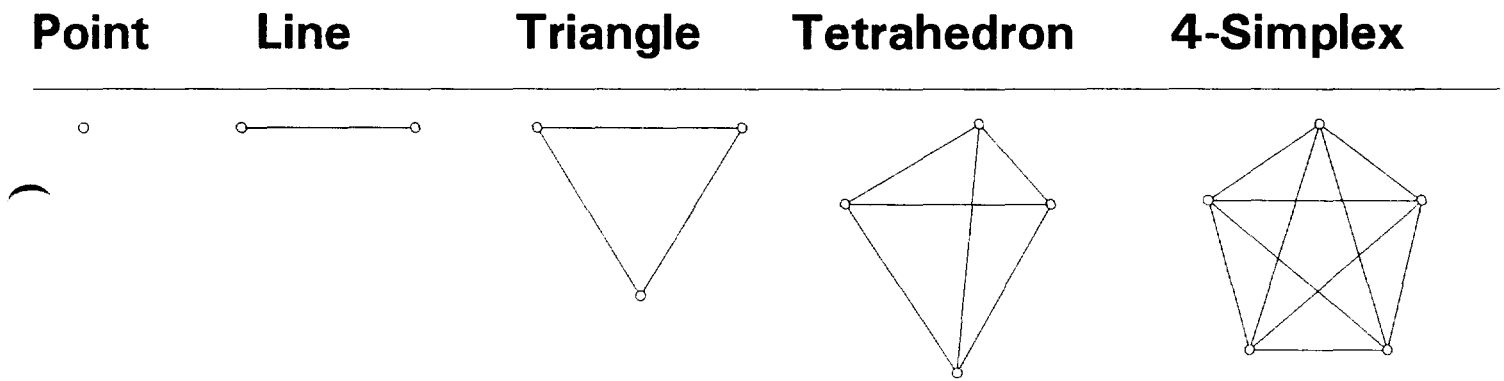
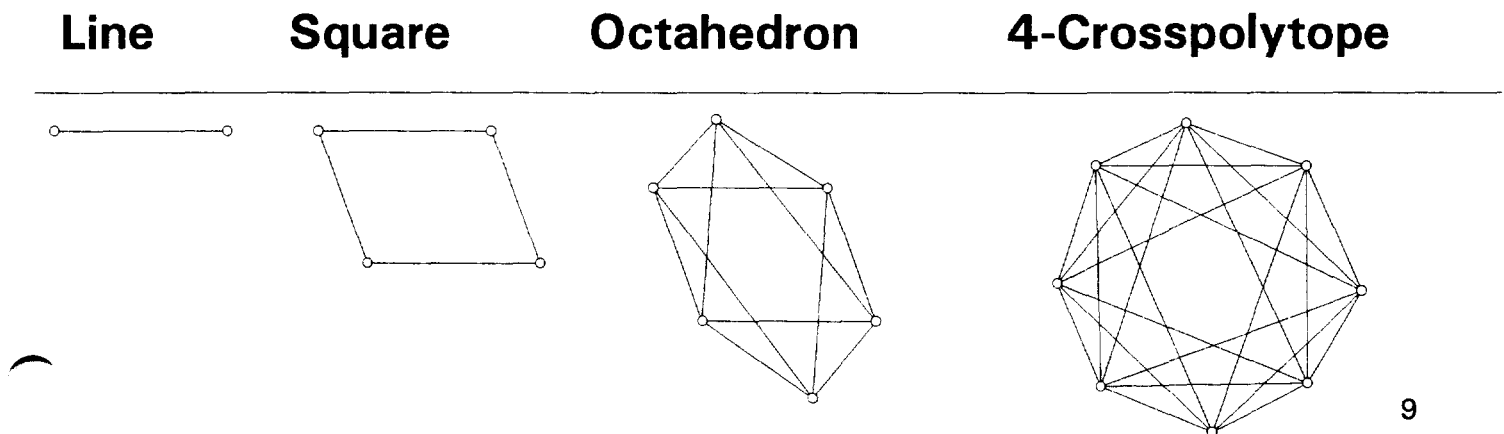


Figure 4



fourth or other dimension, just as the two dimensional figures of the regular polyhedra are shadows. Artists study this concept of impossible figures without realizing the existence of the underlying mathematics.

Mathematics exhibits many patterns. Often these patterns show up in nature. Crystalline structures represent naturally occurring polyhedra. After a short study of regular polytopes it seems appropriate to display some pattern information about the regular polytopes. The formula for calculating the number of vertices of a regular n -cube is relatively simple and straight forward. Each step adds twice as many vertices as before, hence 2^n . Likewise the formula for calculating the number of vertices of a simplex is also simple. Each step adds only one new vertex, hence $n + 1$. Similarly the vertex formula for the crosspolytopes is found by adding two new vertices at each step, hence $2n$. The formulas for the edge count are more detailed. For the cube the formula is

$(n) 2^{(n-1)}$. The simplex formula represents the triangular numbers $\frac{n(n+1)}{2}$.

This also happens to be the sum of the first n numbers. The third term in the sequence is three and is the addition of $1 + 2$, the fifth term is 10 the sum of

$0 + 1 + 2 + 3 + 4$. A proof by mathematical induction proves this formula true for all integers, n . The crosspolytope formula is $2(n-1) n$. These formulas show the pattern and development in further dimensions of the regular polytopes.

Dimension	0	1	2	3	4	5	6	7	n
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Cube

Vertices	1	2	4	8	16	32	64	128	2^n
Edges	0	1	4	12	32	80	192	448	$(n) 2^{(n-1)}$

Simplex

Vertices	1	2	3	4	5	6	7	8	$n + 1$
Edges	0	1	3	6	10	15	21	28	$\frac{n(n+1)}{2}$

Crosspolytope

Vertices	0	2	4	6	8	10	12	14	$2n$
Edges	0	0	4	12	24	40	60	84	$2(n-1) n$

Mathematics encompasses a variety of topics. The world can be modeled in the language of mathematics. Patterns for example represent a piece of that language. Dimensions conjure up different thoughts for people. Regular polytopes should now come to mind. All of the higher dimensional polytopes are physically impossible in our boring 3-dimensional world, yet seem to naturally unfold in higher dimensions just

like the straight jacket did. What if the dog had a four dimensional backyard? A Dutch artist once wrote:

Drawing is deception; it suggests three dimensions when there are but two! And no matter how hard I try to convince you about the deception, you persist in seeing three-dimensional objects.

--M.C. Escher

Studying the regular polytopes and higher dimensions can lead to interesting discoveries in the field of applied mathematics.

Some mathematicians believe that all mathematics will have value in the real world; regular polytopes have some application outside of the academic community. The word *application* means the capacity of being usable, relevance. Certain topics in mathematics have found applications. Others are considered pure or theoretical in nature and exist only to the academician. It is of interest then to ask what is the motivation for studying mathematics?

Is the goal of mathematics to create or discover strictly for the beauty of mathematics? Of course mathematicians believe in the beauty of mathematics, but some only comprehend the usefulness of those topics which can readily be made *applied*. Studying higher dimensions falls in the middle of theory and application. The abstract nature of the subject defines it as theoretical, while there are some applications in use today. The following examines applications of high dimensions and a use for studying regular polytopes.

Science fiction writers and movies often hinder the public's interpretation of dimensions past three. These situations make time travel possible and allow terrible earth shattering events to occur. Think of the

television shows that aired not long ago. *Quantum Leap* is one of the strangest space-time-jumping shows. Another example of this would be *Star Trek Generations*. Understanding higher dimensions and polytopes could come from a variety of other places however.

Early this century a man named Albert Einstein developed a new theory, the Theory of Relativity. One that still has the world shocked. The earth is a three dimensional box, how could there exist a space-time continuum? As a result of Einstein's work, most people now relate the fourth dimension with time. This is one simple way to think of it, but there is much more involved in high dimensions than just length, width, height, and time. Humans are trapped in an existence that will not allow us to see the fourth dimension as a whole, and since humans are questioning creatures, they cannot easily or blindly accept the concept of higher dimensions.

The justification for studying higher dimensions varies. The mere beauty of the subject is not a valid justification to some, but for those who demand it, applications do exist. Some *real life* examples of the uses of higher dimensions include networking problems, communication specialists looking for the tightest packing solution, astrophysicists examining patterns in star clusters, quantum theory, Quasi-crystals and art. Architects and artists have studied the fourth dimension. Salvador Dali depicted the crucifix as an unfolded hypercube. In Physics the derivative of acceleration gives jerk. Many other fields utilize the study of these higher dimensions.

Quantum mechanics, for example, uses the idea of the fourth dimension in the discussion of quantum numbers. The atoms are organized by electron configuration and labeled using four quantum

numbers. These four designations are: n -- principle quantum number, l -- orbital quantum number, m -- magnetic quantum number, m_s electron spin. The Pauli Exclusion principle governs the electron configuration and states that no two electrons in an atom can be in the same quantum state; no two electrons can have the same set of values of the quantum number n , l , m , m_s . The quantum numbers represent the energy levels of the atoms.

Ecologists use more than three dimensions to study the Structure of the Ecosystem. A niche is referred to as an n dimensional hypervolume gradient. It is n dimensional based on the number of variants to study, such as temperature, moisture, food supply, and any other environmental variables. Sociologists use dimensions to examine sociological multivariants. Any system that has more than three independent variables is representative of a system using an n dimensional model.

In mathematics different voting methods can be analyzed using n -cubes. In health one can talk about the six dimensions of wellness. These are simple applications, but they greatly enhance the solution to the problem. Models are used in applications to make the problem more understandable and easier. Therefore the study of higher dimensions is worth while for a variety of reasons.

Mathematics truly does encompass a variety of topics. Galileo said, "Mathematics is the language with which God wrote the Universe." The world can be modeled in the language of mathematics. Through further research and study even more topics will become available to the reaches of mathematics. In particular, the study of higher dimensions through hypercubes, crosspolytopes, and simplexes will encourage growth in the before mentioned applications and add new topics to the list. To the

three dimensional world the fourth dimension is unattainable except through the mind, abstraction. Eventhough the two- and three-dimensional models are merely shadows of their fourth dimensional figures, these provide mathematicians with tools to study. Imagination is the only limit to these explorations.

AFTERWORD

This project began as a two semester Undergraduate Research Fellowship at Ball State University. The research has spanned nearly two years. Geometry has always been an interesting topic to me. Our Mathematics Education curriculum requires only two Geometry classes, but I opted to take an additional course, Surface Topology. This piqued my interest in a variety of topics, but definitely in the study of higher dimensions. Since that class I have built several three dimensional models and drawn two dimensional sketches of the various regular polytopes. The sketches were done using two software packages: *Cabri-Geometry* and *Geometer's Sketchpad*. This construction formed the crux of the project, but hours of research preceded the physical fruits of my labor. This thesis documents my research before, during, and after the construction of the models in what I hope is a concise and readable format. I have had the pleasure of presenting my work during four conferences, once at Miami University, twice at Ball State University, and once at Hillsdale College. These preparation periods helped me decide to write this thesis. It was very good experience to be in front of an audience speaking about a topic they might know only very little about. I want to thank the Ball State Mathematical Sciences Department for all the help and support that I received during the Fellowship. I would also like to thank my Fellowship advisor who is doubling as my Thesis advisor, Dr. John Emert. He has a true love of the subject of Mathematics and a talent in the art of educating.

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Appendix A: Cube Family

Transparency: Corresponding connections	A-1
Two squares: One step of the motion process to a cube	A-2
Transparency: Corresponding connections	A-3
Two cubes: One step of the motion process to a hypercube	A-4
Cube	A-5
Hypercube	A-6--A-8
5-Cube	A-9
2 Hypercubes: One step of the motion process to the 5-cube	A-10
4-Hypercubes: One step of the motion process to the 6-cube	A-11
2 5-Cubes: One step of the motion process to the 6-cube	A-12
8 Hypercubes: One step of the motion process to the 7-cube	A-13
4 5-Cube: One step of the motion process to the 7-Cube	A-14

Appendix B: Simplex Family

Coning of a triangle to a tetrahedron	B-1
Tetrahedron (3-simplex)	B-2
Coning of a tetrahedron to a 4 5-Simplex	B-3
4-Simplex	B-4--B-5
5-Simplex	B-6--B-7
6-Simplex	B-8--B-9
7-Simplex	B-10--B-11

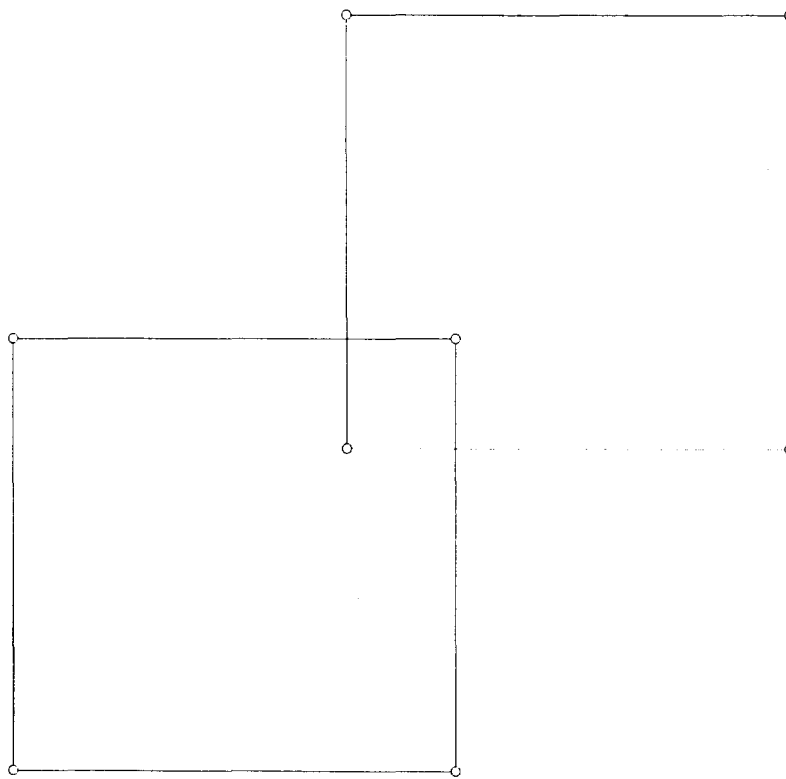
Appendix C: Crosspolytope Family

Suspension of a square producing the octahedron	C-1
Octahedron (3-Crosspolytope)	C-2--C-3
Suspension of a octahedron producing the 4-crosspolytope	C-4
4-Crosspolytope	C-5
5-Crosspolytope	C-6
6-Crosspolytope	C-7
7-Crosspolytope	C-8--C-9

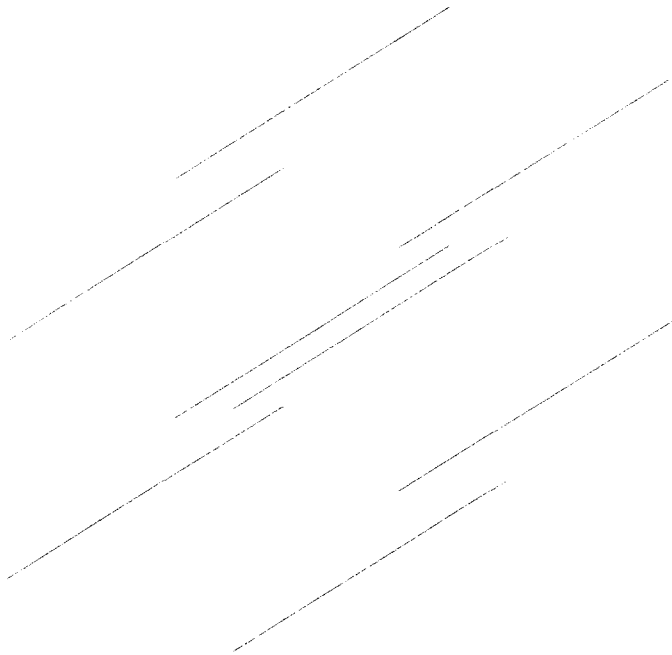
Transparency: Corresponding connections



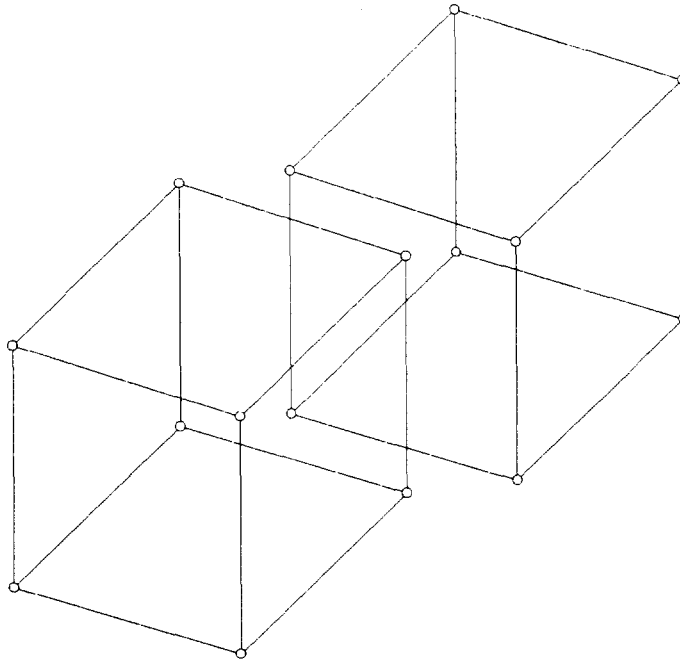
Two squares: One step of the motion process to a cube



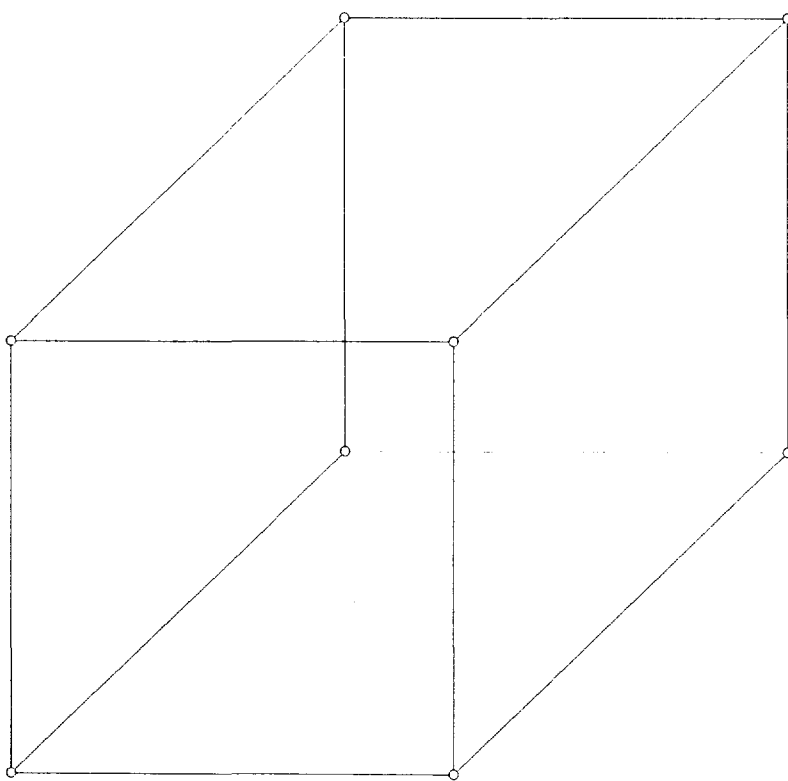
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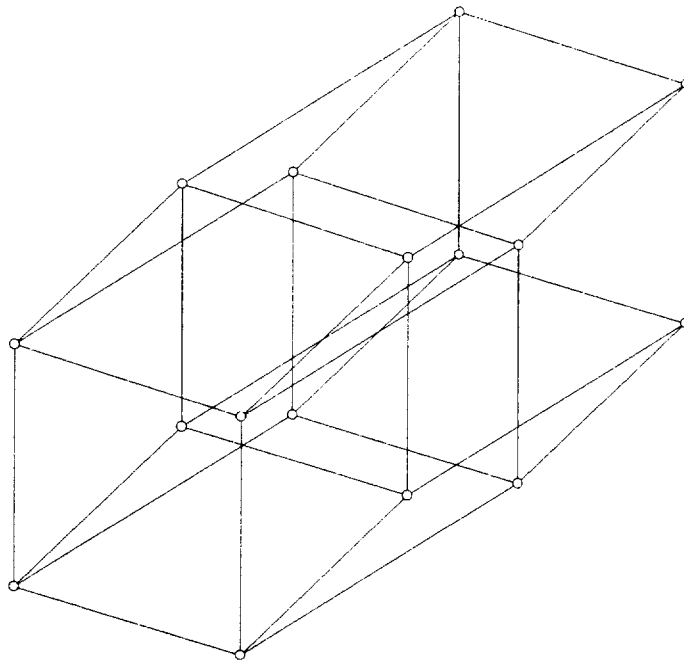
Two cubes: One step of the motion process to a hypercube



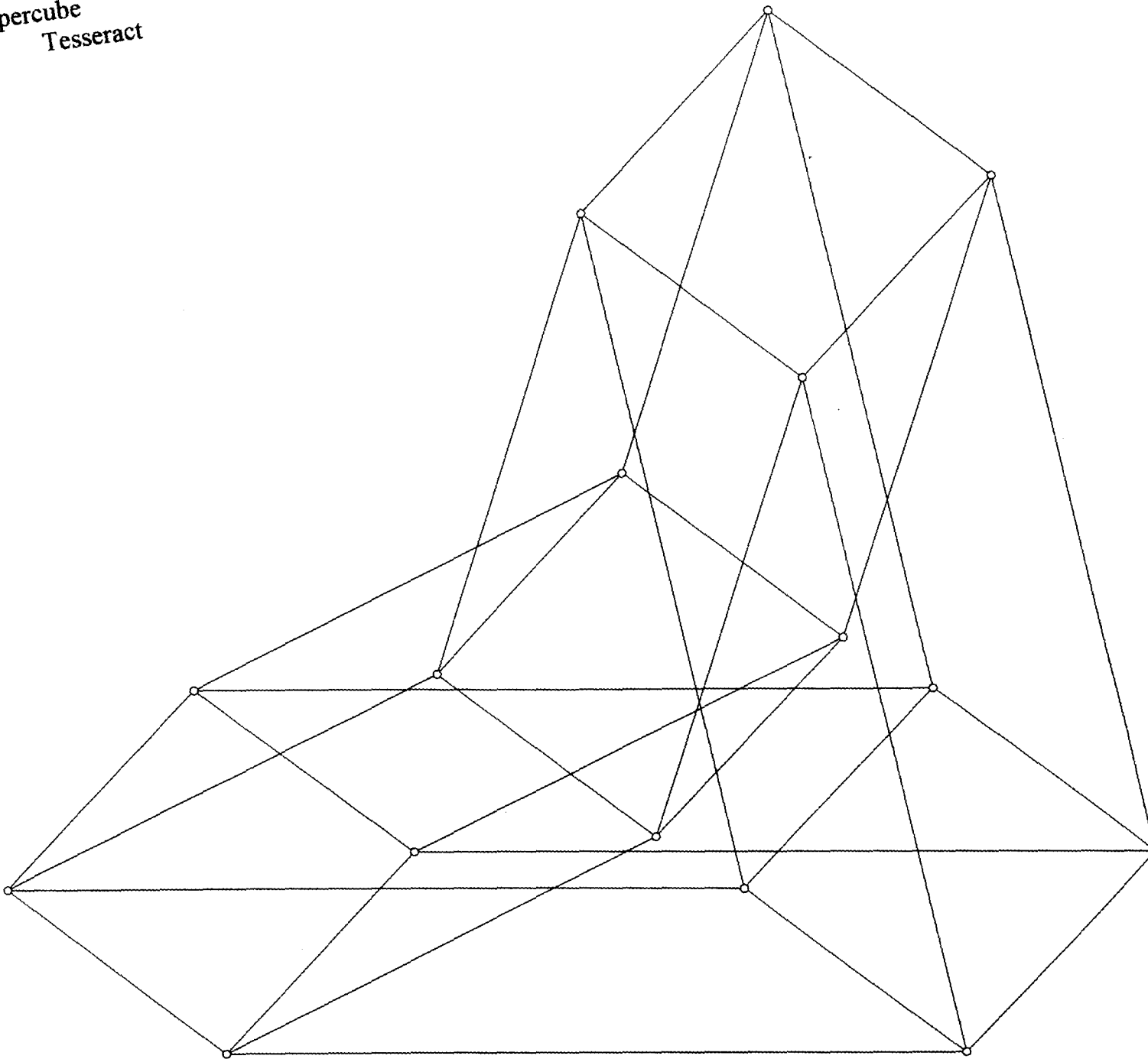
Cube



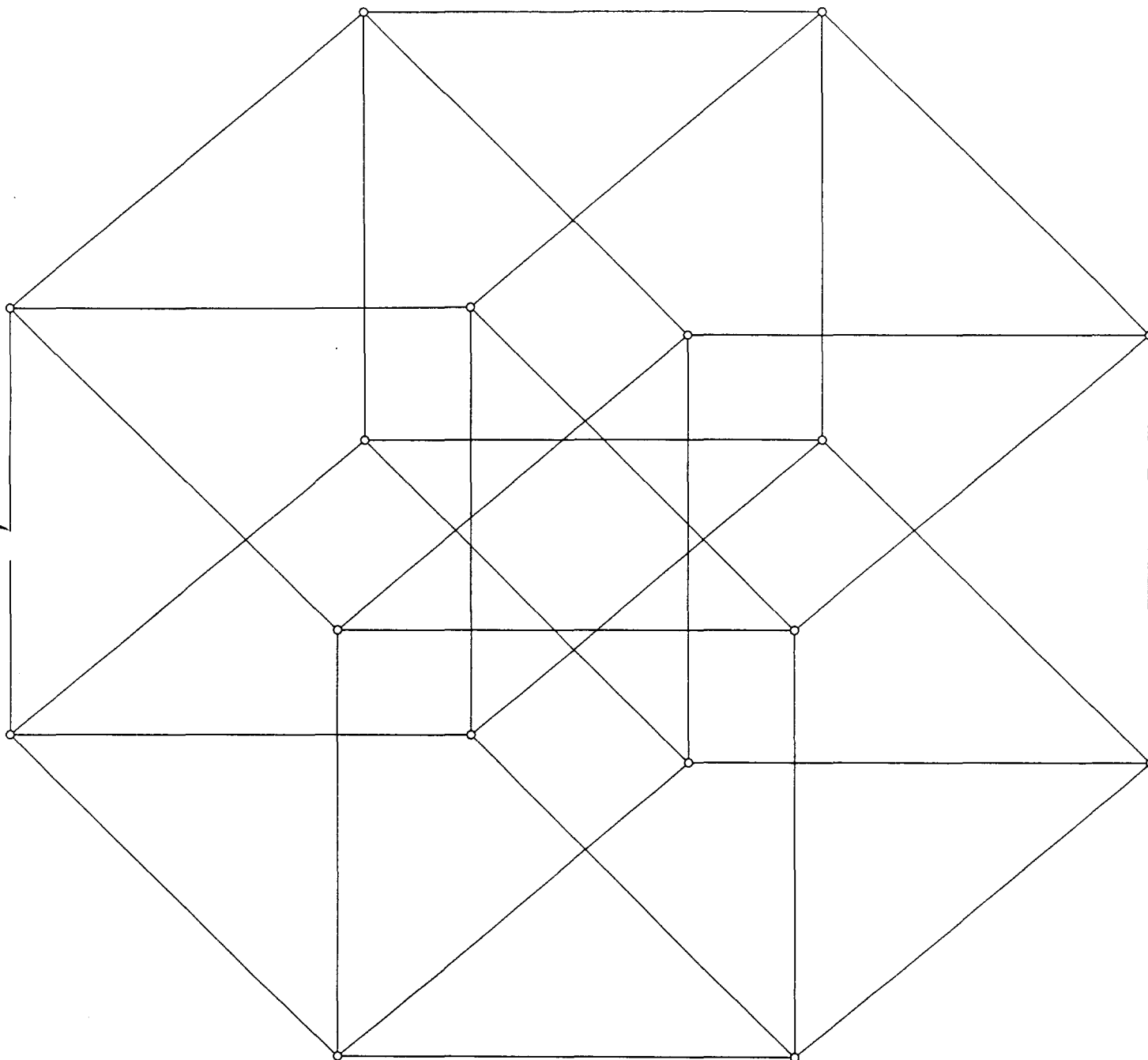
Hypercube



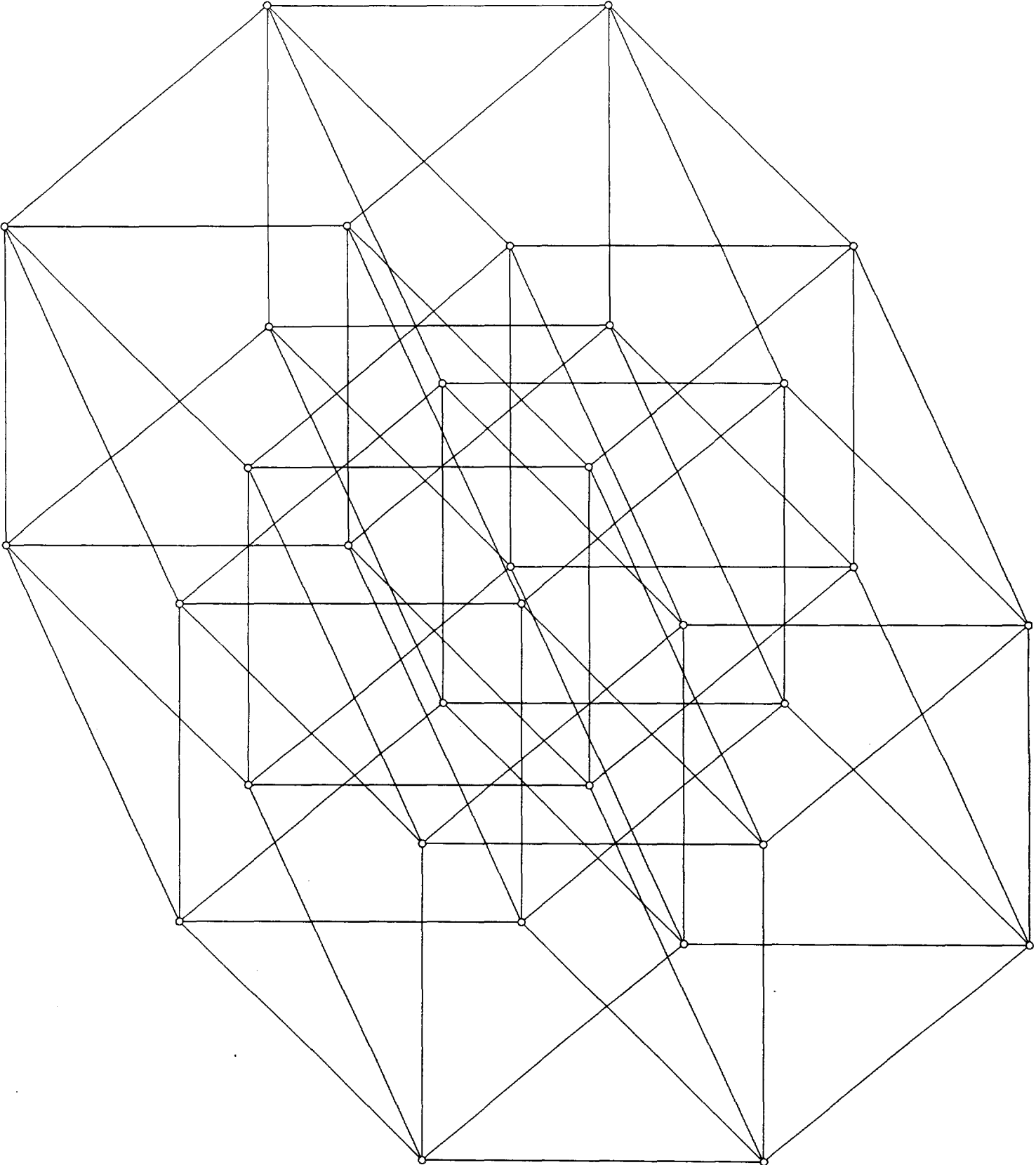
Hypercube
Tesseract



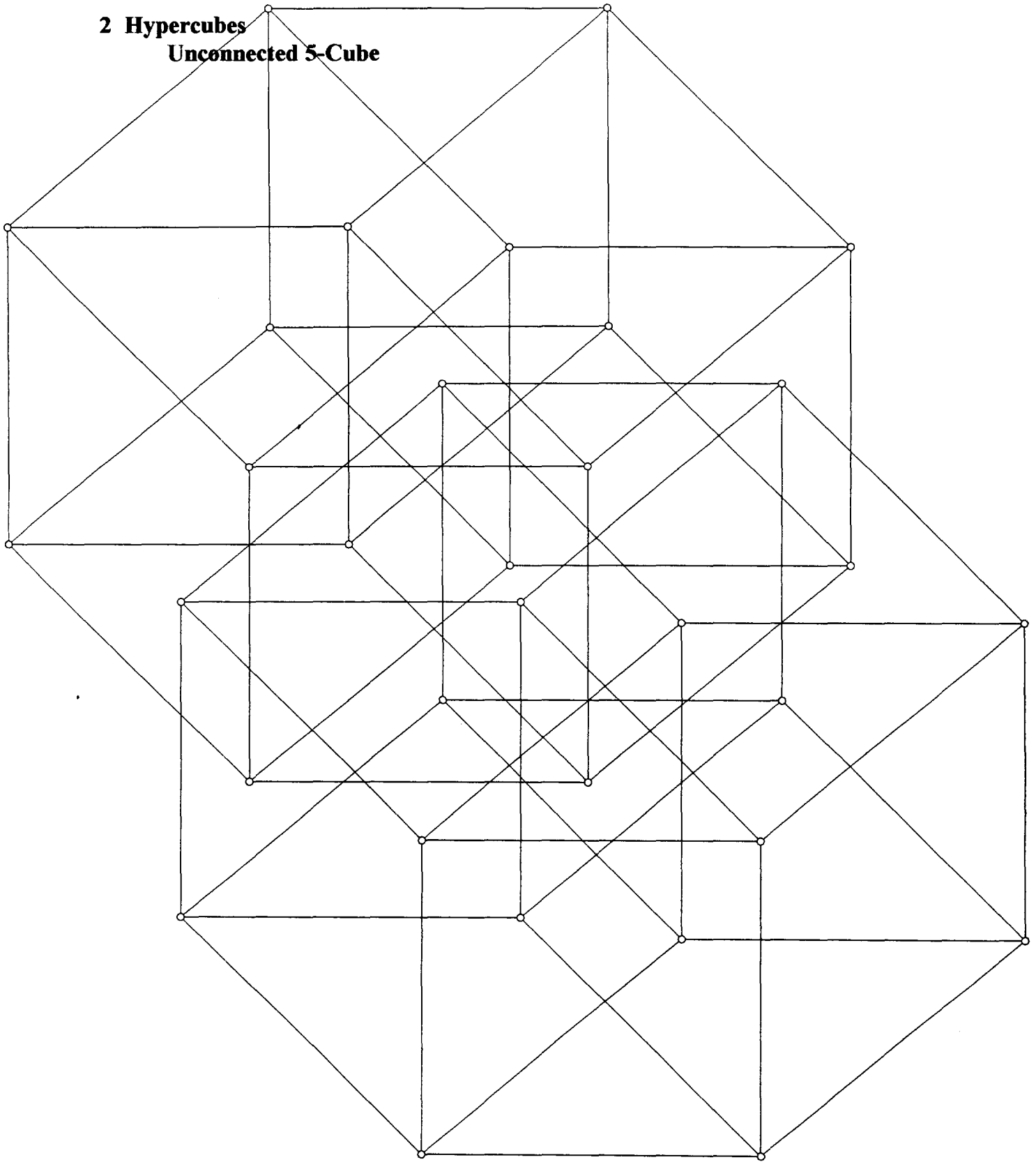
Hypercube
Tesseract



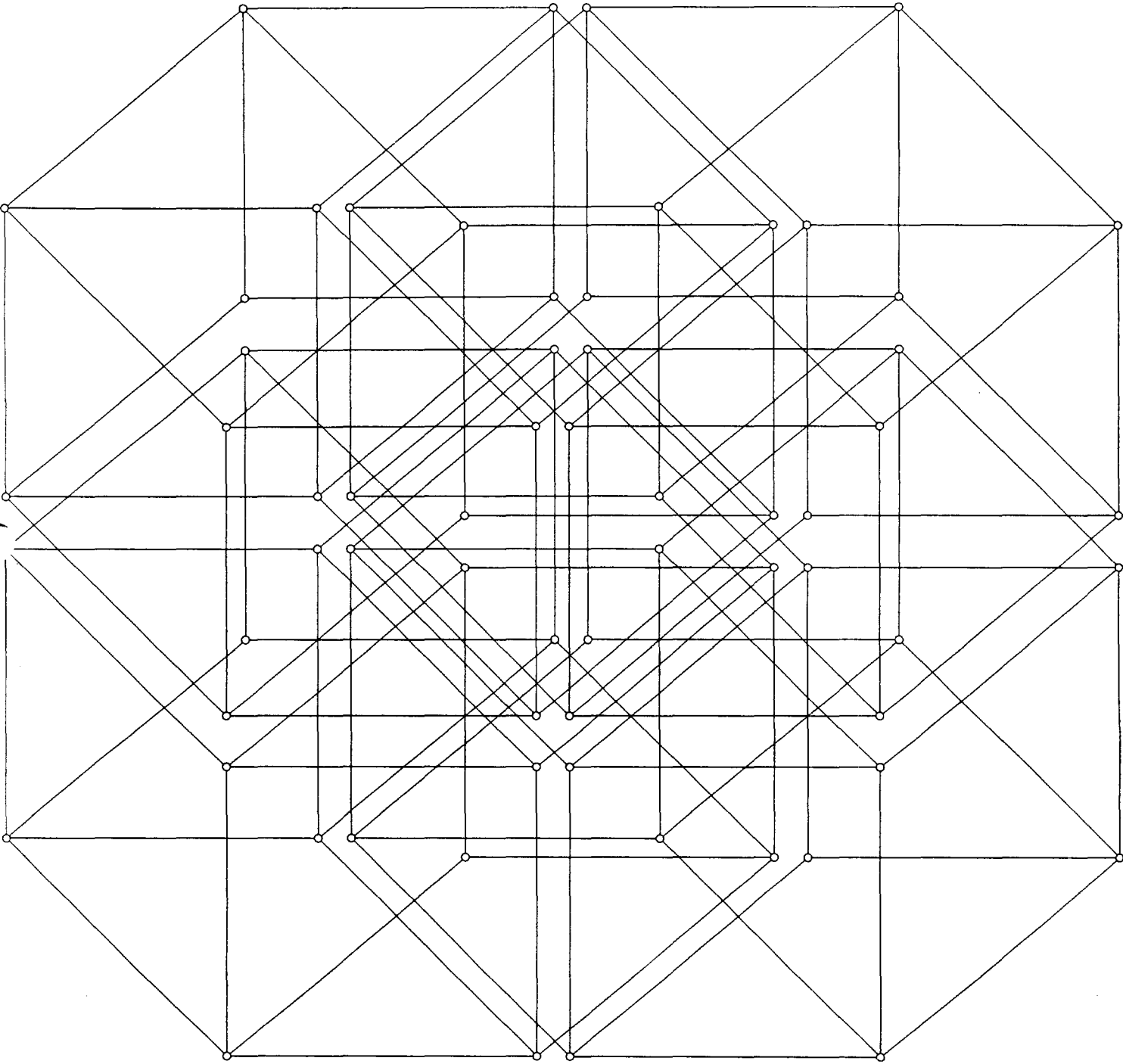
5-Cube



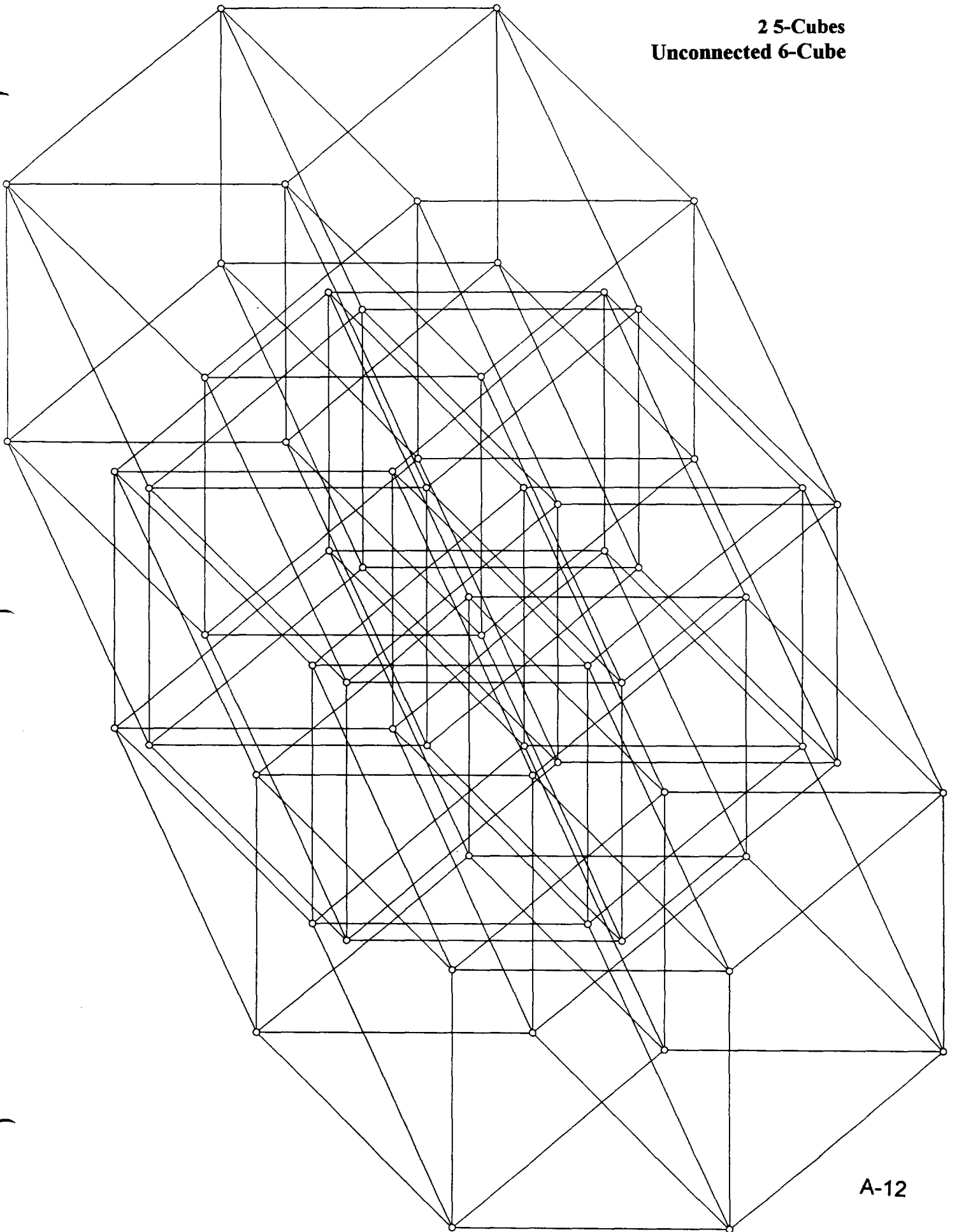
2 Hypercubes
Unconnected 5-Cube



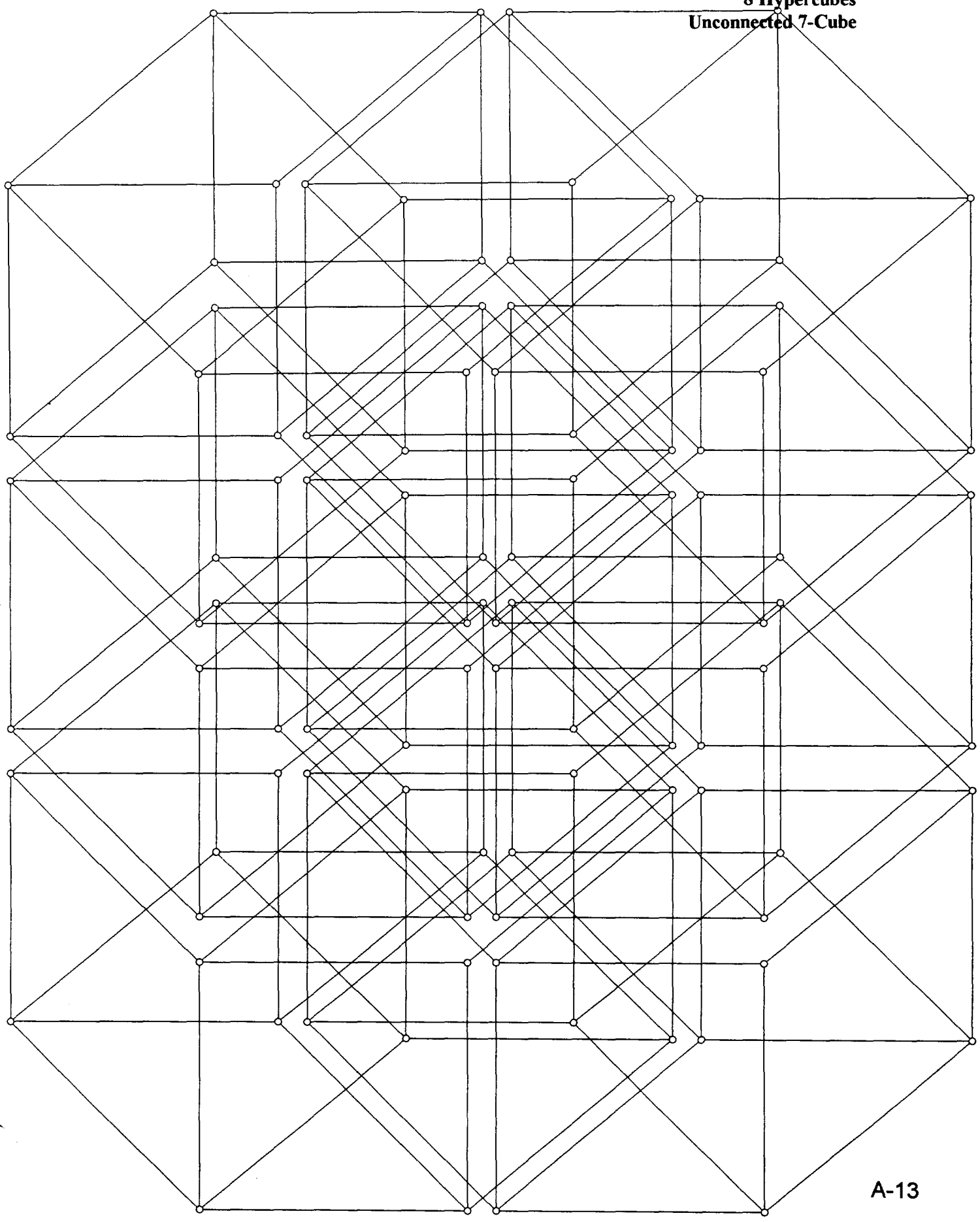
**4 Hypercubes
Unconnected 6-Cube**



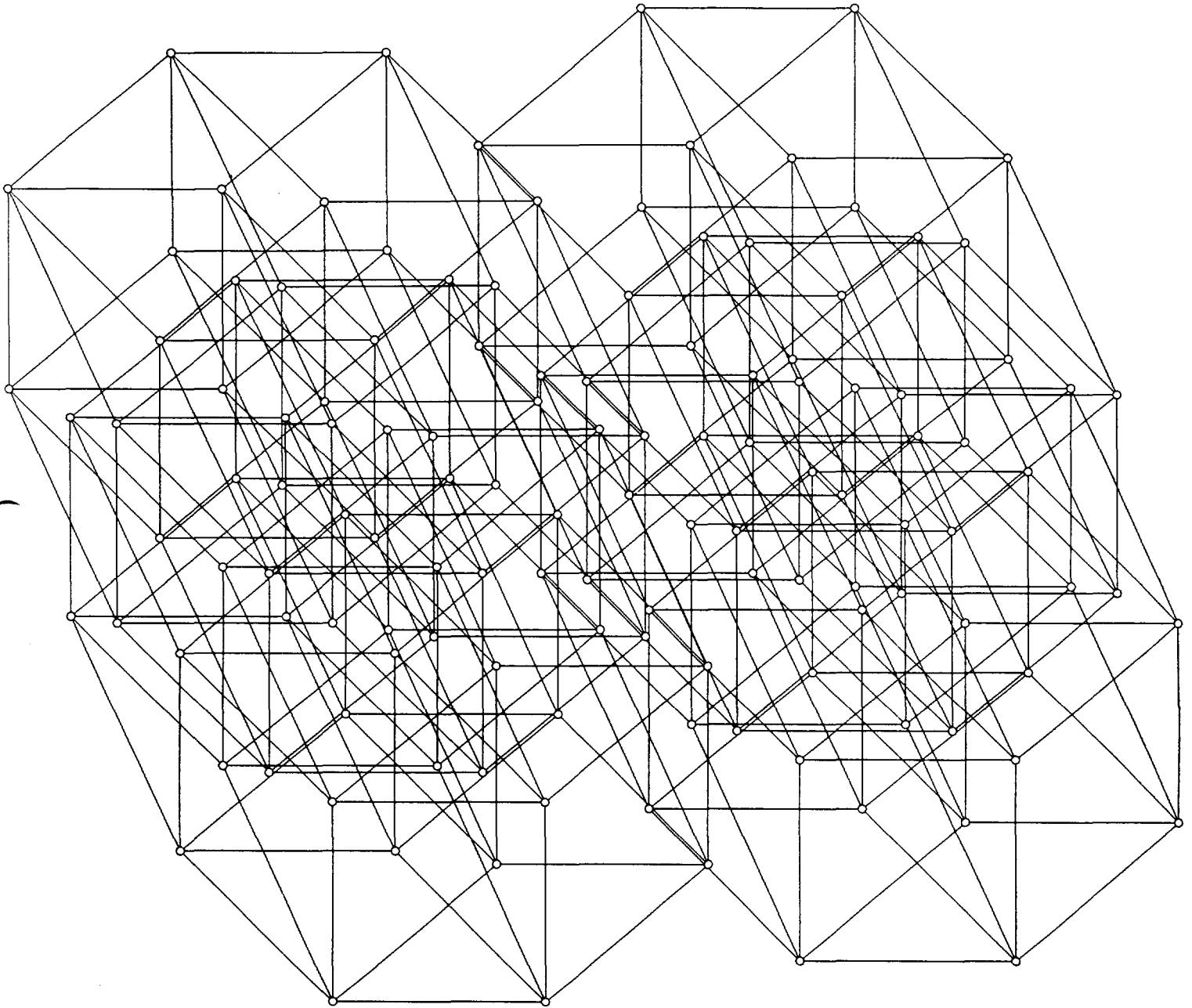
**2 5-Cubes
Unconnected 6-Cube**



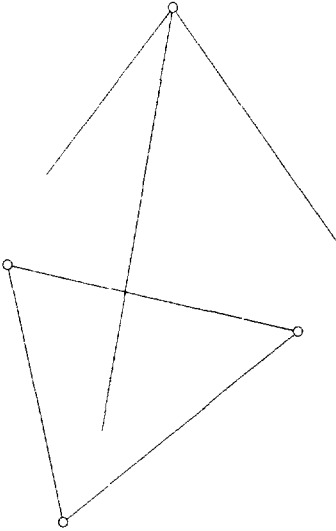
8 Hypercubes
Unconnected 7-Cube



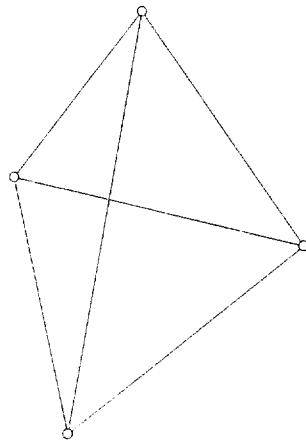
4 5-Cubes
Unconnected 7-Cube



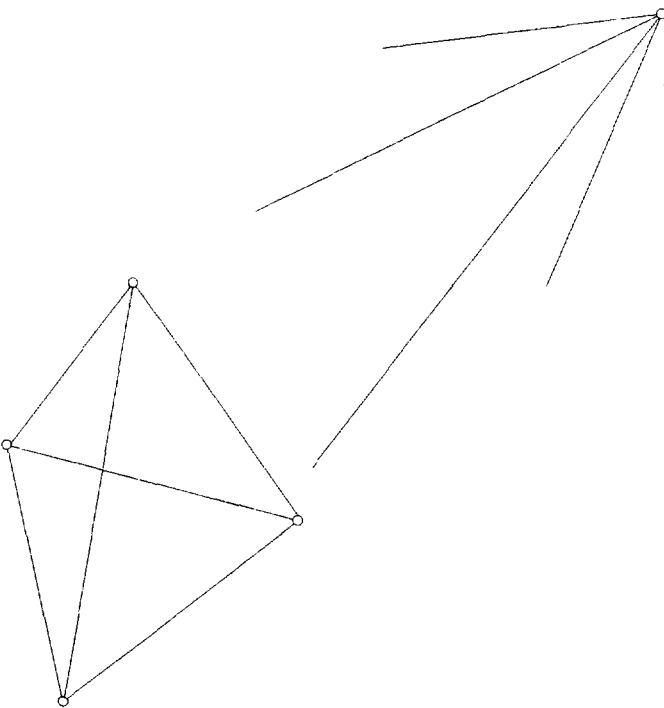
Coning of a Triangle to a Tetrahedron



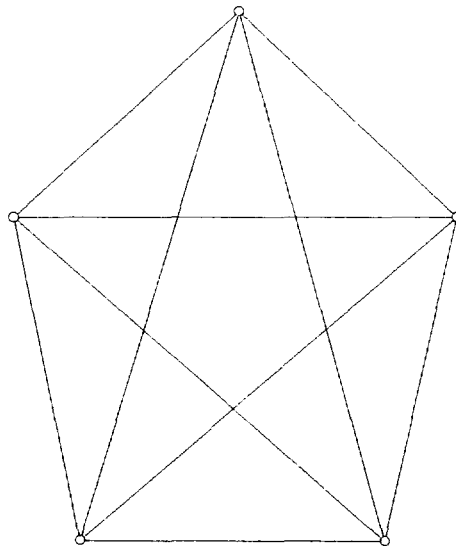
Tetrahedron (3-simplex)



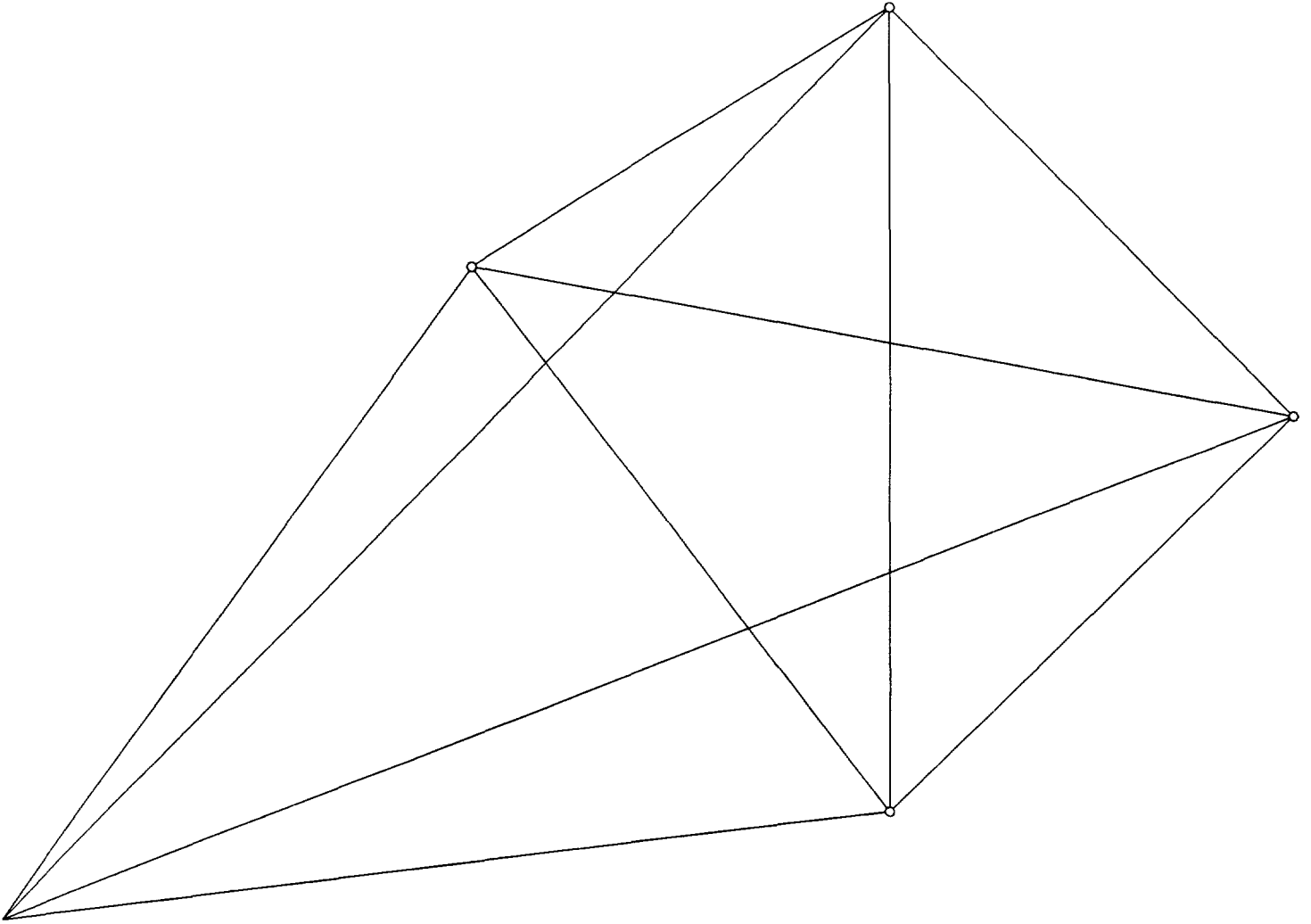
Coning of a Tetrahedron to a 4-Simplex



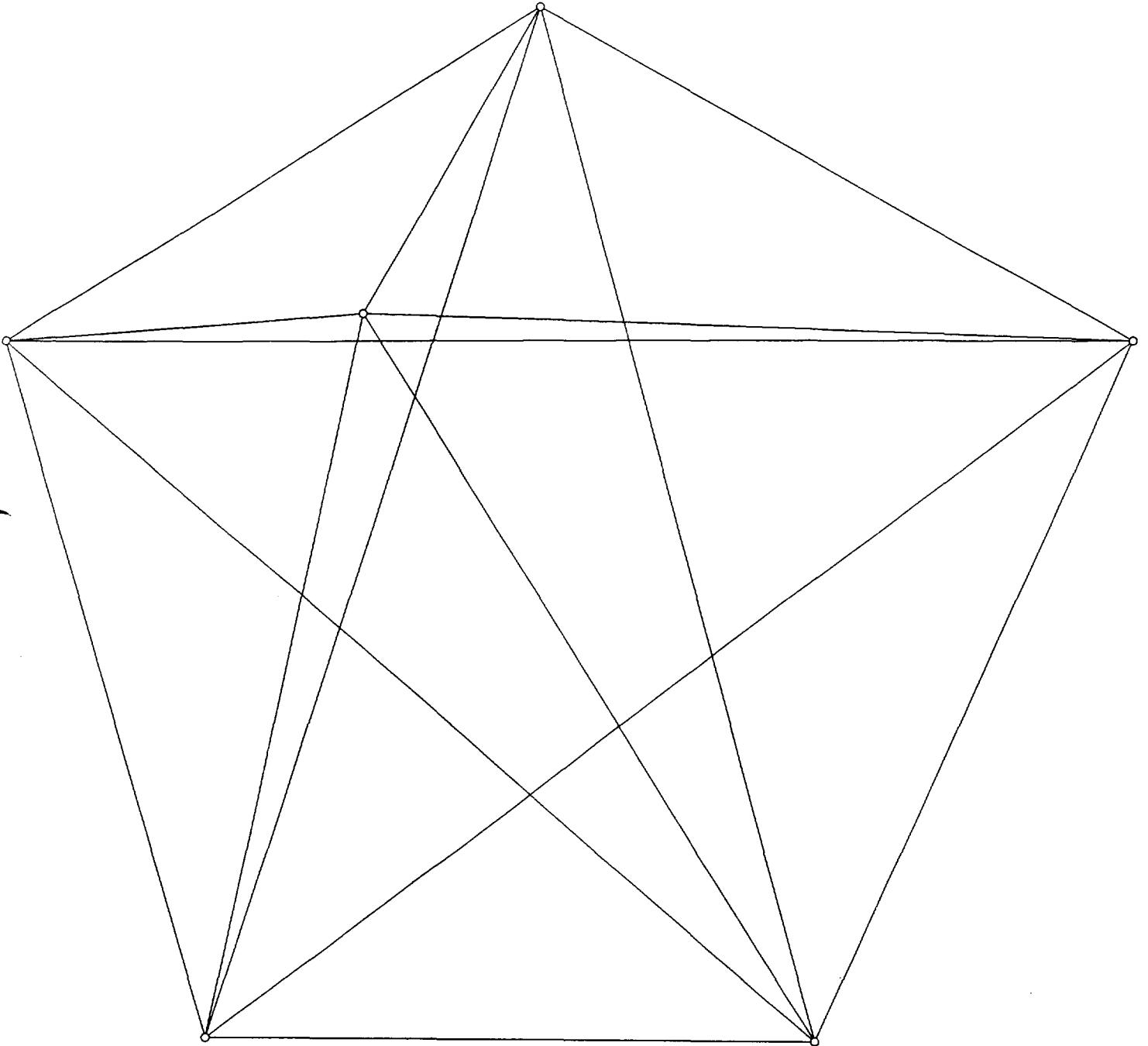
4-Simplex



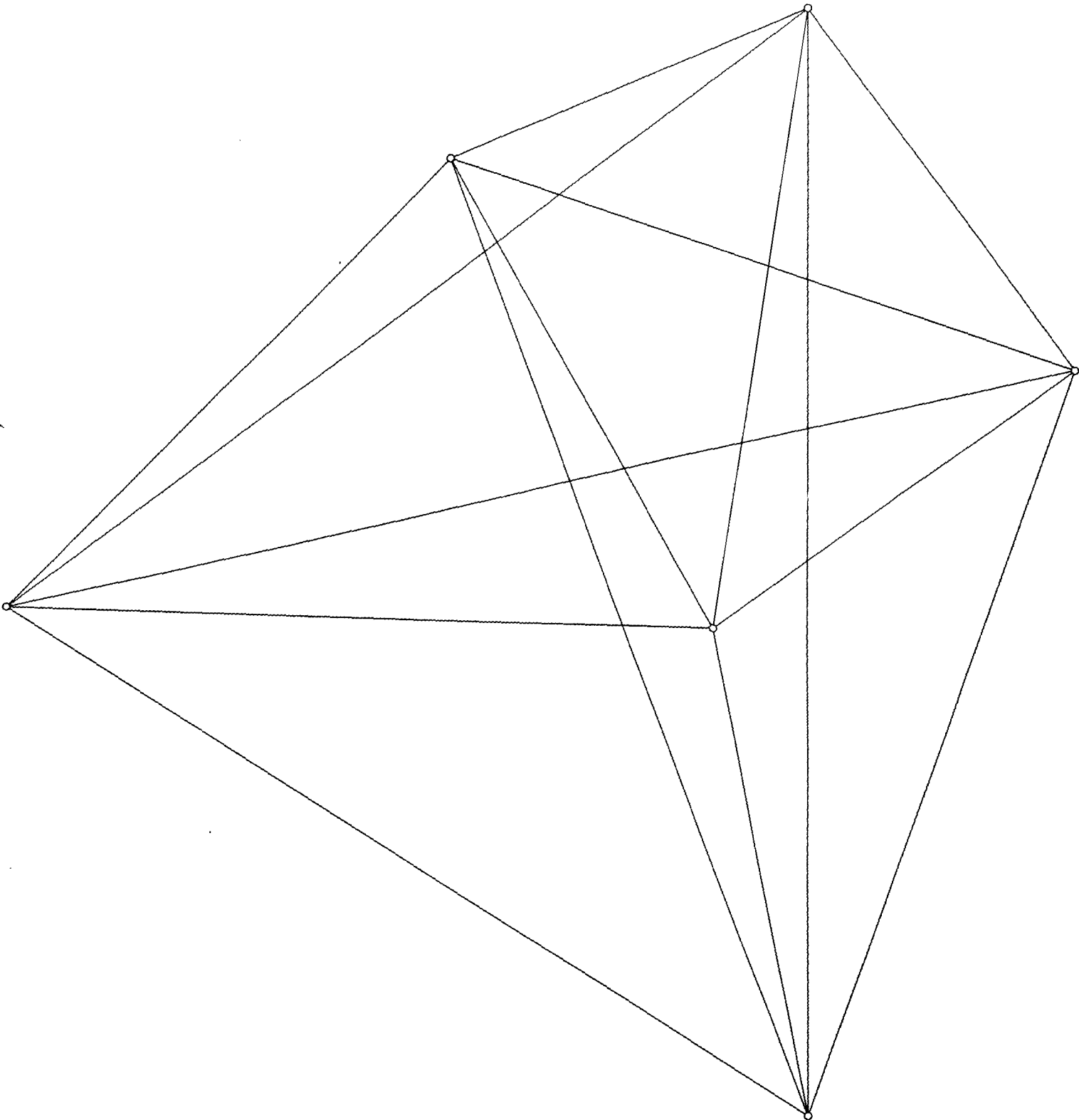
4-Simplex



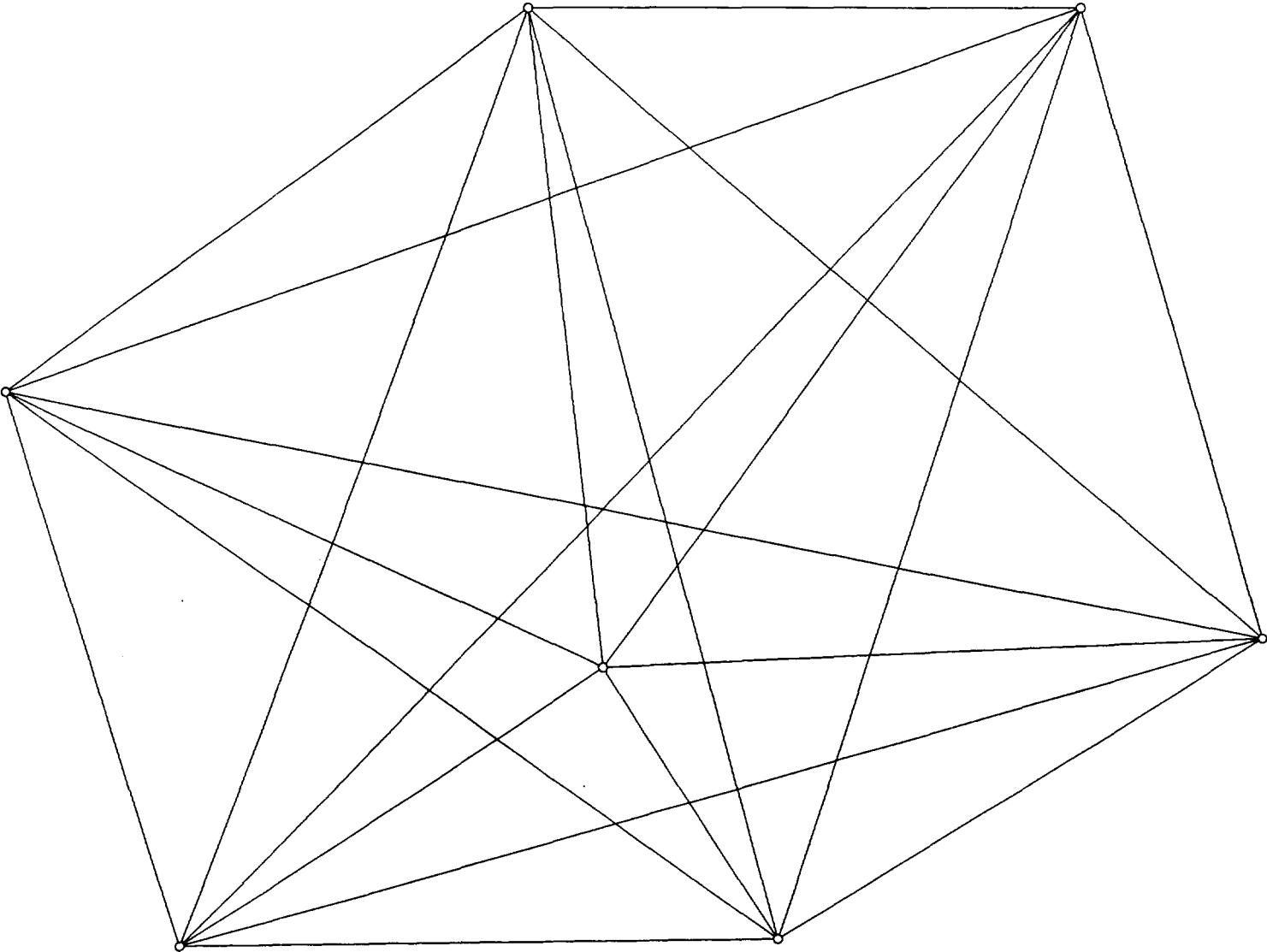
5-Simplex



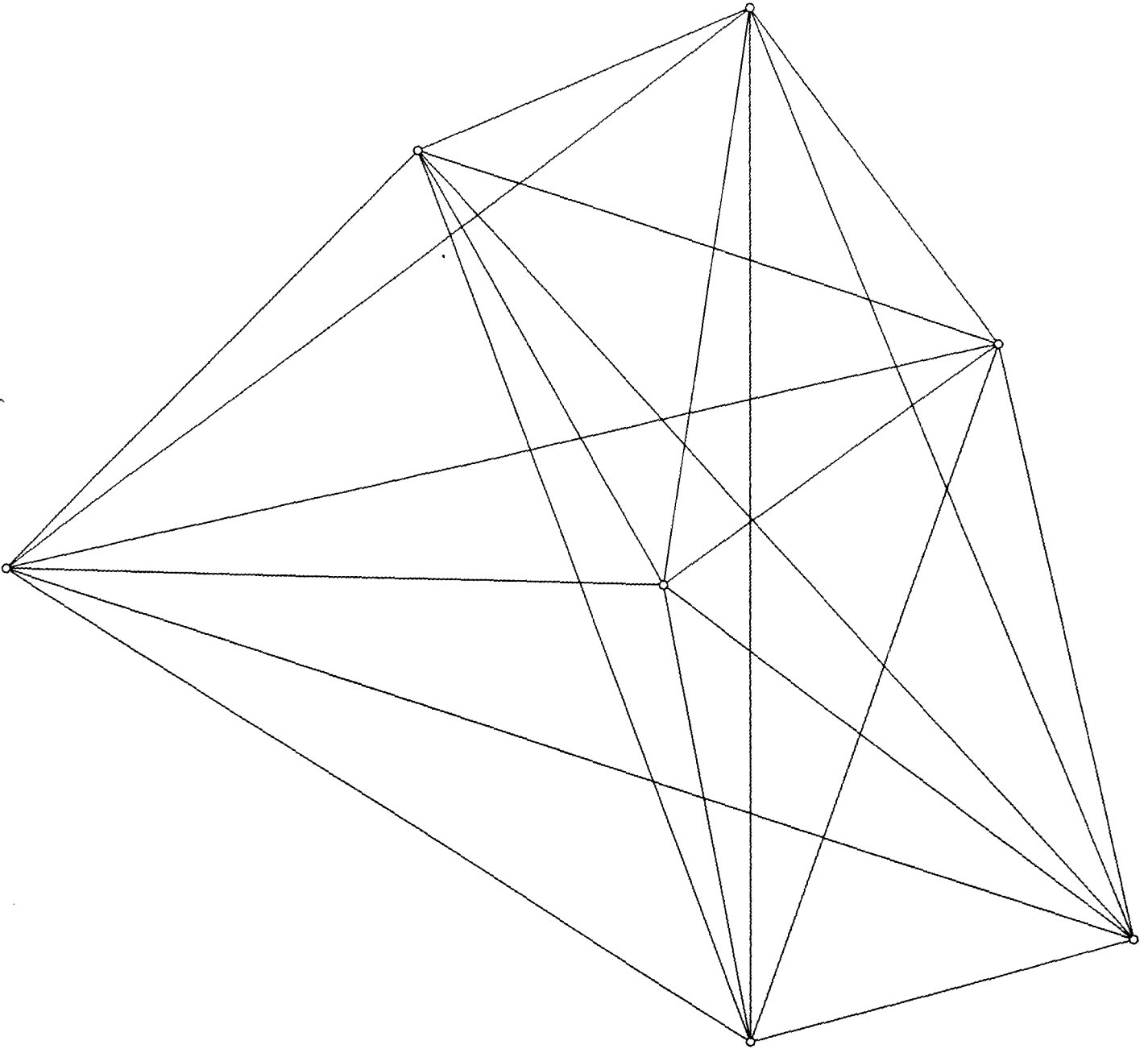
5-Simplex



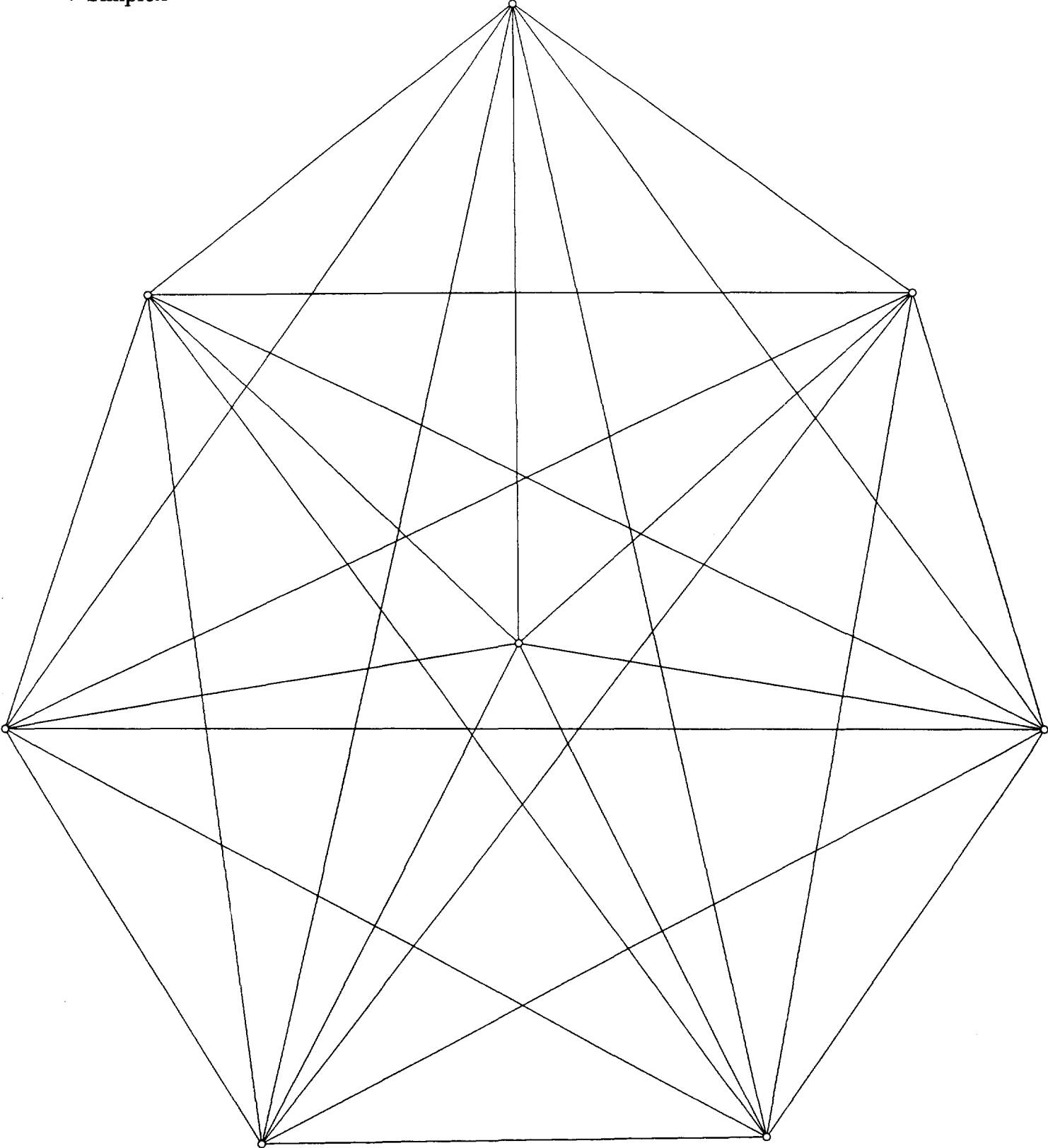
6-Simplex



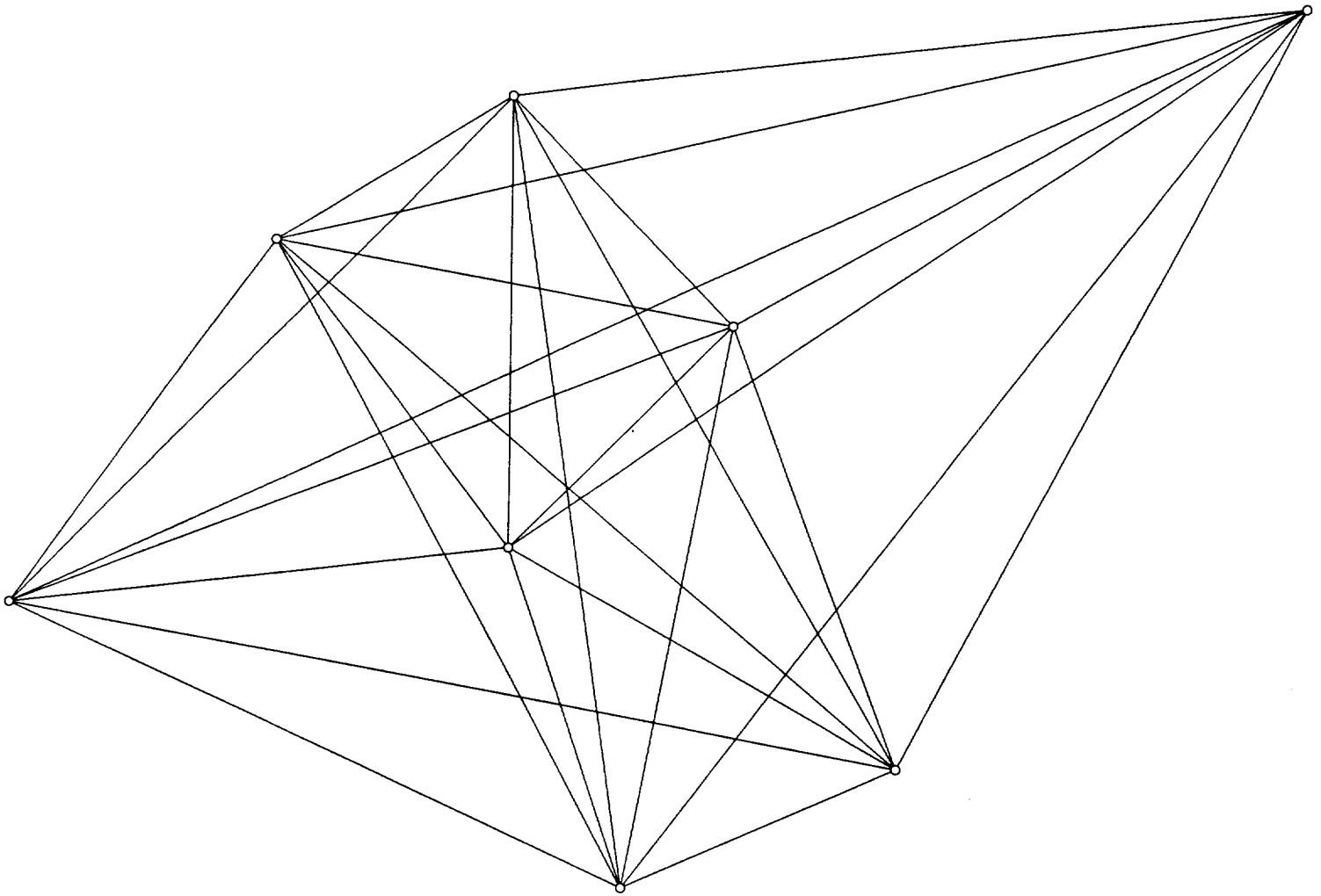
6-Simplex



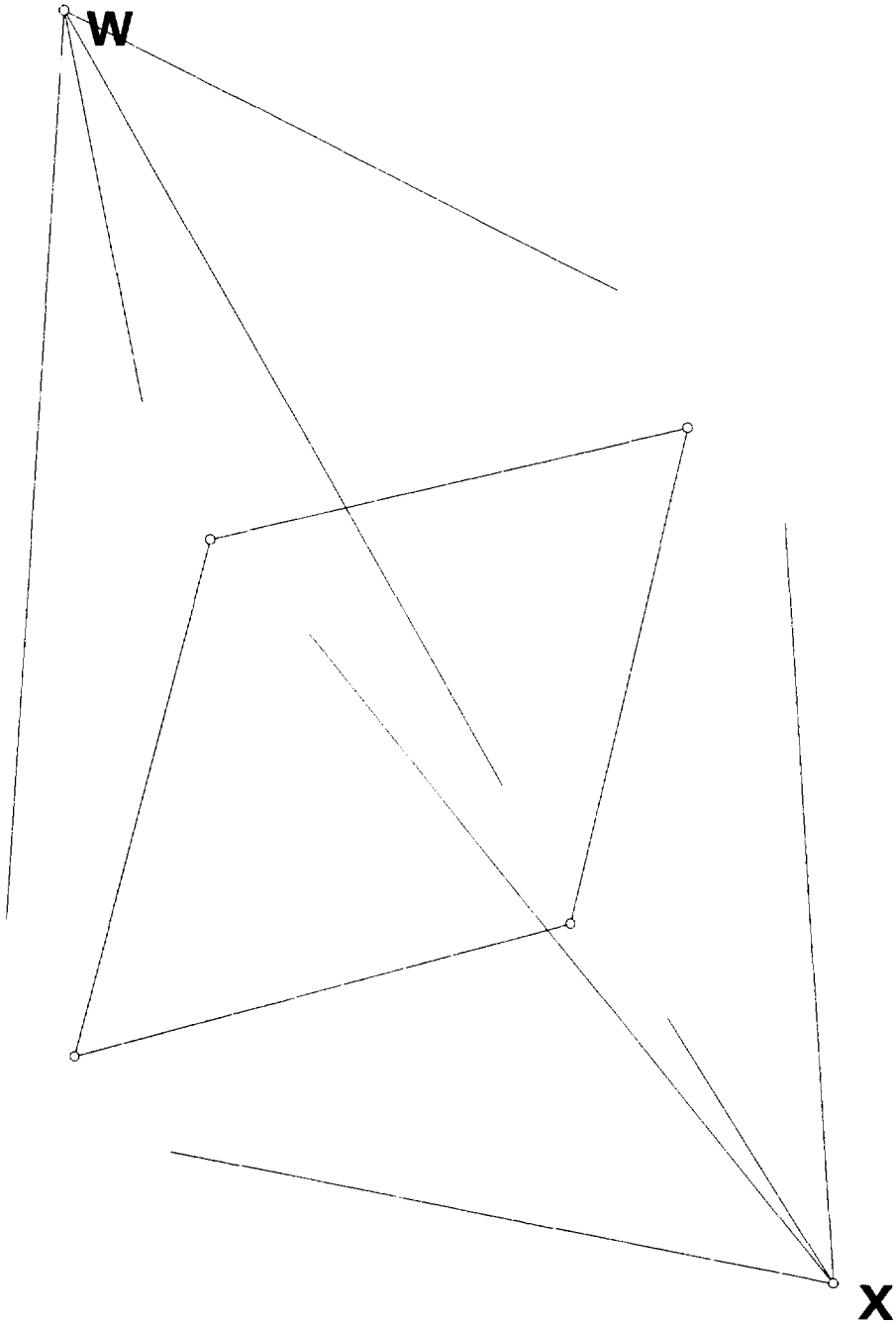
7-Simplex



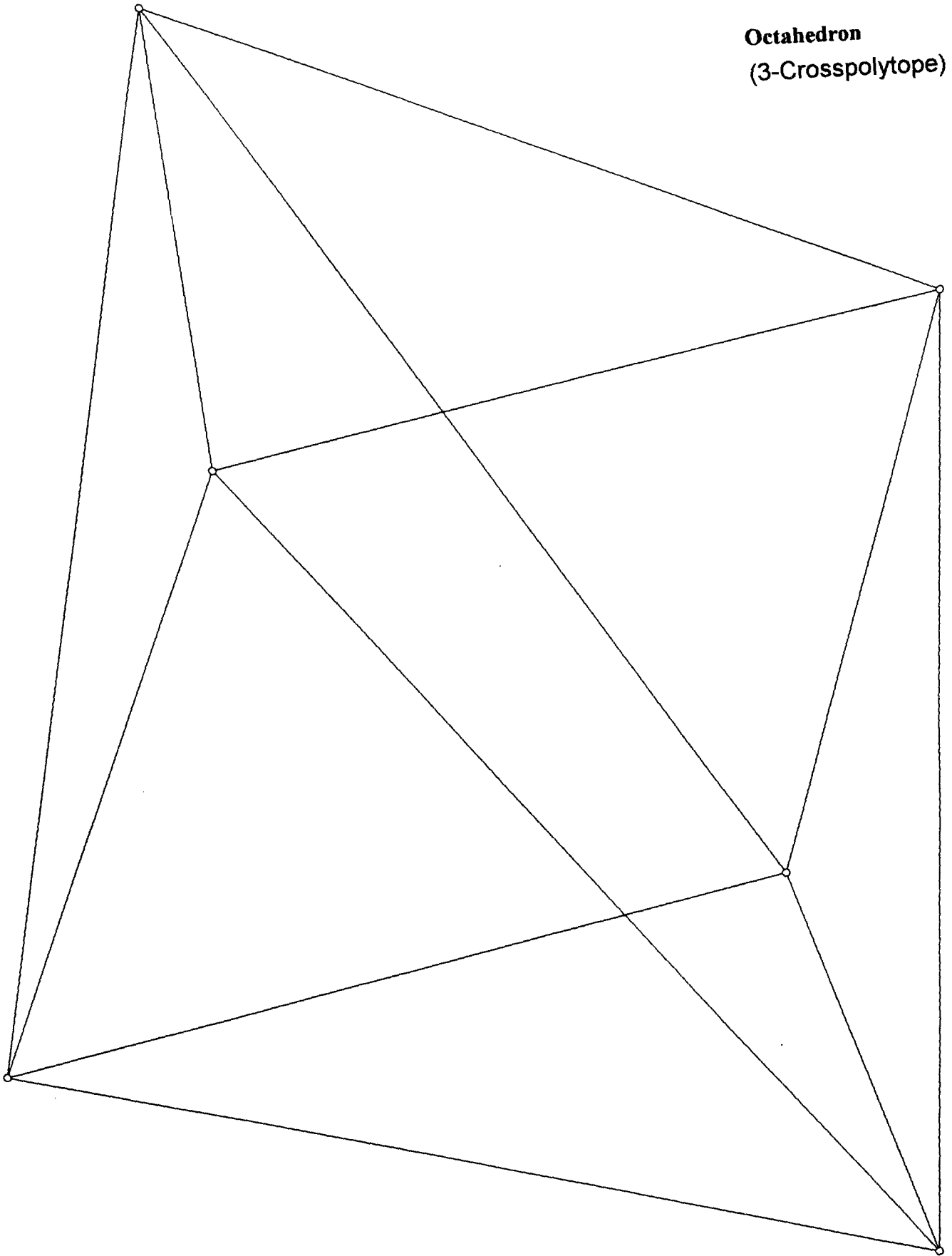
7-Simplex



Suspension of a Square to Octahedron

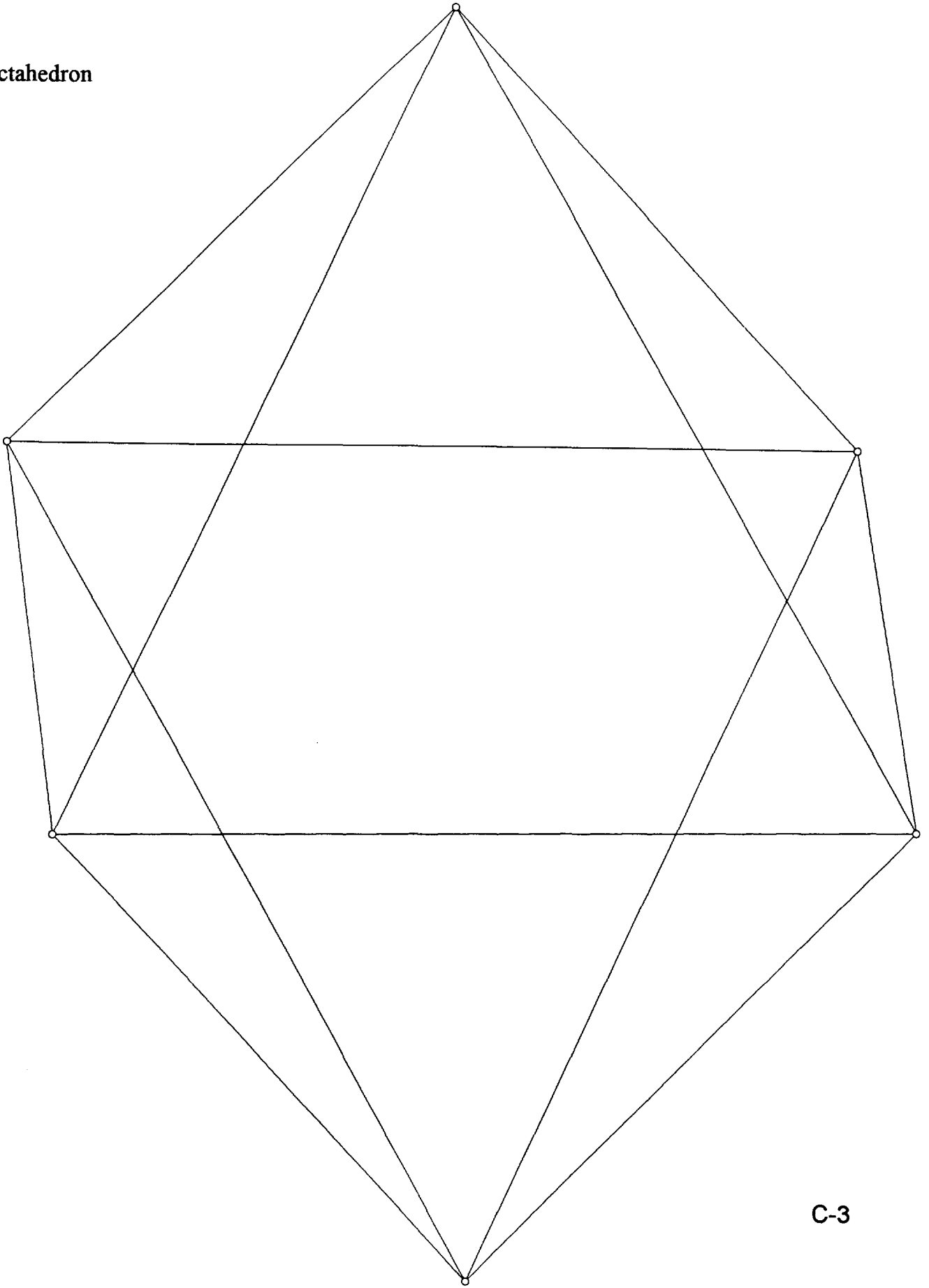


Octahedron
(3-Crosspolytope)



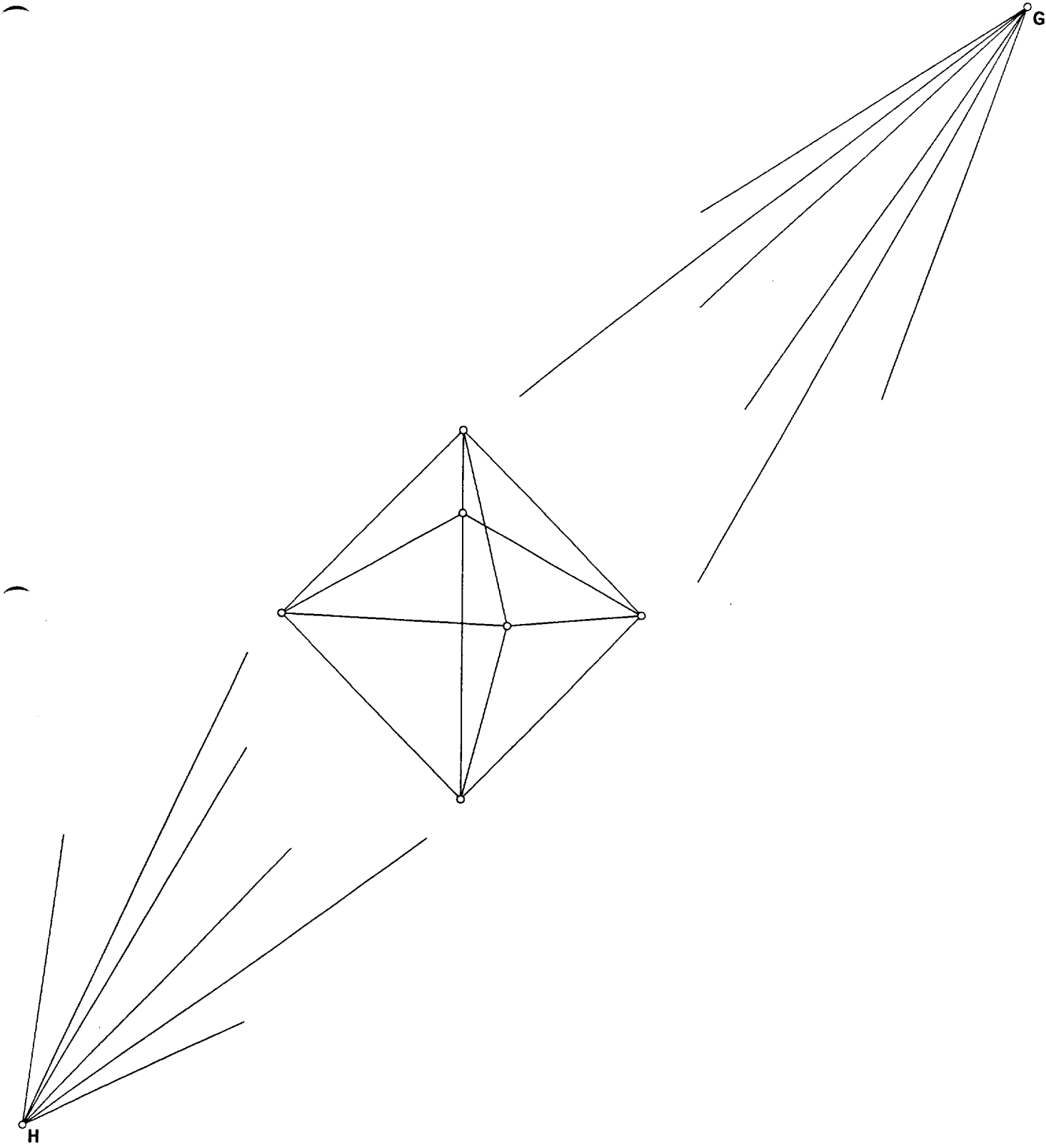
C-2

Octahedron

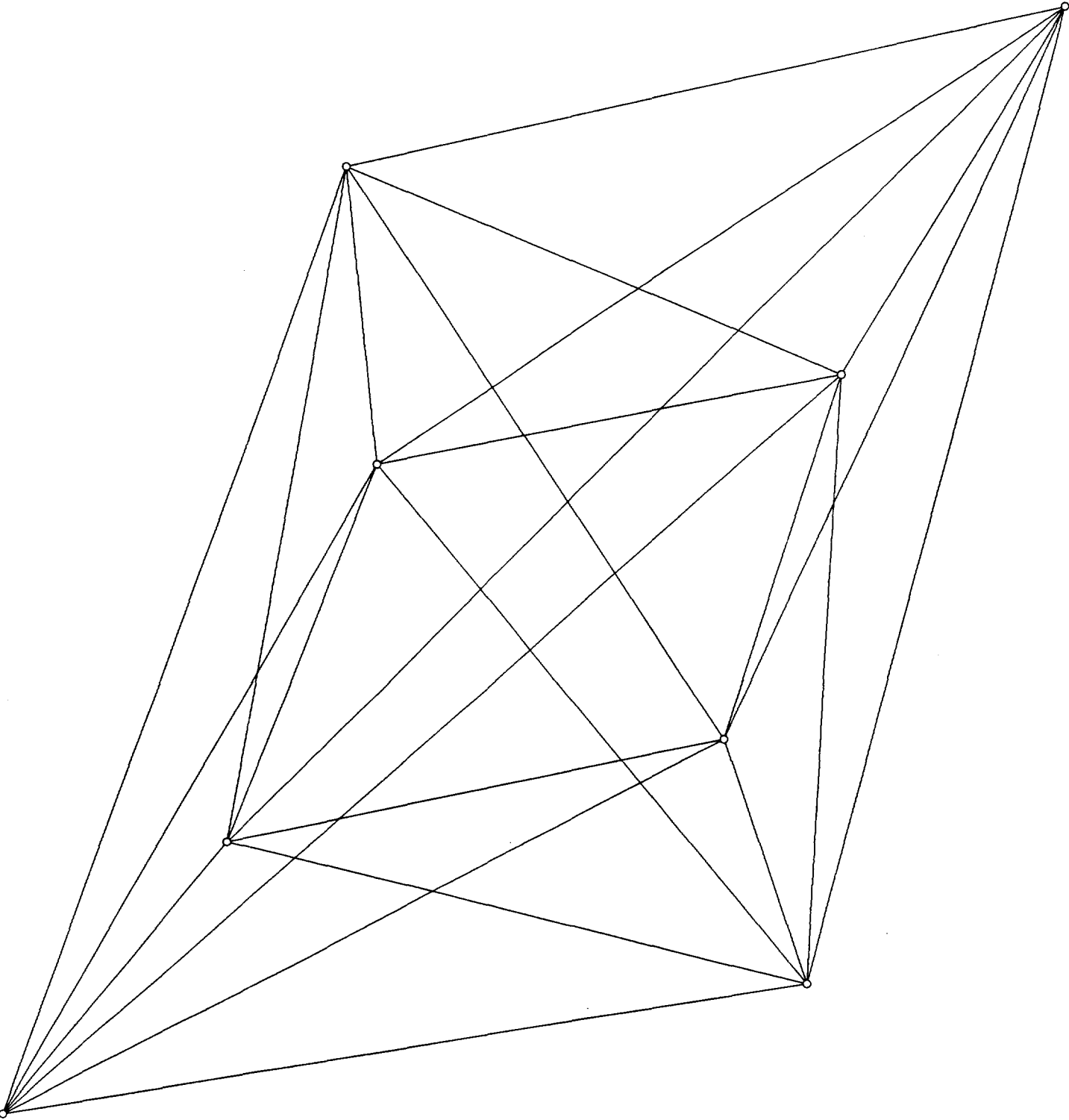


C-3

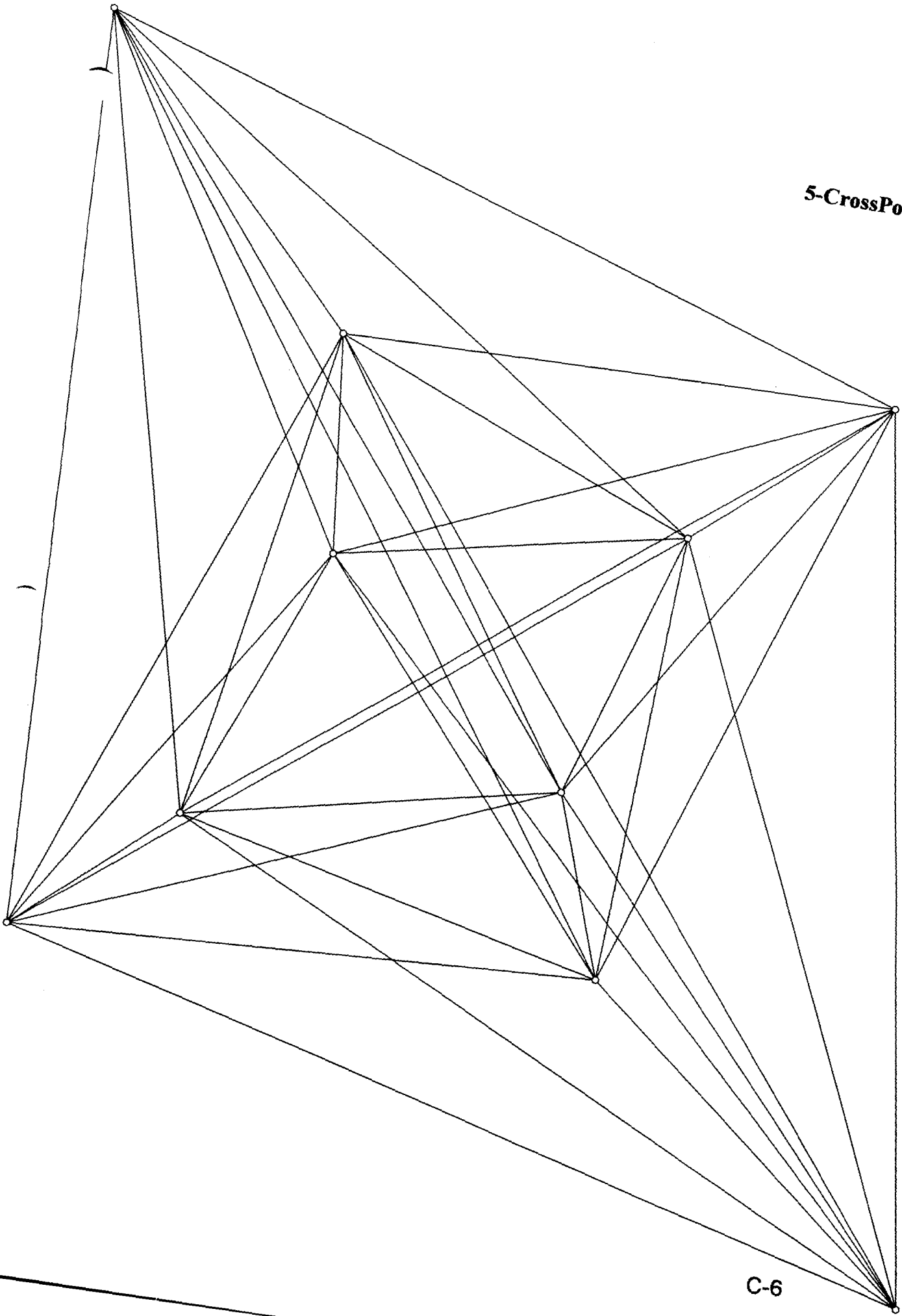
Suspension of Octahedron
4-Crosspolytope



4-CrossPolytope

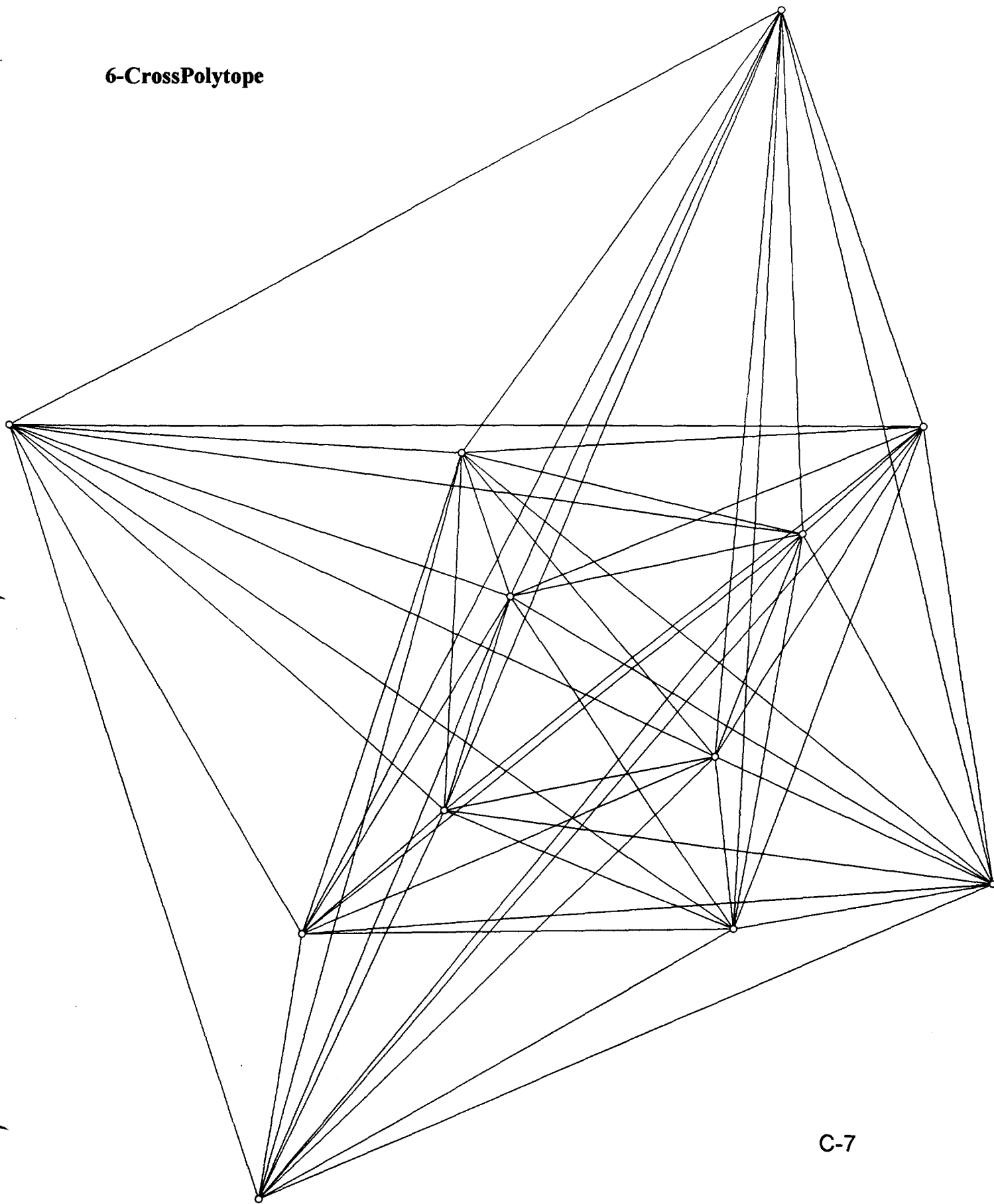


5-CrossPolytope

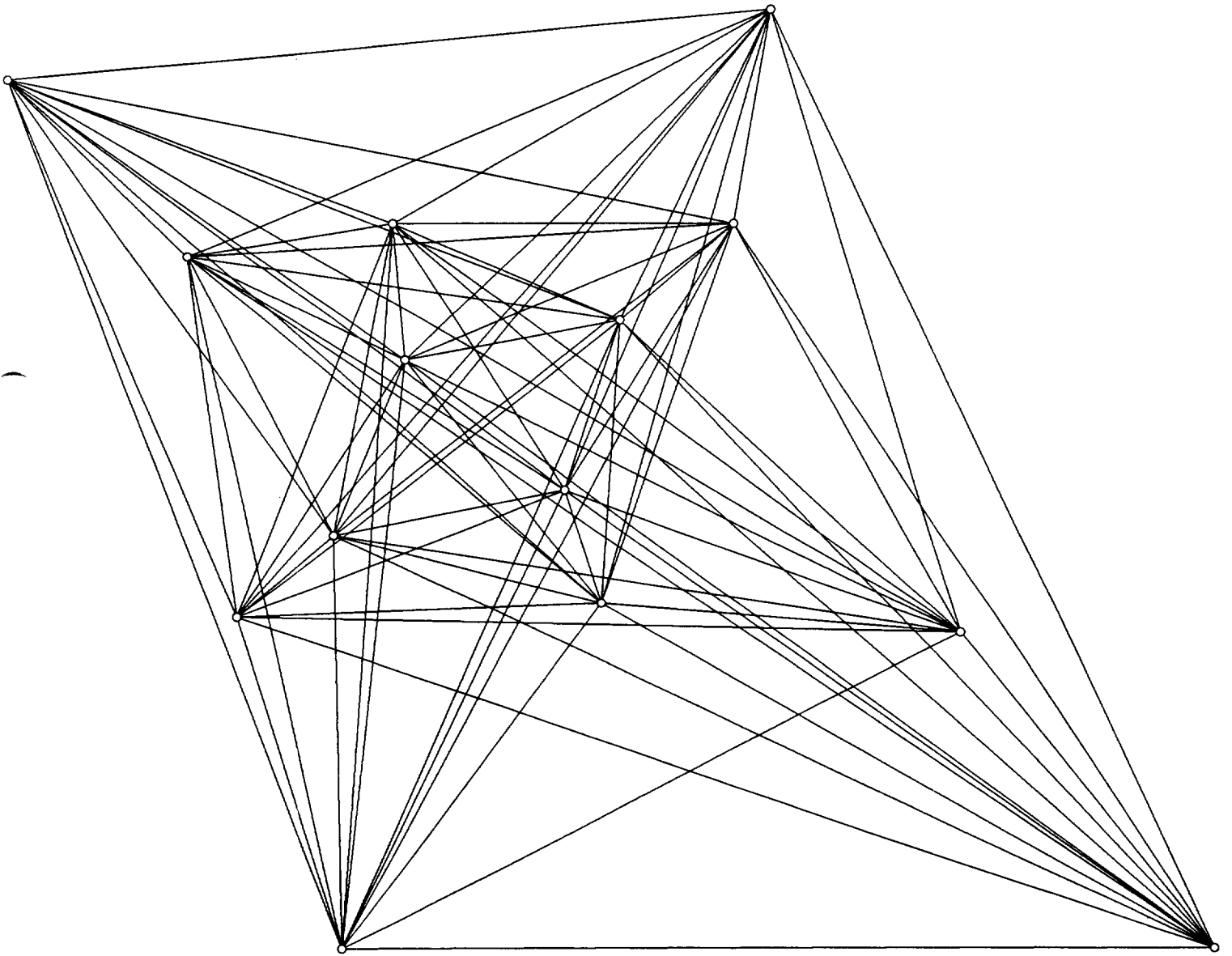


C-6

6-CrossPolytope



7-CrossPolytope



7-CrossPolytope

