Continued Fractions
Uses in Indeterminate Equations of
First and Second Degree

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Continued Fractions: Uses in Indeterminate Equations of First and Second Degree

Continued fractions are a method to obtain an end: they prove to be very useful in higher algebra in solving first and second degree indeterminate equations. It is my purpose in this paper to review the theory of continued fractions, and to demonstrate the major uses of this technique. In some cases I will introduce methods which may be used as supplements to the continued fraction technique, and attempt to compare the procedures and results of both methods.

1. An expression of the form $a+\cfrac{b}{c+\cfrac{d}{e+\cfrac{f}{g+\cdots}}}$ is what is what is known as a continued fraction; where a, b, c, d, e,... are any quantities one may choose. This expression is the most general case of a continued fraction. The abreviated form of this expression is $a+\frac{b}{c+}\frac{d}{e+}\frac{f}{g+}\cdots$. At this early period, however, we shall consider a more useful form of a continued fraction.

2. This is the form $a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}$ where a_1, a_2, and a_3, are any quantities.[1]

3. This expression may be written as $a_1+\frac{1}{a_2+}\frac{1}{a_3+}\cdots$ where the addition sign below the fraction line indicates adding the next fraction as is shown in article (2).

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1. H. S. Hall and S. R. Knight, Higher Algebra, 4th edition, (London, 1932), p. 273.

If the number of quotients $a_1, a_2, a_3, \ldots$ is finite, the continued fraction is a terminating continued fraction; if this is not the case the continued fraction is said to be a non-terminating continued fraction. The latter will be discussed in a later section of the paper.

To convert a given fraction into a continued fraction the following procedure may be used.

4. Suppose $\frac{m}{n}$ is the given fraction; divide m by n. Let a_1 be the quotient and p the remainder; $\frac{m}{n} = a_1 + \frac{p}{n}$ [2], where $p < n$. This quantity may be written $\frac{m}{n} = a_1 + \cfrac{1}{\frac{n}{p}}$, which is identical with the first quantity. One may now divide n by p, allowing a_2 to be the quotient and q the remainder; then $\frac{n}{p} = a_2 + \frac{q}{p}$ or $\frac{n}{p} = a_2 + \cfrac{1}{\frac{p}{q}}$. Divide p by q and let the quotient be a_3; the remainder will be r. This process may be continued indefinitely until the last division yields an integer. The result will eventually be in this form: $\frac{m}{n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$ which may be written $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$.

Any fraction whose numerator and denominator are positive may be converted into a continued fraction.

Example: Convert the fraction $\frac{832}{159}$ into a continued fraction.

$$\frac{832}{159} = 5 + \frac{37}{159} = 5 + \cfrac{1}{\frac{159}{37}}$$

$$5 + \cfrac{1}{\frac{159}{37}} = 5 + \cfrac{1}{4 + \frac{11}{37}} = 5 + \cfrac{1}{4 + \cfrac{1}{\frac{37}{11}}} = 5 + \cfrac{1}{4 + \cfrac{1}{3 + \frac{4}{11}}} = 5 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{\frac{11}{4}}}}$$

$$= 5 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{3}{4}}}} = 5 + \frac{1}{4+}\frac{1}{3+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3}.$$

2. Ibid., p. 274.

To simplify this long procedure one may use the familiar technique of finding the greatest common measure of m and n. The arrows indicate the divisor dividing the dividend, and

4	159	832	5
2	11	37	3
3	3	4	1
		1	

the quotients are placed adjacent to the dividends; the remainders are entered below the dividend and become the new divisors. The dividend is the previous divisor. To illustrate this, dividing 832 by 159 yields 5, with a remainder of 37; dividing 159 by 37 yields 4 and a remainder of 11, and so on. One can observe that by writing the quotients in order the original fraction has been converted into a continued fraction with a minimum of effort.

5. The fractions obtained by stopping at the first, second, third,..., are convergents of the continued fraction.[3] Each successive convergent is closer to the true value of the continued fraction, as will be demonstrated later.

6. The convergents are alternately less and greater than the continued fraction.[4] Assume $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 + \cdots}$ represents the continued fraction. The first convergent is a_1. It is less than the continued fraction because $\frac{1}{a_2 +} \frac{1}{a_3 + \cdots}$ is excluded. The second convergent is $a_1 + \frac{1}{a_2}$; this is too great because the denominator of the second part is too small, with $\frac{1}{a_2}$ being larger than $\frac{1}{a_2 + \frac{1}{a_3 + \cdots}}$ which is the second step in converting a fraction to a continued fraction.

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3. Ibid., p. 274.

4. Ibid., p. 275.

The third convergent, $a_1 + \frac{1}{a_2 +} \frac{1}{a_3}$, is too small because $\frac{1}{a_2 + \frac{1}{a_3}}$ is smaller than the required $\frac{1}{a_2 + \frac{1}{a_3 + \cdots}}$; and so on.

7. The convergents of an odd order are all less than, and the convergents of an even order are all greater than the continued fraction. This is true for all cases; if m is smaller than n assume $a_1 = 0$.

8. To find what the convergent might be corresponding to any particular quotient of the continued fraction, a method of formation will now be developed. Assume $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots$ is the continued fraction. The first three convergents are a_1, $\frac{a_1a_2+1}{a_2}$, $\frac{a_3(a_1a_2+1)+a_1}{a_3 \cdot a_2+1}$, which may be derived by stopping at any quotient. The convergent at a_1 is just a_1;

$$a_1 + \frac{1}{a_2} = \frac{a_2a_1+1}{a_2},$$

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3}} = a_1 + \frac{1}{\frac{a_2a_3+1}{a_3}} = a_1 + \frac{a_3}{a_2a_3+1} = \frac{a_1a_2a_3+a_1a_3}{a_2a_3+1}$$

$$= \frac{a_3(a_2a_1+1)+a_1}{a_2a_3+1}.$$

The numerator of the third convergent is formed by multiplying the numerator of the second by the third quotient, and adding the numerator of the first convergent; the denominator is formed by multiplying the third quotient by the denominator of the second convergent and adding the denominator of the first convergent.

9. Suppose that the successive convergents are formed in a similar manner; assume the numerators are denoted by p_1, p_2, $p_3, \ldots$ and the denominators by q_1, q_2, $q_3, \ldots$. Suppose the process of formation for convergents holds true for the n^{th} convergent, where $n \geq 3$. Let $p_n = a_np_{n-1} + p_{n-2}$, $q_n = a_nq_{n-1} + q_{n-2}$ for some value of n. The $(n+1)^{th}$ convergent differs from the n^{th}

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5. Ibid., p. 275.

convergent only in having the quotient $a_n+\frac{1}{a_{n+1}}$ in place of a_n.[6] The $(n+1)^{th}$ convergent, if a law is to be established in the formation of convergents, should follow the previously introduced pattern. The $(n+1)^{th}$ convergent equals,

$$\frac{\left(a_n+\frac{1}{a_{n+1}}\right)p_{n-1}+p_{n-2}}{\left(a_n+\frac{1}{a_{n+1}}\right)q_{n-1}+q_{n-2}} = \frac{(a_na_{n+1}+1)p_{n-1}+a_{n+1}p_{n-2}}{(a_na_{n+1}+1)q_{n-1}+a_{n+1}q_{n-2}}$$

$$= \frac{a_{n+1}(a_np_{n-1}+p_{n-2})+p_{n-1}}{a_{n+1}(a_nq_{n-1}+q_{n-2})+q_{n-1}}$$

$$= \frac{a_{n+1}p_n+p_{n-1}}{a_{n+1}q_n+q_{n-1}}.$$

One. may observe that the $(n+1)^{th}$ convergent follows the pattern assumed above for some n (9). Consequently the law of the formation of successive convergents is shown to be true by the use of mathematical induction.

Example: Find the successive convergents of the continued fraction $2+\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{11+}\frac{1}{2}$.

$\frac{p_1}{q_1}=\frac{2}{1}$; $\frac{p_2}{q_2}=\frac{2\cdot6+1}{6}=\frac{13}{6}$; $\frac{p_3}{q_3}=\frac{1(2\cdot6+1)+2}{1\cdot6+1}=\frac{15}{7}$;

$\frac{p_4}{q_4}=\frac{1\cdot15+13}{1\cdot7+6}=\frac{28}{13}$; $\frac{p_5}{q_5}=\frac{11\cdot28+15}{11\cdot13+7}=\frac{153}{74}$;

$\frac{p_6}{q_6}=\frac{2\cdot153+28}{2\cdot74+13}=\frac{306+28}{148+13}=\frac{334}{161}$.

10. If will be convenient to call a_n the n^{th} partial quotient; the complete quotient at this stage being $a_n+\frac{1}{a_{n+1}+}\frac{1}{a_{n+2}+}\cdots$. The complete quotient at any stage may be denoted by k. The partial quotient at any stage is $\frac{p_n}{q_n}=\frac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}}$; if the continued fraction is represented by x, then x differs from $\frac{p_n}{q_n}$ only in taking the complete quotient k instead of the partial quotient a_n.[7]

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6. Ibid., p. 276.
7. Ibid., p. 276

11. If the complete quotient k is taken instead of $\frac{p_n}{q_n}$, x takes the form $x = \frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}}$.[8]

12. If $\frac{p_n}{q_n}$ is the n^{th} convergent of a continued fraction, then $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$.[9] To prove this important property of convergents first assume a continued fraction is represented by $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots$; then by substituting in the values for p_n and q_n we have $p_n q_{n-1} - p_{n-1} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(a_n q_{n-1} + q_{n-2})$

$$= a_n p_{n-1} q_{n-1} + p_{n-2} q_{n-1} - p_{n-1} a_n q_{n-1} - p_{n-1} q_{n-2}$$
$$= (-1)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$$
$$= (-1)^2 (p_{n-2} q_{n-3} - p_{n-3} q_{n-2})$$

, in a like, manner, ...

$= (-1)^{n-2}(p_2 q_1 - p_1 q_2)$. But $p_2 q_1 - p_1 q_2 = (a_1 a_2 + 1) - a_1 \cdot a_2 = 1 = (-1)^2$. Therefore $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$. If the continued fraction is less than unity, the result will hold true. Here $a_1 = 0$, and the first convergent is zero.

13. Each convergent must be in lowest terms, because if p_n and q_n had a common divisor it would divide $p_n q_{n-1} - p_{n-1} q_n$, or unity, which is impossible.[10]

14. The difference between two successive convergents is a fraction whose numerator is unity; $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n-1}} = \frac{1}{q_n q_{n-1}}$ by article (12).

15. Each convergent is closer to the value ofthe continued fraction than any of the preceding convergents.[11] Suppose x denotes the continued fraction, and $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, and $\frac{p_{n+2}}{q_{n+2}}$ are three successive convergents; then by (11), x differs from $\frac{p_{n+2}}{q_{n+2}}$ only in taking the complete $(n+2)^{th}$ quotient instead of a_{n+2}.

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8. Ibid., p. 276.
9. Ibid., p. 277
10. Ibid., p. 277.
11. Ibid., p. 278.

Denote this by k: then $x=\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}$, $x-\frac{p_n}{q_n}=\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}-\frac{p_n}{q_n}$

$$=\frac{kp_{n+1}q_n+p_nq_n-kq_{n+1}p_n-q_np_n}{(kq_{n+1}+q_n)q_n}=\frac{k(p_{n+1}q_n-q_{n+1}p_n)}{q_n(kq_{n+1}+q_n)}$$

$$=\frac{k(-1)^{n+1}}{q_n(kq_{n+1}+q_n)};$$

$$\frac{p_{n+1}}{q_{n+1}}-x=\frac{p_{n+1}q_n-p_nq_{n+1}}{q_{n+1}(kq_{n+1}+q_n)}=\frac{(-1)^{n+1}}{q_{n+1}(kq_{n+1}+q_n)}$$

Now k is greater than unity, and q_n is less than q_{n+1}; therefore one may observe that in both cases the difference between $\frac{p_{n+1}}{q_{n+1}}$ and x is numerically less than the difference between $\frac{p_n}{q_n}$ and x. Consequently every convergent is closer to the value of the continued fraction than any of the preceding convergents.

16. By considering articles (6) and (14) together one may observe that the convergents of an odd order continually increase but are always less than the continued fraction; and convergents of an even order decrease continually, but their values are always greater than the continued fraction.

17. Limits of error may be established in taking any convergent for the value of the continued fraction.[12] Again suppose $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, and $\frac{p_{n+2}}{q_{n+2}}$ are three successive convergents of a continued fraction. Let k denote the complete $(n+2)^{th}$ quotient. As before, $x=\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}$, $x-\frac{p_n}{q_n}=\frac{k}{q_n(kq_{n+1}+q_n)}$ (14)

$$=\frac{1}{\frac{q_n(kq_{n+1}+q_n)}{k}}=\frac{1}{q_nq_{n+1}+\frac{q_n}{k}}.$$

Since k is greater than unity the difference between x and $\frac{p_n}{q_n}$ is less than $\frac{1}{q_nq_{n+1}}$, and greater than $\frac{1}{q_n(q_{n+1}+q_n)}$ since $q_n > \frac{q_n}{k}$. Also since $q_{n+1}>q_n$ the error in taking $\frac{p_n}{q_n}$ for x is less than $\frac{1}{2q_{n+1}^2}$ since $\frac{1}{2q_{n+1}^2}$ is less than $\frac{1}{q_n(q_{n+1}+q_n)}$.

12. Ibid., p. 279.

From the last article it is apparent that the error in taking $\frac{p_n}{q_n}$ for the continued fraction x is less than $\frac{1}{q_n q_{n+1}}$, or $\frac{1}{q_n(a_{n+1}q_n+q_{n-1})}$, since $q_{n+1}=a_{n+1}q_n+q_{n-1}$. Consequently $x-\frac{p_n}{q_n}<\frac{1}{a_{n+1}q_n^2}$, and finally the larger a_{n+1} is the closer $\frac{p_n}{q_n}$ approximates the continued fraction.

One may conclude, then, that any convergent which immediately precedes a large quotient is a near approximation of the continued fraction.

The properties of continued fractions enable us to find two small integers whose ratio closely approximates to that of two incommensuralbe quanitities, or to that of two quanitities whose exact ratio can be expressed only with large integers.

18. Any convergent of a continued fraction is a nearer approximation of the continued fraction than any other fraction with a smaller denominator.[13] To show this let x be the continued fraction and $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ be two consecutive convergents. Assume the fraction $\frac{r}{s}$ with denominator s less than q_n is nearer x than $\frac{p_n}{q_n}$; then $\frac{r}{s}$ must be nearer x than $\frac{p_{n-1}}{q_{n-1}}$ (15). Because x lies between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$ (15), $\frac{r}{s}$ must be between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$. Therefore $\frac{r}{s}-\frac{p_{n-1}}{q_{n-1}}<\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}$ which is less than $\frac{1}{q_n q_{n-1}}$ (16). Then $\frac{r}{s}-\frac{p_{n-1}}{q_{n-1}}<\frac{1}{q_n q_{n-1}}$, or $rq_{n-1}-sp_{n-1}<\frac{s}{q_n}$. But $\frac{s}{q_n}$ is a fraction since s was assumed to be $<q_n$; $rq_{n-1}-sp_{n-1}$ is an integer. An integer cannot be less than a fraction, and consequently, our assumption is false.

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13. Ibid., p. 280.

By indirect proof, then, $\frac{p_n}{q_n}$ must be nearer the continued fraction than $\frac{r}{s}$.

19. If $\frac{p}{q}$ and $\frac{p'}{q'}$ are two consecutive convergents to a continued fraction x, then $\frac{pp'}{qq'}$ is greater or less than x^2, according as $\frac{p}{q}$ is greater than or less than $\frac{p'}{q'}$.[14] Choose k to be the complete quotient corresponding to the convergent immediately following p. Then one may set $x = \frac{kp'+p}{kq'+q}$ by (11).

Therefore $\frac{pp'}{qq'} - x^2 = \frac{pp'}{qq'} - \left(\frac{kp'+p}{kq'+q}\right)^2$

$$= \frac{pp'(kq'+q)^2 - qq'(kp'+p)^2}{qq'(kq'+q)^2}$$

$$= \frac{pp'(k^2q'^2 + 2kqq' + q^2) - qq'(k^2p'^2 + 2kpp' + p^2)}{qq'(kq'+q)^2}$$

$$= \frac{pp'k^2 + q'^2 + pp'q^2 - qq'p^2 - qq'k^2p'^2}{qq'(kq'+q)^2}$$

$$= \frac{pp'k^2q'^2 + pp'q^2 - qq'k^2p'^2 - qq'p^2}{qq'(kq'+q)^2}$$

The numerator may be factored into $\frac{(k^2p'q' - pq)(pq' - p'q)}{qq'(kq'+q)}$.

The factor $k^2p'q' - pq$ is positive because $p' > p$, $q' > q$, since each successive numerator is greater than the one before, as are successive denominators. But k is greater than one; therefore $\frac{pp'}{qq'} >$ or $< x^2$, according as $pq'-p'q$ is positive or negative, which depends on whether $\frac{p}{q} >$ or $< \frac{p'}{q'}$.

20. By investigation one may show that the expressions $pq'-p'q$, $pp'-qq'x^2$, $p^2-q^2x^2$, and $q'^2x^2-p'^2$ have the same sign.[15] This investigation follows: (A) If $\frac{p}{q} > \frac{p'}{q'}$, then $pq' > p'q$

14. Ibid., p. 281.

15. Ibid., p. 281.

and the expressions are positive. From (19) then, $\frac{pp'}{qq'} > x^2$, (B) from which $pp' > qq'x^2$, or $pp' - qq'x^2 > 1$. (C) Also if $\frac{pp'}{qq'} > x^2$ and $\frac{p}{q} > \frac{p'}{q'}$, then $\frac{p^2}{q^2} > \frac{pp'}{qq'}$, from which $\frac{p^2}{q^2} > x^2$, $p^2 > q^2x^2$, $p^2 - q^2x^2 > 1$.

In a similar manner one may show the expressions to be negative when $\frac{p}{q} < \frac{p'}{q'}$. With the now existing proofs and laws the theory of continued fractions becomes a valuable aid in solving indeterminate equations of first and second degree. The next portion of this paper will be devoted to using the theory of terminating continued fractions to solve and discover techniques which are useful in working with linear equations of the first degree.

Indeterminate equations of the first degree are of the form $ax \pm by = \pm c$. Any linear equation involving two integral unknowns x and y may be reduced to this form. An equation of this form, as will be seen, has an unlimited number of solutions.

Because of the unlimited number of solutions we shall consider only positive integral solutions. One may easily adjust the following procedures to find all solutions. It is evident that $ax + by = -c$ has no positive solutions. Consequently we will confine our discussion to $ax \pm by = c$; however, the theory could easily be applied to $ax \pm by = -c$.

21. One may find solutions in positive integers of the equation $ax - by = c$.[16] Suppose $\frac{a}{b}$ is converted into a continued fraction; let $\frac{p}{q}$ be the convergent just preceeding $\frac{a}{b}$. By article (12) $aq - bp = \pm 1$.

Case I.

If $aq - bp = 1$, the given equation $ax - by = c$ may be written:

16. Ibid., p. 285.

$$ax-by=c(aq-bp)$$
$$ax-by=caq-cbp$$
$$ax-caq=by-cbp$$
$$a(x-cq)=b(y-cp)$$

By the original supposition a and b have no common divisor. Therefore x-cq must be divisible by b, and y-cp must be divisible by a. Therefore $\frac{x-cq}{b} = t = \frac{y-cp}{a}$, or $x-cq=bt$, $y-cp=at$; hence $x=bt+cq$, $y=at+cp$. Positive integral values may be obtained by giving t any poitive integral value, or any negative value less than the smaller of the two quantities $\frac{cq}{b}$, $\frac{cp}{a}$. The negative values must be smaller because cq must be greater than bt, and cp must be greater than at in order for x and y to be positive; zero also could be used as a value of t.

Case II.

If $aq-bp=-1$,

$$ax-by=-c(aq-bp)$$
$$ax-by=-caq+cbp$$
$$ax+caq=by+cbp$$
$$a(x+cq)=b(y+cp)$$

Using the same reasoning as was used in Case I, b must divide x+cq, and a must divide y+cp. Therefore $\frac{x+cq}{b}=\frac{y+cp}{a}=t$, some integer. From this one can solve again for x and y: $x+cq=bt$, $x=bt-cq$; $y=at-cp$. Positive integral solutions may be obtained by giving to t positive integral values which are greater than the smallest of the quantities, $\frac{cq}{b}$ and $\frac{cp}{a}$. Once agin it may be observed that the number of solutions is unlimited. Also t may be given a negative value if this value is numerically less than the smaller of $\frac{cq}{b}$ and $\frac{cp}{a}$.

It is obvious that one need not restrict the solutions to positive values, but to simplify maters we will continue to find solutions in terms of positive integral numbers.

Case III.

If either a or b is unity, the fraction $\frac{a}{b}$ cannot be converted into a continued fraction and the above method will not work. When this occurs, however, the solutions may be written down by inspection. If, say, $a = 1$, the equation is $x - by = c$, and $x = by + c$; by giving y different values one easily finds x. In the same manner if $b = 1$, the equation is $ax - y = c$, from which $y = ax - c$. Positive values may be found by giving x values such that $ax > c$.

Example: Find solutions in positive integers of $29x - 42y = 5$. First convert $\frac{42}{29}$ into a continued fraction. $\frac{42}{29} = 1 + \frac{13}{29}$

$$\frac{49}{29} = 1 + \frac{1}{2+}\frac{1}{4+}\frac{1}{3}$$

The convergents of this continued fraction are, $\frac{1}{1}, \frac{3}{2}, \frac{13}{9}, \frac{42}{29}$.

The convergent immediately preceeding $\frac{42}{29}$ is $\frac{13}{9}$.

$$13 \cdot 29 - 9 \cdot 42$$
$$= 377 - 378 = -1.$$

Therefore $29(x + 5 \cdot 13) = 42(y+5)$

$$29(x+65) = 42(y+45)$$

$$\frac{x+65}{42} = \frac{y+45}{29} = t, \text{ an integer.}$$

The algebriac solution is then, $x = 42t - 15$, $y = 29t - 45$. Letting $t = 2$, one finds that $x = 19$, $y = 13$ are solutions of the equation.

Example: Solve the equation $755x - 711y = 1$. The continued fraction is $\frac{755}{711} = 1 + \frac{1}{16+}\frac{1}{6+}\frac{1}{3+}\frac{1}{2}$. The convergents of this continued fraction are $\frac{1}{1}, \frac{17}{16}, \frac{103}{97}, \frac{326}{307}, \frac{755}{711}$.

The convergent immediately preceeding $\frac{a}{b}$ is $\frac{326}{307}$.

Therefore $\quad 755 \cdot 307 - 711 \cdot 326 = \pm 1$

$$231785 - 231786 = -1$$

$$x = 711t - 307$$

$$y = 755t - 326$$

Let $t = 1$, $x = 404$, $y = 429$.

One may observe now an important use of continued fractions. Continued fractions prove to be a useful technique in solving linear equations which possess large coefficients.

22. If one solution of the equation $ax - by = c$ is given, a general solution for all positive integers may be developed.[17] Suppose h, k is a solution of $ax-by = c$. Then $ah-bk = c$. Therefore, $ax-by = ah-bk$, and $a(x-h) = b(y-k)$. As before $\frac{x-h}{b} = \frac{y-k}{a} = t$, an integer. The general solution is then $x = h + bt$, $y = k + at$.

23. One may also find solutions of the equation $ax + by = c$, if there are any.[18] Again let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent just preceeding $\frac{a}{b}$; then $aq-bp = \pm 1$ (12).

Case I.

If $aq-bp = 1$, $\quad ax + by = c(aq-bp)$

$$ax + by = caq-cbp$$

$$a(cq-x) = b(y + cp)$$

Since a and b have no common divisor $\frac{(cq-x)}{b} = \frac{(y+cp)}{a} = t$, some integer. Therefore $x = bt-cq$, $y = cp-at$. Positive integral solutions may be obtained by giving to t positive integral values greater than $\frac{cp}{a}$ and less than $\frac{cq}{b}$.

17. Ibid., p. 286.

18. Ibid., p. 287.

The number of positive solutions is limited, but if one disreguards the signs of the solutions, the number of solutions is unlimited.

Case II.

If $aq-bp=-1$,
$$ax+by=-c(aq-bp)$$
$$ax+by=-caq+cbp$$
$$a(x+cq)=b(cp-y)$$

$\frac{x+cq}{b}=\frac{cp-y}{a}=$ t, an integer, and $x=bt-cq$, $y=cp-at$. Positive integral solutions may be obtained by giving t positive integral values greater than $\frac{cq}{b}$ and less than $\frac{cp}{a}$. Again the number of positive solutions is limited, but if the signs of the solutions are disreguarded, the number is unlimited.

Case III.

If either a or b is unity, the solutions may be found by inspection as in Case III of (21).

24. If one solution in positive integers of the equation $ax+by=c$ has been found, one may establish a general solution.[19] Let h, k be a solution of the equation $ax+by=c$; then $ah+bk=c$, and
$$ax+by=ah+bk$$
$$ax-ah=bk-by$$
$$a(x-h)=b(k-y)$$
$$\frac{x-h}{b}=\frac{k-y}{a}= \text{ t, an integer.}$$
Therefore $x=bt+h$, $y=k-at$, the general solution.

25. At times it will be helpful to find the number of solutions in positive integers of the equation $ax+by=c$.[20]

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19. Ibid., p. 287.

20. Ibid., p. 288

To find this number let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent just preceeding $\frac{a}{b}$; then $aq-bp = \pm 1$.

Case I.

If $aq-bp = 1$, the general solution is then $x = cq-bt$, $y = at-cp$, by article (24). Positive solutions may be found by giving t positive integral values less than $\frac{cq}{b}$, and greater than $\frac{cp}{a}$. Suppose $\frac{c}{a}$ and $\frac{c}{b}$ are not integers. In order for $\frac{cp}{a}$ and $\frac{cq}{b}$ to be integers a must divide c, and b must divide c since p is smaller than a, and q is smaller than b; the numerator and denominator of a convergent is always larger than the previous convergent's numerator and denominator. Then $\frac{cp}{a} = m + f$, and $\frac{cq}{b} = n + g$, where m and n are positive integers, and f and g are proper fractions. The least value of t is $m + 1$ because t must be greater than $\frac{cp}{a}$ and be an integer. The greatest value of t is then n as t cannot be greater than $\frac{cq}{b}$ and must be an integer.

A. $$\frac{cq}{b} - \frac{cp}{a} = n - m + g - f$$
$$n - m = \frac{cq}{b} - \frac{cp}{a} + f - g = \frac{acq - bcp}{ab} + f - g$$
$$= \frac{c(aq-bp)}{ab} + f - g$$
$$= \frac{c}{ab} + f - g \quad (12).$$

Since n-m is an integer, $\frac{c}{ab} + f - g$ is an integer. Then $n-m = \frac{c}{ab} \pm$ some fraction, according as f is greater or less than g. Therefore the number of solutions is the integer nearest $\frac{c}{ab}$, greater or less according to whether $f > g$ or $f < g$.

B. If $\frac{c}{b}$ is an integer, $g = 0$ and one value is $x = 0$.

$$\frac{ax}{b} + y = \frac{c}{b}$$

If $\frac{c}{b}$ is an integer, then $x = 0$ when $y = \frac{c}{b}$. Since $g = 0$, the number of solutions is $\frac{c}{ab} + f$, which must be an integer. Therefore the number of solutions is the greatest integer in $\frac{c}{ab} + 1$, or $\frac{c}{ab}$, depending on whether $x = 0$, $y = \frac{c}{b}$ is counted as a positive solution.

C. Suppose that $\frac{c}{a}$ is an integer. In this case $f = 0$, and one value of y is zero.

If $x = \frac{c}{a}$, $y = 0$. Including this solution would make the least value of t in $x = cq-bt$, $y = at-cp$, m; and the greatest value would be n. Since $n = \frac{cq}{b}$, and $m = \frac{cp}{a}$, if t was m or n one would have

$$\frac{x}{b} = \frac{cq}{b} - t \quad , \quad y = t - \frac{cp}{a}$$

$$\frac{x}{b} = n - n \quad , \quad \frac{y}{a} = m - m$$

$$x = 0 \quad , \quad y = 0 .$$

Consequently if the range of t is $m \leq t \leq n$ one would have all possible positive solutions, which would total $n-m+1$, because m must be included as a solution.

D. If $\frac{c}{a}$ and $\frac{c}{b}$ are both integers, $f = 0$, and $g = 0$; therefore both x and y would have a zero value. In this case the least value of t is m, and the greatest value is n; hence the number of solutions is again $n-m+1$, or $\frac{c}{ab} + 1$.

CaseII.

If $aq-bp = -1$, the general solution is $x = bt-cq$, $y = cp-at$. Similar results will be obtained in this case.

Thus far the theory of continued fractions has helped to solve equations of the form $ax \pm by = \pm c$. Also by using the elementary theory of convergents to a continued fraction the number of possible solutions to the above equation may be established.

Example: Find the number of positive solutions of the equation $11x + 15y = 1031$. First rewrite the equation as $by + ax = 1031$. Then convert $\frac{b}{a}$ into a continued fraction. $\frac{15}{11} = 1 + \frac{1}{2+} \frac{1}{1+} \frac{1}{3}$

The convergents to this continued fraction are $\frac{1}{1}, \frac{3}{2}, \frac{4}{3}, \frac{15}{11}$.

Then $\frac{p}{q}$ is $\frac{4}{3}$, and $aq-bp = 1$.

$15y + 11x = 1031(15.3-4\cdot 11)$

$11x + 1031\cdot 4\cdot 11 = 1031\cdot 15\cdot 3 - 15y.$

$11(x+4124) = 15(3093-y)$

$$\frac{x+4124}{15} = \frac{3093-y}{11}$$

$x = 15t-4124$, $y = 3093-11t$.

If $t = 280$, $x = 76$, $y = 13$ is a solution. The total number of positive integral solutions may be found by finding $\frac{c}{ab}$, and then setting $n-m = \frac{c}{ab} + f-g$. (25-A)

Example: $n-m = \frac{c}{ab} + \frac{14}{15} - \frac{2}{11}$

$$= 6 + \frac{41}{165} + \frac{124}{165}$$

$$m-n = 7.$$

The number of positive integral solutions to the equation $11x + 15y = 1031$ is 7.

It is comparatively easy theoretically to solve three equations of three unknowns. One merely eliminates by a series of operations two of the unknowns; this is known as solving equations simultaneously. This method does not work, however, when one needs to solve two equations of three unknowns. One of the important functions of the theory of continued fractions is to provide a method of finding solutions to two equations of three unknowns.

26. First consider the solution in positive integers of equations of the form $ax + by + cz = d$. By adding $-cz$ to both sides of the equation one may obtain $ax + by = d-cz$.

One may then give z successive values of 0,1,2,..., from which equations of the form $ax + by = c'$ evolve. Equations of this form may be solved by the method already discussed (23).

Example: Find solutions of the equation $7x + 4y + 19z = 18$.

Let $z = 1$, $7x + 4y = 18 - 19z$.

By (23-II), convert $\frac{7}{4}$ into a continued fraction. $\frac{7}{4} = 1 + \frac{1}{1+} \frac{1}{3}$.

$$7\cdot 1 - 4\cdot 2 = -1$$
$$7x + 4y = 1(7-8)$$
$$4y + 4\cdot 2 = 7 - 7x$$
$$4(y+2) = 7(1-x)$$
$$\frac{y+2}{7} = \frac{1-x}{4}$$

$y+2 = 7t$, $1-x = 4t$
$y = 7t - 2$, $1 - 4t = x$

It is obvious that one cannot obtain integral solutions where x and y are both positive. Solutions where x and y are of different signs may easily be obtained by giving t various values. Remembering that z was set at 1, if $t = 1$ one has $x = -3$, $y = 5$ and $z = 1$ as a solution.

27. If two simultaneous equations are involved, $ax + by + cz = d$, and $a'x + b'y + c'z = d'$, one may find positive solutions by proceeding in the following manner.[21] Eliminate one of the unknowns, z say; this will produce an equation of the form $Ax + By = C$. Assume by (23) that $x = h$, $y = g$ is a solution; then the general solution can be written $x = f + Bs$, $y = g - As$ by article (24). Call these equations, equations #1. By substituting in either of the given equations values of x and y one may obtain an equation of the form $Fs + Gz = H$.

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21. Ibid., p.289

$$a(f + Bs) + b(g - As) + cz = d$$
$$af + aBs + bg - bAs + cz = d$$
$$(aB-bA)s + cz = d - af - bg,$$

or $Fs + Gz = H$, as explained. Assume h, k is a solution of the equation. The form by which one may obtain solutions of this equation is then $s = h + Gt$, $z = k - Ft$ by article (24). Substituting for s in #1 one obtains $x = f + Bh + BGt$, $y = g - Ah - Agt$. The values of x, y, z are obtained by giving to t suitable integral values.

Example: Solve the simultaneous equations $12x - 11y - 4z = 22$, and $-4x + 5y + z$ 17. Multiply the second equation by 3.

$$\begin{array}{r} 12x - 11y + 4z = 22 \\ \underline{-12x + 15y + 3z = 51} \\ 4y + 7z = 73, \end{array} \text{ or } 7z + 4y = 73.$$

Convert $\frac{7}{4}$ into a continued fraction. The continued frac- is $\frac{7}{4} = 1 + \frac{1}{1 + \frac{1}{3}}$. The convergents are $\frac{1}{1}, \frac{2}{1}, \frac{7}{4}$.

$$aq - bp = 7 - 8 = -1 \qquad (23 - II).$$
$$7z + 4y = -73(7\cdot 1 - 4\cdot 2)$$
$$7z - 7\cdot 73 = 4\cdot 73\cdot 2 - 4y$$
$$7(z - 73) = 4(146 - y)$$
$$\frac{z+73}{4} = \frac{146-y}{7} = t$$
$$z = 4t - 73, \quad y = 146 - 7t.$$

The positive solutions must be from $\frac{73}{4} < t < \frac{146}{7}$

Suppose $t = 20$. $z = 80 - 73 = 7$
$y = 146 - 140 = 6$

One may find the general solution by (24).

$$7z + 4y = 7\cdot 3 + 4\cdot 13$$
$$7(z - 3) = 4(13 - y)$$
$$\frac{z-3}{4} = \frac{13-y}{7}$$
$$z = 4s + 3, \quad y = 13 - 7s.$$

Substitute these values in either of the given equations.

$$-4x + 5(13-7s) + 3 + 4s = 17$$
$$-4x + 65 - 35s + 3 + 4s = 17$$
$$-4x - 31s = -51$$
$$31s + 4x = 51.$$

Using (23) again, first convert $\frac{31}{4}$ into a continued fraction, $\frac{31}{4} = \frac{7}{1+}\frac{1}{1+}\frac{1}{3}$. The convergents are $\frac{7}{1}, \frac{8}{1}, \frac{31}{4}$

$$31 \cdot 1 - 8 \cdot 4 = -1$$
$$31s + 4x = -51 \cdot 31 + 51 \cdot 8 \cdot 4$$
$$31(s+51) = 4(51 \cdot 8 - x)$$
$$\frac{s+51}{4} = \frac{408-x}{31}, \quad s = 4t - 51, \quad x = 408 - 31t.$$

Positive values of s and x will be obtained when .

Let $t = 13$, $s = 52-51$, $x = 408-403$
$s = 1$ $x = 5.$

Substitute the general form of s in the general forms of y and z;

$$y = 13-7(4t-51) \qquad z = 4(4t-51) + 3$$
$$y = 13 - 28t + 357 \qquad z = 16t - 204 + 3$$
$$y = 370 - 28t \qquad z = 16t - 201$$

Setting $t = 13$, $y = 370 - 364$ $z = 208 - 201$
$y = 6$ $z = 7$

A solution of the simultaneous equations $12x - 11y + 4z = 22$ and $-4x + 5y + z = 17$ is $x = 5$, $y = 6$, $z = 7$.

28. If one solution in positive integers of the equations $ax + by + cz = d$, and $a'x + b'y + c'z = d'$ is found, the general solution may be obtained.[22] Often times, however, one positive integral solution is all that satisfies both equations. Suppose f,g,h is a solution; then $af + bg + ch = d$, and $a'f + b'g + c'h = d'$.

By subtraction $a(x-f) + b(y-g) + c(z-h) = 0$

and $a'(x-f) + b'(y-g) + c'(z-h) = 0$

Multiply the first equation by b' and the second by b.

$$ab'(x-f) + bb'(y-g) + b'c\ (z-h) = 0$$
$$ab(x-f) + bb'(y-g) + bc'(z-h) = 0$$
$$\overline{(ab'-ab)\ (x-f) + (b'c'-b'c)\ (z-h) = 0}$$

22. Ibid., p.290.

$$(ab'-a'b)(x-f)-(bc'-b'c)(z-h)=0$$
$$(ab'-a'b)(x-f)=(bc'-b'c)(z-h)$$
$$\frac{x-f}{bc'-b'c}=\frac{z-h}{ab'-a'b}=\frac{t}{k}$$

where t is some integer and k is the greatest common denominator. The third unknown is y, and it is found from the expression $\frac{y-g}{ca'-c'a}$ in the same manner as x and y. Thus the general solution is $x=f+(bc'-b'c)\frac{t}{k}$, $y=g+(ca'-c'a)\frac{t}{k}$, $z=h+(ab'-a'b)\frac{t}{k}$.

The more important uses of terminal continued have now been pointed out and demonstrated. Another phase of continued fractions which proves useful at times will now be introduced.

It has been shown that a terminating continued fraction has rational quotients and may be reduced to an ordinary fraction with an integral numerator and denominator; a terminating fraction therefore cannot be equal to an irrational. A quadratic irrational, however, can be expressed as an infinite continued fraction whose quotients recur. To assist in understanding the theoretical conversion of a quadratic irrational to a recurring continued fraction, an example will first be presented.

Example: Express $\sqrt{19}$ as a continued fraction, and find a series of fractions approximating its value. First find the greatest integer in $\sqrt{19}$. $\sqrt{19}=4+(\sqrt{19}-4)$ now multiply $(\sqrt{19}-4)$ by its conjugate.

$$\sqrt{19}=4+(\sqrt{19}-4)\ \frac{(\sqrt{19}+4)}{(\sqrt{19}+4)}$$
$$\sqrt{19}=4+\frac{(19-16)}{\sqrt{19}+4}=4+\frac{3}{\sqrt{19}+4}=4+\frac{1}{\frac{\sqrt{19}+4}{3}}.$$

Continue this process with $\frac{\sqrt{19}+4}{3}$. $\frac{\sqrt{19}+4}{3} = 2 + \left(\frac{\sqrt{19}+4}{3} - 2\right)$
$= 2 + \frac{\sqrt{19}+4-6}{3} = 2 + \frac{(\sqrt{19}-2)(\sqrt{19}+2)}{3(\sqrt{19}+2)} = 2 + \frac{5}{3(\sqrt{19}+2)}$
$= 2 + \frac{1}{\frac{\sqrt{19}+2}{5}}$.

Repeat for $\frac{\sqrt{19}+2}{5}$, and so on. $\frac{\sqrt{19}+2}{5} = 1 + \left(\frac{\sqrt{19}+2}{5} - 1\right) = 1 + \frac{\sqrt{19}-3}{5}$
$= 1 + \frac{(\sqrt{19}-3)(\sqrt{19}+3)}{5(\sqrt{19}+3)} = 1 + \frac{2}{\sqrt{19}+3} = 1 + \frac{1}{\frac{\sqrt{19}+3}{2}}$

$\frac{\sqrt{19}+3}{2} = 3 + \left(\frac{\sqrt{19}+3}{2} - 3\right) = 3 + \frac{\sqrt{19}-3}{2} = 3 + \frac{5}{2(\sqrt{19}+3)}$

$\frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5} = 1 + \frac{3}{\sqrt{19}+2}$

$\frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3} = 2 + \frac{1}{\sqrt{19}+4}$, $\sqrt{19}+4 = 8 + (\sqrt{19}-4)$.

After 8 the quotients 2, 1, 3, 1, 2, 8, recur; therefore $\sqrt{19} =$
$4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{8+}\cdots$

It is of importance to note that after 8 the quotients recur, and that 8 is twice the first quotient of 4; it will be shown later that the quotients do recur as soon as the quotient which is double the first is obtained.

The first seven convergents are found just as before, and would be, by article(8), $\frac{4}{1}$, $\frac{9}{2}$, $\frac{13}{3}$, $\frac{48}{11}$, $\frac{61}{4}$, $\frac{170}{39}$, $\frac{1421}{321}$, . . . If the last quotient was to be taken as the value of $\sqrt{19}$, the error would be less than $\frac{1}{(326)^2}$, (17) or less than $\frac{1}{(326)^2}$ which equals $\frac{1}{102,400}$, finally the error is less than .00001. Thus the seventh convergent gives the value correct to at least four decimal places.

With this introduction on how to form a recurring continued fraction, we may more easily understand the theoretical conversion of a quadratic irrational into a recurring continued fraction.

29. One may always convert a quadratic irrational into a recurring continued fraction.

Suppose N is a positive integer which is not an exact square and a_1^2 is the greatest exact square in N; then $\sqrt{N} = a_1 + (\sqrt{N} - a_1) = a_1 + \frac{r_1}{\sqrt{N} + a_1}$, if $r_1 = N - a_1^2$. Let b_1 be the greatest integer contained in $\frac{\sqrt{N} + a_1}{r_1}$; then $\frac{\sqrt{N} + a_1}{r_1} = b_1 + \frac{\sqrt{N} - b_1 r_1 + a_1}{r_1} = b_1 + \frac{\sqrt{N} - a_2}{r_1} = b_1 + \frac{r_2}{\sqrt{N} + a_2}$ where $a_2 = b_1 r_1 - a_1$ and $r_1 r_2 = N - a_2^2$. Similarly $\frac{\sqrt{N} + a_2}{r_2} = b_2 + \frac{\sqrt{N} - a_3}{r_2} = b_2 + \frac{r_3}{\sqrt{N} + a_3}$ where $a_3 = b_2 r_2 - a_2$ and $r_2 r_3 = N - a_3^2$. In general $\frac{\sqrt{N} + a_{n-1}}{r_{n-1}} = b_{n-1} + \frac{\sqrt{N} - a_n}{r_{n-1}} = b_{n-1} + \frac{r_n}{\sqrt{N} + a_n}$, where $a_n = b_{n-1} r_{n-1} - a_{n-1}$ and $r_{n-1} \cdot r_n = N - a_n^2$. Therefore $\sqrt{N} = a_1 + \frac{1}{b_1 +} \frac{1}{b_2 +} \frac{1}{b_3 +} \frac{1}{b_4 +} \cdots$, and consequently $\sqrt{N}$ can be expressed as a continued fraction. It has not been shown that the periods recur, but this will be shown later.

Since every recurring continued fraction or periodic continued fraction occurs from a quadratic irrational, it would be safe to assume that, given the periodic continued fraction, one could find the quadratic irrational.

30. Every periodic continued fraction is equal to one of the roots of a quadratic equation whose coefficients are real.[23] Assume x represents the continued fraction, and y the recurring part, that is $x = a + \frac{1}{b+} \frac{1}{c+} \frac{1}{d+} \cdots \frac{1}{h+} \frac{1}{k+} \frac{1}{y}$, $y = m + \frac{1}{n+} \cdots \frac{1}{u+} \frac{1}{v+} \frac{1}{y}$, where a, b, c, ..., h, m, n, ..., u, v, are positive integers. Suppose $\frac{p}{q}$, $\frac{p'}{q'}$ are the convergents to x corresponding to the successive quotients h, k respectively; since y is the complete fraction, $x = \frac{p'y + p}{q'y + q}$ by article (11).

23. Ibid., p. 293.

Then $x(q'y+q) = p'y+p$

$$xq'y+qx = p'y+p$$

$$xq'y-p'y = p-qx$$

$$y(xq'-p') = p-qx, \quad y = \frac{p-qx}{q'x-p'}.$$

Let $\frac{r}{s}, \frac{r'}{s'}$ be the convergents successive to the quotients u, v respectively; then $y = \frac{r'y+r}{s'y+s}$ because y is the complete quotient. (11) Then $\frac{p-qx}{q'x-p'} = \frac{r'y+r}{s'y+s}$,

$$(p-qx)(s'y+s) = (r'y+r)(q'x-p')$$

$$\left[p-q\left(\frac{p'y+p}{q'y+q}\right)\right](s'y+s) = (r'y+r)\left[q'\left(\frac{p'y+p}{q'y+q}\right)-p'\right]$$

$$(pq'y+pq-qp'y-pq)(s'y+s) = (r'y+r)(q'p'y+q'p-p'q'y-pq)$$

$$[y(pq'-qp')](s'y+s) = (r'y+r)(q'p-p'q)$$

But $pq' - qp' = \pm 1$ by (12).

$$y(s'y+s) = r'y+r$$

$$y(s'y+s) - r'y-r = 0$$

$$s'y^2+(s-r')y-r = 0$$

This equation gives the value of y. One can see that the roots are real and of opposite signs; if the positive value of y be substituted into $x = \frac{p'y+p}{q'y+q}$, by rationalizing the denominator the value of x is of the form $\frac{A+\sqrt{B}}{C}$ where A, B, and C are integers, with B being positive since the value of y is real. One has found the value of the irrational which had been converted into a recurring fraction. An irrational of this form will not always follow the normal irrational when converted into a continued fraction. The second period does not necessarily begin with $\frac{1}{2a_1}$, nor does the periodic part necessarily begin with the second partial quotient. One can observe in the example that the second periodic cycle does not start with $\frac{1}{2a_1}$; however, when only an irrational square root is involved, the pattern will be as previously described.

Example: Express $x = 1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\cdots$ as an irrational.

y = the periodic part, or $y = \frac{1}{2+}\frac{1}{3+y}$; $x-1 = \frac{1}{2+}\frac{1}{3+y}$.

but x = y+1, therefore y = x-1.

$$x-1 = \frac{1}{2+}\frac{1}{3+x-1}$$

$$x-1 = \frac{1}{2+\frac{1}{3+x-1}} = \frac{1}{\frac{6+2x-2+1}{x+2}}$$

$$x-1 = \frac{x+2}{2x+5} =$$

$$2x^2-3x-5 = x+2$$

$$2x^2-2x-7 = 0.$$

$x = \frac{--2+\sqrt{60}}{4} = \frac{\sqrt{15}--1}{2}$, the irrational sought.

Example: Find the difference of the squares of the continued fractions $a + \frac{b}{2a+}\frac{b}{2a+}\frac{b}{2a+\cdots}$, and $c + \frac{d}{2c+}\frac{d}{2c+}\frac{d}{2c+\cdots}$.

$$x = a + \frac{b}{2a+}\frac{b}{2a+}\frac{b}{2a+}\cdots, \quad y = \frac{b}{2a+}\frac{b}{2a+}\cdots.$$

$$y = x-a$$

$$x-a = \frac{b}{2a+x-a}$$

$$x-a = \frac{b}{x+a}$$

$$x^2-a^2 = b, \text{ and } x = \sqrt{a^2+b}.$$

In a similar manner, $x' = c + \frac{d}{2c+}\frac{d}{2c+}\frac{d}{2c+}\cdots$

$$x'-c = \frac{d}{x'+c}$$

$$(x')^2-c^2 = d$$

$$(x')^2 = c^2+d$$

$$x' = \sqrt{c^2+d}$$

The difference of the squares of the continued fractions

$x = a + \frac{b}{2a+}\frac{b}{2a+}\frac{b}{2a+}\cdots$, and $x' = c + \frac{d}{2c+}\frac{d}{2c+}\frac{d}{2c+}\cdots$,

is a^2+b-c^2-d.

It was stated earlier that a recurring continued fraction consists of recurring periods; thus far this has been assumed.

This fact will, after some necessary preliminary material has been established, be proved. It is evident that the period will begin whenever any complete quotient is first repeated. (29)

From (29) it appears that the quantities a_1, r_1, b_1, b_2, $b_3 \ldots$, are positive integers. This is true because a_1 is the greatest positive integer in N. Therefore $\sqrt{N} > a_1$, and $N^2 - a_1^2$ is positive. Since $N^2 - a_1^2$, is r_1, r_1 is positive. Also b_1, b_2, $b_3 \ldots$, are positive for the same reason a_1 is positive.

31. It will now be proven that the quantities a_2, a_3, $a_4 \ldots$, r_2, r_3, $r_4 \ldots$, are also positive integers.[24] Suppose $\frac{p}{q}$, $\frac{p'}{q'}$, and $\frac{p''}{q''}$ are three consecutive convergents to $\sqrt{N}$; let $\frac{p''}{q''}$ be the convergent corresponding to the partial quotient b_n. The complete quotient as defined earlier is $\frac{\sqrt{N}+a_n}{r_n}$; hence

$$\sqrt{N} = \frac{\left(\frac{\sqrt{N}+a_n}{r_n}\right)p' + p}{\left(\frac{\sqrt{N}+a_n}{r_n}\right)q' + q} \qquad (11)$$

$$\sqrt{N} = \frac{p'\sqrt{N} + a_n p' + r_n p}{q'\sqrt{N} + a_n q' + r_n q}$$

$$q'N = a_n q'\sqrt{N} + r_n q\sqrt{N} = p'\sqrt{N} + a_n p' + r_n p.$$

Equating rationals and irrationals one has

#1 $\quad a_n p' + r_n p = q'N$, $\qquad$ #2 $\quad p'\sqrt{N} = a_n q'\sqrt{N} + r_n q\sqrt{N}$

$$p' = a_n q' + r_n q$$

$$r_n = \frac{p' - a_n q'}{q}$$

Substituting for r_n in #1 one has,

$$a_n p' + \left(\frac{p' - a_n q'}{q}\right)p = q'N$$

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24. Ibid., p. 296.

$$a_n p'q + pp' - a_n q'p = qq'N$$
$$a_n(pq' - p'q) = pp' - qq'N$$

From #1

$$a_n p' = q'N - r_n p$$
$$a_n = \frac{q'N - r_n p}{p'}$$

Substituting in #2

$$p' = (q')^2 - q' r_n p + r_n q$$
$$(p')^2 = (q')^2 N - pq' r_n + r_n p'q$$
$$r_n(pq' - p'q) = N(q')^2 - (p')^2$$

But $pq' - p'q = \pm 1$, and $pq' - p'q$, $pp' - qq'N$, and $N(q')^2 - (p')^2$ have the same sign (20). Therefore a_n and r_n are positive integers. Since two convergents precede the complete quotient $\frac{\sqrt{N} + a_n}{r_n}$ the investigation holds for all values of n greater than one.

32. It may now be proven that complete and partial quotients recur.[25] From article (27) one can see that $r_n r_{n-1} = N - a_n^2$. Also r_n and r_{n-1} are positive integers; therefore a_n must be less than $\sqrt{N}$, and a_n cannot be greater than a_1. Consequently a_n cannot have any values except 1, 2, 3, ..., a_1; that is, the number of different values cannot exceed a_1.

Also, $a_{n+1} = r_n b_n - a_n$; $r_n b_n = a_n + a_{n+1}$, and therefore $r_n b_n$ cannot be greater than $2a_1$. Thus r_n cannot have any values except 1, 2, 3..., $2a_1$; or, the number of different values of r_n cannot exceed $2a_1$. Consequently the complete quotient $\frac{\sqrt{N} + a_n}{r_n}$ cannot have more than $2a_1^2$ different values.

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25. Ibid., p. 296.

Therefore, some one complete quotient, and therefore all subsequent ones, must recur. Also b_n is the greatest integer in $\frac{\sqrt{N}+a_n}{r_n}$; hence the partial quotients must recur, and the number of partial quotients in each period cannot be greater than $2a_1^2$.

33. It will be shown that $a_1 < a_n + r_n$.[26] $a_{n-1} + a_n = b_n - r_{n-1}$. Therefore $a_{n-1} + a_n \geq r_{n-1}$, since b_{n-1} is a positive interger, and $\sqrt{N} + a_n > r_{n-1}$. But $N - a_n^2 = r_n r_{n-1}$ (29). Therefore $\sqrt{N} - a_n < r_n$, and $a_1 - a_n < r_n$.

34. We have assumed that the period of a recurring continued fraction begins with the second partial quotient and terminates with a partial quotient double the first.[27] This will now be proven.

In (32) it is shown that a recurrence must take place; suppose the $(n+1)^{th}$ complete quotient recurs at the $(s+1)^{th}$ complete quotient; the $a_s = a_n$, $r_s = r_n$, and $b_s = b_n$; if it can be shown that $a_{s-1} = a_{n-1}$, $r_{s-1} = r_{n-1}$, and $b_{s-1} = b_{n-1}$ we will have shown that if the $(n+1)^{th}$ quotient recurs, so must the (n^{th}) quotient and the $(n-1)^{th}$, and so on.

One may establish $r_{s-1} r_s = N - a_s^2 = N - a_n^2 = r_{n-1} r_n$, since it is assumed that $a_s = a_n$, $r_s = r_n$, and $b_s = b_n$ (28).

Again $a_{n-1} + a_n = b_{n-1} r_{n-1}$, $a_{s-1} + a_s = b_{s-1} r_{s-1} = b_{s-1} r_{n-1}$;

therefore $a_{n-1} - a_{s-1} = b_{n-1} r_{n-1} - b_{s-1} r_{n-1}$,

$$a_{n-1} - a_{s-1} = r_{n-1} (b_{n-1} - b_{s-1}) .$$

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26. Ibid., p. 297.

27. Ibid., p. 297.

Consequently $\frac{a_{s-1}-a_{n-1}}{r_{n-1}} = b_{n-1}-b_{s-1}$ which equals zero or an integer. But by article (33) $a_1 - a_{n-1} < r_{n-1}$, and $a_1 - a_{s-1} < r_{s-1}$; that is $a_1 - a_{s-1} < r_{n-1}$; therefore $a_{n-1} - a_{s-1} < r_{n-1}$, and $\frac{a_{n-1} - a_{s-1}}{r_{n-1}}$ must be less than unity and must be zero since it is either zero or an interger. Consequently $a_{s-1} = a_{n-1}$, and also $b_{s-1} = b_{n-1}$. If, then, the $(n+1)^{th}$ complete quotient recurs, the n^{th} complete quotient must recur; therefore the $(n-1)^{th}$ complete quotient recurs, and so on. This proof holds as in (31), if a_n is not less than two. Hence the complete quotients recur beginning with the second quotient $\frac{\sqrt{N}+a_1}{r_1}$.

From this one can see that the recurrence begins with the second partial quotient b_1. It remains to be shown that the cycle ends with a partial quotient equal to $2a_1$. Suppose $\frac{\sqrt{N}+a_n}{r_n}$ is the complete quotient which just precedes the second complete quotient $\frac{\sqrt{N}+a_1}{r_1}$ when it recurs; $\frac{\sqrt{N}+a_n}{r_n}$ and $\frac{\sqrt{N}+a_1}{r_1}$ are then consecutive complete quotients; and, $a_n + a_1 = r_n b_n$, $r_n r_1 = N - a_1^2$. But $N - a_1^2 = r_1$; therefore r_n must equal one. Again $a_1 - a_n < r_1$, or < 1; hence $a_1 - a_n = 0$, or $a_n = a_1$. Also $a_n + a_1 = r_n b_n = b_n$; therefore $b_n = 2a_1$, and the hypothesis is proven.

35. It may now be shown that in any period the partial quotients which are equidistance from beginning and end are equal, excluding the last partial quotient.[28] If the last complete quotient is $\frac{\sqrt{N}+a_n}{r_n}$ then $r_n = 1$, $a_n = a_1$, and $b_n = 2a_1$.

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28. Ibid., p. 298.

We shall prove $r_{n-1} = r_1$, $a_{n-1} = a_2$, $b_{n-1} = b_1$,

$r_{n-2} = r_2$, $a_{n-2} = a_3$, $b_{n-2} = b_2$.

If this can be established we will have proved the partial quotients which are equidistant from beginning and end are equal.

$$r_{n-1} = r_n r_{n-1} = N - a_n{}^2 = N - a_1{}^2 = r_1 \quad (34).$$

also $a_{n-1} + a_1 = a_{n-1} + a_n = r_{n-1} b_{n-1} = r_1 b_{n-1}$, and $a_1 + a_2 = r_1 b_1$; therefore $a_1 + a_2 - a_{n-1} - a_n = r_1 b_1 - r_1 b_{n-1}$, $a_2 - a_{n-1} = r_1(b_1 - b_{n-1})$. Hence $\frac{a_2 - a_{n-1}}{r_2} = b_1 - b_{n-1}$ which equals zero or an integer. But $\frac{a_2 - a_{n-1}}{r_1} < \frac{a_1 - a_{n-1}}{r_1}$, that is $< \frac{a_1 - a_{n-1}}{r_{n-1}}$ which is less than unity (33); thus $a_2 - a_{n-1} = 0$, $a_2 = a_{n-1}$, and $b_{n-1} = b_n$. Similarly $r_{n-2} = r_2$, $a_{n-2} = a_3$, $b_{n-2} = b_2$, and so on.

36. At times the penultimate convergents, or the next to last convergents of recurring periods prove of use.[29] They may be obtained in the following manner. If n is the number of partial quotients in the recurring period, the penultimate convergents of the recurring periods would be the n^{th}, $2n^{th}$, $3n^{th}$, ..., convergents. Suppose these are denoted by $\frac{p_n}{q_n}, \frac{p_{2n}}{q_{2n}}, \frac{p_{3n}}{q_{3n}}, \frac{p_{4n}}{q_{4n}}, \ldots$, respectively. Since $\sqrt{N} = a_1 + \frac{1}{b_1 +}\frac{1}{b_2 +}\frac{1}{b_3 +} \cdots \frac{1}{b_{n-1} +}\frac{1}{2a_1 + \cdots}$, the partial quotient corresponding to $\frac{p_{n+1}}{q_{n+1}}$ is $2a_1$. Therefore $\frac{p_{n+1}}{q_{n+1}} = \frac{2a_1 p_n + p_{n-1}}{2a_1 q_n + q_{n-1}}$ (8). The complete quotient at the same stage is the period $2a_1 + \frac{1}{b_1 +}\frac{1}{b_2 +} \cdots \frac{1}{b_{n-1} + \cdots}$ ...;

29. Ibid., p. 299.

this is equal to $a_1+\sqrt{N}$, since $\sqrt{N} = a_1 + \frac{1}{b_1+} \frac{1}{b_2+} \frac{1}{b_3+} \cdots \frac{1}{b_{n-1}+} \cdots$. Consequently $\sqrt{N} = \dfrac{(a_1+\sqrt{N})\,p_n + p_{n-1}}{(a_1+\sqrt{N})\,q_n + q_{n-1}}$ (11)

Clearing of fractions,

$$\sqrt{N}\left[(a_1+\sqrt{N})q_n + q_{n-1}\right] = (a_1+N)p_n + p_{n-1},$$

$$q_n\sqrt{N}a_1 + Nq_n + \sqrt{N}q_{n-1} = a_1p_n + \sqrt{N}p_n + p_{n-1}.$$

Equating rationals and irrationals one has

$$q_n\sqrt{N}a_1 + \sqrt{N}\,q_{n-1} = \sqrt{N}\cdot p_n$$

#1 $a_1q_n + q_{n-1} = p_n$
and
#2 $N\,q_n = a_1p_n + p_{n-1}.$

Also $\frac{p_{2n}}{q_{2n}}$ can be obtained from $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$ by taking for the quotient $2a_1 + \frac{1}{b_1+}\frac{1}{b_2+}\frac{1}{b_3+}\cdots\frac{1}{b_{n-1}}$ which is equal to $a_1 + \frac{p_n}{q_n}$.

Then $$\frac{p_{2n}}{q_{2n}} = \frac{\left(a_1+\frac{p_n}{q_n}\right)p_n + p_{n-1}}{\left(a_1+\frac{p_n}{q_n}\right)q_n + q_{n-1}} \quad (11)$$

which equals $$\frac{a_1p_n + \frac{p_n}{q_n}\cdot p_n + p_{n-1}}{a_1q_n + \frac{p_n}{q_n}q_n + q_{n-1}} = \frac{Nq_n + \frac{p_n}{q_n}\cdot p_n}{p_n + q_n\cdot\frac{p_n}{q_n}}$$ from equations #1 and #2.

Therefore $$\frac{p_{2n}}{q_{2n}} = \frac{Nq_n}{2p_n} + \frac{\frac{p_n}{q_n}\cdot p_n}{p_n\left(1+\frac{q_n}{q_n}\right)} = \frac{1}{2}\left(N\frac{q_n}{p_n} + \frac{p_n}{q_n}\right)$$ where $\frac{p_{2n}}{q_{2n}}$ is the second periodic period.

General Examples:

A. Convert $\sqrt{a^2+1}$ into a continued fraction, and check by employing article (30).

$$\sqrt{a^2+1} = a(\sqrt{a^2+1} - a)$$
$$= a + \frac{(\sqrt{a^2+1}-a)(\sqrt{a^2+1}+a)}{\sqrt{a^2+1}+a}$$
$$= a + \frac{a^2+1-a^2}{\sqrt{a^2+1}+a} = a + \frac{1}{\sqrt{a^2+1}+a}$$

$$\sqrt{a^2+1}+a = 2a + (\sqrt{a^2+1} + a - 2a)$$

$$= 2a + (\sqrt{a^2+1} - a)$$
$$= 2a + \frac{(\sqrt{a^2+1} - a)(\sqrt{a^2+1} + a)}{\sqrt{a^2+1} + a}$$
$$= 2a + \frac{1}{\sqrt{a^2+1} + a}$$

One can see that the next quotient will also be 2a, and so on. Consequently the continued fraction is $a + \frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\cdots$ Use (30) to see if the correct continued fraction has been found. Let x equal the continued fraction.

$$x = a + \frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\cdots,$$
$$y = \frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\cdots,$$

Therefore $y = x-a$; then $x-a = \frac{1}{2a+x-a}$,

$$x - a = \frac{1}{x+a}$$
$$x^2 - a^2 = 1$$
$$x^2 = a^2+1$$
$$x = \sqrt{a^2+1}$$, the original irrational.

B. Show the fourth convergent of the irrational $\sqrt{a^2+1}$ is $\frac{8a^4+8a^2+1}{8a^4+4a}$. By (8) one finds the first convergents to be $\frac{a}{1}$, $\frac{2a^2+1}{2a}$, $\frac{2a(2a^2+1)+a}{2a(2a^2)+1} = \frac{4a^3+3a}{4a^3+1}$,

The fourth is then $\frac{2a(4a^3+3a)+2a^2+1}{2a(4a^3+1)+2a} = \frac{8a^3+6a^2+2a^2+1}{8a^3+2a+2a}$

$$= \frac{8a^3+8a^2+1}{8a^4+4a},$$

the convergent being sought.

The theory of continued fractions is quite useful in solving indeterminate equations of the second degree, although probably not as useful as it was in solving indeterminate equations of the first degree.

A second degree indeterminate equation is one of the form,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c, f, g, and h are integers.

In many cases one need not use continued fractions to find positive integral values of x and y. The following method may often be applied.

37. Solving the equation $ax^2+2hxy+by^2+2gx+2fy+c = 0$ as a quadratic one has $ax^2+x(2hy+2g)+(by^2+2fy+c) = 0$; by the quadratic formula $x = -2hy-2g \pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}/2a$

$$x = -hy-g \pm\sqrt{h^2y^2+2hyg+g^2-ahy^2-2a+y+ac}/a$$

$$ax+hy+g = \pm\sqrt{y^2(h^2-ab)+2y(hg-af)+(g^2-ac)}.$$ [30]

If the values of x and y are to be positive integers, the expression under the radical, which may be denoted as $py^2+2qy+r$, must be a perfect square; or $py^2+2qy+r = z^2$. Solving this equation as a quadratic in the same manner as above, one has $py+q = \pm\sqrt{q^2-pr+pz^2}$; and, as above, the expression under the radical must be a perfect square. Suppose this is equal to t^2; then $q^2-pr+pz^2= t^2$

$$q^2-pr = t^2-pz^2,$$ where t and z are variables, and p, q, and r are constants. Unless this equation can be solved in positive integers, the original equation has no positive integral solutions. This is true because $py+q$ would contain negative or imaginary roots if the solutions are negative, as would $ax+hy+g$. If a, b, and h are all positive, it is obvious that the number of solutions is limited, because for large values of x and y the sign of the expression on the left in the original equation depends upon that of $ax^2+2xy+by^2$ and thus cannot be zero for large positive integral values of x and y.

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30. Ibid., p.303

where N and a are positive integers.[31] It is obvious that the equation $x^2+Ny^2=-a$ has no real roots, and the equation $x^2+Ny^2=a$ has only a limited number of solutions which may be obtained by trial and error. Consequently we shall limit ourselves to equations of the form $x^2-Ny^2=\pm a$. Again we will find certain parts of the theory of continued fractions useful in obtaining solutions to equations of this form.

39. The equation $x^2-Ny^2=1$ can always be solved in terms of positive integers.[32] Suppose the $\sqrt{N}$ is converted into a continued fraction; let $\frac{p}{q}$, $\frac{p'}{q'}$, and $\frac{p''}{q''}$ be any three consecutive integers. Suppose that $\frac{\sqrt{N}+r_n}{r_n}$ is the complete quotient corresponding to $\frac{p''}{q''}$; one then has $r_n(pq'-p'q)=Nq'^2-p'^2$ (31). But $r_n=1$ at the end of any period (35). Therefore $p'^2-Nq'^2=p'q-pq'$, with $\frac{p'}{q'}$ being the penultimate convergent of the recurring period. If the number of quotients in the period is even, $\frac{p'}{q'}$ is an even convergent, and is therefore greater than $\sqrt{N}$, and likewise greater than $\frac{p}{q}$; therefore $p'q-pq'=1$. (7) Then $p'^2-Nq'^2=1$, and hence $x=p'$, $y=q'$ is a solution of the equation $x^2-Ny^2=1$.

Suppose the number of quotients in the period is odd, then the penultimate convergent of the first period is odd, but the penultimate convergent of the second period is an even convergent. Consequently integral solutions will be found by putting $x=p'$, $y=q'$, where $\frac{p'}{q'}$ is the penultimate convergent in the second, fourth, sixth,... recurring periods.

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31. Ibid., p. 304.

32. Ibid., p. 304.

One can see that the number of solutions in either case is unlimited.

40. We shall now consider a method to obtain solutions in positive integers of the equation $x^2-Ny^2=-1$.[33] Using the same procedure as (39), one has $p'^2-Nq'^2=p'q-pq'$. If the number of partial quotients is odd, and if $\frac{p'}{q'}$ is an odd penultimate convergent in any recurring period, $\frac{p'}{q'} < \frac{p}{q}$ (7). Therefore $p'q-pq'=-1$. In this case, then, $p'^2-Nq'^2=-1$, and integral solutions may again be obtained by letting $x=p'$, $y=q'$, where $\frac{p'}{q'}$ is the penultimate convergent in the first, third, fifth,....., recurring periods.

Example: Solve in positive integers $x^2-13y^2=\pm 1$.

$$\sqrt{13} = 3 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots$$

The number of quotients in the period is odd; consequently the penultimate convergent of the first period is a solution of $x^2-13y^2=-1$. The convergents are $\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5},\ldots$; consequently $x=18$, $y=5$ is a solution of $x-13y=-1$. By article (36) the penultimate convergent in the second recurring period is $\frac{1}{2}\left(\frac{18}{5}+\frac{5}{18}\times 13\right)=\frac{649}{180}$. Therefore $x=649$, $y=180$ is a solution of $x-13y=1$.

41. If one solution in positive integers of the equation $x^2-Ny^2=1$ has been found, others may be obtained by employing the following method.[34] Suppose that $x=h$, $y=k$ is a solution of the equation; then $(h^2-Nk^2)^n=1$, where n is any positive integer. Then $x^2-Ny^2=(h^2-Nk^2)=1=(h^2-Nk^2)^n=1$.

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33. Ibid., p. 305.

34. Ibid., p. 306.

Therefore $(x+y\sqrt{N})(x-y\sqrt{N}) = (h+k\sqrt{N})^n(h-k\sqrt{N})^n$

(A). Set $(x+y\sqrt{N}) = (h+k\sqrt{N})^n$, and (B),

set $(x-y\sqrt{N}) = (h-k\sqrt{N})^n$.

Setting these factors equal will not provided all of the possible solutions, but it will supply many. Adding (A) and (B) one has $2x = (h+k\sqrt{N})^n-(h-k\sqrt{N})^n$. By ascribing to n values of 1, 2, 3, 4, 5,..., one may find unlimited number of values for x and y which satisfy the equation.

One may obtain values for $x^2-Ny^2= -1$ in the same way. If n is any odd positive integer $x^2-Ny^2 = (h^2-Nk^2) = -1 = (h^2-Nk^2)^n$. This method is the same as the previous one except that n is restricted to odd values.

42. If one puts $x=ax'$, $y=ay'$, the equations $x^2-Ny^2 = \pm a^2$, or $x'^2-Ny'^2 = \pm 1$, which may be solved as in (40) and (41).[35]

43. It has been shown in article (39) that $p'^2-Nq'^2 = -r_n(pq'-p'q) = \pm r_n$. Therefore if a in the equation $x^2-Ny^2= \pm a$ is a denominator of a complete quotient which occurs in converting $\sqrt{N}$ into a continued fraction, and if p'/q' is the convergent obtained by stopping short of the complete quotient, one of the equations $x^2-Ny^2 = \pm a$ is satisfied by the values $x=p'$, $y=q'$.[36]

Again, the odd convergents are all less than N, and the even are all greater than N; consequently if p'/q' is an even convergent, $x=p'$, $y=q'$ is a solution of $x^2-Ny^2 = -a$. This applies only if a is a complete quotient.

44. The method explained in the preceding article enables

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35. Ibid., p. 306

36. Ibid., p. 306

us to find a solution of one of the equation $x^2-Ny^2 = \pm a$, only when a is one of the denominators which occurs in the process of converting $\sqrt{N}$ into a continued fraction.[37]

Example: If the $\sqrt{7}$ was converted into a continued fraction one would find this result. $\sqrt{7} = 2 + (\sqrt{7}-2) = 2 + \frac{3}{\sqrt{7}+2}$

$$\frac{\sqrt{7}+2}{3} = 1 + \left(\frac{\sqrt{7}+2}{3} - 1\right) = 1 + \frac{\sqrt{7}-1}{3} = 1 + \frac{2}{\sqrt{7}+1},$$

$$\frac{\sqrt{7}+1}{2} = 1 + \left(\frac{\sqrt{7}+1}{2} - 1\right) = 1 + \left(\frac{\sqrt{7}-1}{2}\right) = 1 + \frac{3}{\sqrt{7}+1}$$

$$\frac{\sqrt{7}+1}{3} = \ldots$$ and so on.

The denominators of the complete quotients are 3, 2, 3, 1.

The successive convergents are $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48}, \ldots$

If we take the cycle of equations $x-7y = -3$, $x-7y = 2$, $x-7y = -3$, $x-7y = 1$, we shall find that they are satisfied by taking for x the values 2, 3, 5, 8, 48,... .

45. Apparently then, the number of cases in which solutions in integers of equations of the form $x^2-Ny^2 = \pm a$ is limited.[38] In an actual numerical example, however, often times a positive integral solution can be discovered by trial and error if a does not fit into any of the discussed categories. As an example one can easily see that the equation $x^2-7y^2 = 53$ is satisfied by $y = 2$, $x = 9$.

If one solution is found other solutions may easily be obtained by the method now to be explained.

46. Suppose that $x = f$, $y = g$ is a solution of the equation $x^2-Ny^2 = a$; let $x = h$, $y = k$ be any solution of the equation $x^2-Ny^2 = 1$. Then $x^2-Ny^2 = (f^2-Ng^2)(h^2-Nk^2)$.

38. Ibid., p. 307.

By adding and subtracting 2Nfghk and 2Nfghk one has

$x^2-Ny^2 = f^2h^2-Ng^2h^2-Nk^2f^2+N\ g^2k^2 \pm 2Nfghk\mp 2N\ fghk$

$= (fh \pm Ngk)^2 - N\ (fk \pm gh)^2.$

If $x^2-Ny^2 = (fk \pm Ngk)^2 - N\ (fk \pm gh)^2$, then $x^2 = (fh \pm Ngk)^2$, and $x = \pm\ (fh\pm Ngk)^2$. Hence $y = \pm\ (fk\pm gh)$, and we have the general solution of the equation.

Example: As shown above a solution of $x^2-7y^2 = 53$ is $x=9$, $y=2$. By article (39) a solution of $x^2-7y^2 = 1$ is $x=8$, $y=3$. Therefore $x=114$, $y=43$ is another solution of $x^2-7y^2 = 53$.

47. **Up** to now we have assume N is not a perfect square; if, however, N is a perfect square, the form of the equation and the solution are as follows.[39] Solve the equation $x^2-n^2y^2 = a$. Suppose $a=bc$, where b and c are two positive integers of which b is the larger; then $(x+ny)(n-ny) = bc$. Set $x+ny = b$, and $x-ny = c$. If the values of x and y found from these equations are integers a solution has been found. Other solutions may be found by giving b and c other values such that bc still equals a.

Example: Find two positive integers whose squares differ by sixty. Suppose x and y are the two integers; then $x^2-y^2 = 60$, or $(x+y)(x-y) = 60$. The product of the following pairs of integers is 60.

1, 60; 2, 30; 3, 20; 4, 15; 5,12; 6,10.

The values sought are obtained from the equations $x+y = 30$, $x-y = 2$; $x+y = 10$, $x-y = 6$. The other equations give fractional values of x and y.

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39. Ibid., p. 308

The solutions are $x=16$, $y=14$, and $x=8$. $y=6$.

By a similar process one may obtain the solution in positive integers of $ax^2+2hxy+by^2+2gx+2gy+c=k$ if the left hand member can be resolved into two rational linear factors.

48. If, in the general equation $ax^2+2hxy+by^2+2gx+2fy+c=k$ either a or b, or both, are zero, instead of employing the method explained in article (37), one may use the simpler method demonstrated in the following example.[40]

Example: Solve in positive integers $2xy-4x^2+12x-5y=11$. Expressing y in terms of x one has $y=\frac{4x^2-12x+11}{2x-5}=2x-1+\frac{6}{2x-5}$. If y is to be an integer $\frac{6}{2x-5}$ must be an integer; therefore $2x-5$ must be equal to $\pm1, \pm2, \pm3$, and ±6. The cases ±2, ±6 may be rejected since they do not yield integers when substituted for x in the expression $\frac{6}{2x-5}$. Consequently the acceptable values of x are obtained from $2x-5=\pm1$, and $2x-5=\pm3$; the values of x are then 3, 2, 4, and 1. Taking the values in succession we obtain $x=3$, $y=11$; $x=2$, $y=-3$; $x=4$, $y=9$; $x=1$, $y=-1$. The positive integral solutions are then selected from this group of solutions.

49. The principles already explained are a help in discovering for what values of the variables given linear or quadratic functions of x and y become perfect squares.[41]

Example: Find the general expressions for two positive integers which are such that if their product is subtracted

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40. Ibid., p. 309.

41. Ibid., p. 309.

from the sum of their squares the difference is a perfect square.

Assume the equation is $x^2-xy+y^2 = z^2$. Then $x(x-y)=z^2-y^2$. This equation is satisfied by the suppositions $mx=n(z+y)$, $n(x-y)=m(z-y)$, where m and n are positive integers.
Hence $mx-ny-nz=0$ (A), $nx+(m-n)y-mz=0$ (B).
From (A) $nz=mx-ny$, $z=\frac{mx-ny}{n}$; $mx-ny-mx+ny$

$$= nx+(m-n)y-\frac{m}{n}(mx-ny)=0;$$
$$n^2x+n(m-n)y-m^2x-mny=0,$$
$$m^2x-n^2x=2mny-n^2y$$
$$x(m^2-n^2)=y(2mn-n^2)$$
$$\frac{x}{2mn-n^2}=\frac{y}{m^2-n^2}=\frac{z}{m^2-mn+n^2}$$

The third ratio in this proportion, $\frac{z}{m^2-mn+n^2}$ is found in a similar manner as $\frac{x}{2mn-n^2}$ and $\frac{y}{m^2-n^2}$. Since the given equation is homogeneous the general solution may be written as follows. $x=2mn-n^2$, $y=m^2-n^2$, $z=m^2-mn+n^2$. Again m and n are any two positive integers, m being the larger; thus if $m=7$, $n=4$ one has $x=4$, $y=33$, $z=37$.

Example: Find the general expression for three positive integers in an arithmetic progression so that the sum of every two is a perfect square. Let the arithmetic progression be denoted by $x-y$, x, $x+y$, and let $2x-y=p^2$, $2x=q^2$, $2x+y=r^2$. Therefore $p^2+r^2=2q^2$, or $r^2-q^2=q^2-p^2$.
This equation is satisfied by the suppositions $m(r-q)=n(q-p)$, $n(r+q)=m(q+p)$. From these equations we obtain as in the previous example, $\frac{p}{n^2+2mn-m^2}=\frac{q}{m^2+n^2}=\frac{r}{m^2+2mn-n^2}$ and $p=n^2+2mn-m^2$, $q=m^2+n^2$, $r=m^2+2mn-n^2$. Since $x=\frac{q^2}{2}$,

$$x=\frac{1}{2}(m^2+n^2)^2,$$

$y=r^2-2x$, $y=(m^2+2mn-n^2)^2-(m^2+n^2)^2 = 4mn(m^2-n^2)$.

Three integers can now be found which satisfy the original conditions. From the value of x it is clear that m and n are either both even or both odd; also their values must be such that x is greater than y, that is

$$(m^2+n^2)^2 > 8mn(m^2-n^2)$$

$$m^4+2m^2n+n^4-8m^3n+8mn^3 > 0$$

$$m^3(m-8n)+2m^2n^2+8mn^3+n^4 > 0$$

which is true if $m > 8n$. If $m=9$, then $x=3362$, $y=2880$, and the numbers are 482, 3362, and 6242. The sums of these taken in pairs are 3844, 6774, 9604, which are the squares of 62, 82, and 98 respectively.

In this paper a few of the important uses of the theory of continued fractions have been demonstrated. Equations which are extremely difficult to solve without the use of continued fractions and related material have been presented.

There are, however, many other important areas where the theory of continued fractions is of great importance. These areas involve a more general form of the continued fraction; in this more general form the numerator of the partial quotient need not be unity. This in itself is another segment of the theory of continued fractions. Important uses are found in the areas of transcendental numbers, theory of numbers, and infinite series.

One can abserve, then that only a small, but important, area of continued fractions has been discussed in this paper. In the final analysis, one must conclude, that to the higher mathematician, continued fractions are of great importance.

Appendix

The following examples are problems whose solutions may be obtained by using the techniques introduced in this paper.

1. Solve in positive integers, $x^2-14y^2 = 1$.

$$\sqrt{14} = 3 + (\sqrt{14}-3) = 3 + \frac{5}{\sqrt{14}+3}$$

$$\frac{\sqrt{14}+3}{5} = 1 + \left(\frac{\sqrt{14}+3}{5} - 1\right) = 1 + \frac{2}{\sqrt{14}+2}$$

$$\frac{\sqrt{14}+2}{2} = 2 + \left(\frac{\sqrt{14}+2}{2} - 2\right) = 2 + \frac{\sqrt{14}-2}{2} = 2 + \frac{5}{\sqrt{14}+2}$$

$$\frac{\sqrt{14}+2}{5} = 1 + \left(\frac{\sqrt{14}+2}{5} - 1\right) = 1 + \frac{1}{\sqrt{14}+3}$$

$$\frac{\sqrt{14}+3}{1} = 6 + (\sqrt{14}+3-6) = 6 + \frac{5}{\sqrt{14}+3}$$

$$\sqrt{14} = 3 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{6+}\cdots,$$

The convergents are $\frac{3}{1}, \frac{4}{1}, \frac{11}{3}, \frac{15}{4}, \frac{101}{27}, \ldots$.

Then x=15, y=14 by (39). By (36) the second penultimate convergent is $\frac{1}{2}\left(\frac{15}{4} + \frac{4}{15} \times 14\right) = \frac{1}{2}\left(\frac{225+224}{60}\right) = \frac{449}{120}$.

Then x=449, y=120 is another solution.

2. Solve: xy-2x-y 8. By article (48) one has x(y-2) = 8+y. $x = \frac{8}{y-2} + \frac{y}{y-2}$

It appears as though y could be 4, 7, 12 and still yield integral values of x.

If y=4, x=6; y=7, x=3; y=12, x=2.

3. Solve 3x+3xy-4y = 14. Once again y(3x-4) = 14-3x

$$y = \frac{14}{3x-4} - \frac{3x}{3x-4}.$$

A value of x which yields an integral positive solution is x=2, from which y=4.

4. Find the irrational root of the continued fraction

$$\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots$$

Let x equal the continued fraction. $x = \frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+x}$.

$$x = \frac{1}{3+\dfrac{1}{2+\dfrac{1}{1+x}}},$$

$$x = \frac{1}{3+\dfrac{x+1}{2x+3}},$$

$$x = \frac{1}{3+\dfrac{x+1}{2x+3}} = \frac{1}{\dfrac{10+7x}{2x+3}} = \frac{2x+3}{7x+10}$$

$$7x^2+10x = 3+2x$$

$$7x^2+8x-3=0$$

$$x = \frac{-8+\sqrt{64+84}}{14}$$

The irrational root is

$$x = \frac{-8+2\sqrt{37}}{14} = \frac{1}{7}\left(-4+\sqrt{37}\right)$$

5. Prove: If $x = \frac{a}{b+}\,\frac{c}{d+}\,\frac{a}{b+}\,\frac{c}{d+}\cdots,$

and $y = \frac{c}{d+}\,\frac{a}{b+}\,\frac{c}{d+}\,\frac{a}{b+}\cdots,$

then $bx-dy=a-c$.

$$x = \frac{a}{b+y}$$

#1 $xy+bx=a$

$$y = \frac{c}{d+x}$$

#2 $xy+dy=c$

Subtract #2 from #1.

$$\begin{array}{l} xy+bx=a \\ \underline{xy+dy=c} \\ \end{array}$$

Then $bx-dy=a-c$, and the proof is complete.

List of References

1. H. S. Hall and S. R. Knight, Algebra for Colleges and Schools, (New York, 1947).

2. H. S. Hall and S. R. Knight, Higher Algebra, 4th edition (London, 1932), pp. 273-311, 359-372.