

THESIS: An Analysis of the Henstock-Kurzweil Integral

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Let $I = [a, b] \subseteq \mathbb{R}$ be a compact interval. Let $f : I \rightarrow \mathbb{R}$ be a real-valued function. We investigate the so-called Henstock-Kurzweil or \mathcal{HK} integral and compare it to the integrals of Riemann and Lebesgue. In the classic version of the Fundamental Theorem of Calculus, given a compact interval $I = [a, b] \subseteq \mathbb{R}$ and functions $f : I \rightarrow \mathbb{R}$ and $F : I \rightarrow \mathbb{R}$ with $F'(x) = f(x)$ for all $x \in I$, neither the Riemann, or \mathcal{R} , integral nor the Lebesgue, or \mathcal{L} , integral guarantees that

$$\int_a^b f = F(b) - F(a).$$

However, the Henstock-Kurzweil integral integrates every derivative, making integration and differentiation truly inverse processes. We look at some of the consequences of this result. In addition, we investigate specific properties of the Henstock-Kurzweil integral which serve to illustrate how a relatively small change in the definition of the Riemann integral can have far reaching consequences, and how the Lebesgue integral can be seen as a special case of the Henstock-Kurzweil integral. In particular, we show that

$$\mathcal{R}(I) \subsetneq \mathcal{L}(I) \subsetneq \mathcal{HK}(I)$$

where $\mathcal{R}(I)$, $\mathcal{L}(I)$ and $\mathcal{HK}(I)$ denote the classes of Riemann integrable, Lebesgue integrable and Henstock-Kurzweil integrable functions over I , respectively. Finally, we discuss the consequences of using the Henstock-Kurzweil integral in various areas of applied mathematics.