An Exploration of Relative Frame Potential

An Honors Thesis (HONR 499)

by

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Abstract

A frame for a vector space is a set of vectors which behaves in some ways like a basis for the vector space while having a more flexible structure that allows for frames to be constructed to fit a variety of application situations, such as in signal processing. The frame potential function, developed in 2001, takes a frame as input and returns its "frame potential value". This value reveals important information about the properties of the input frame. In this paper, we adapt the frame potential function to create a "relative frame potential function", which takes one frame as input with respect to another frame. We analyze the behavior of this function to determine what information the relative frame potential value can reveal about the relationship between the two frames.

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Process Analysis Statement

Summary of Project

Suppose we are sending a signal through a communication system, e.g. sending a text message. *Frames* are sequences of mathematical objects called vectors that can be used to accomplish this task. In order to be a frame, this sequence of vectors must satisfy a particular restriction on its structure. This restriction is necessary for our signal to be broken down, sent through the system, and reconstructed on the receiving end with considerable speed. Aside from this restriction, frames actually have a very flexible structure; specifically, they can contain as large a number of vectors as we want, and those vectors don't have to be of a particular length. This means that we can construct our frame to have more vectors than necessary to break down and reconstruct our signal. Then, if some of the signal is lost during transmission, the lost part can be reconstructed on the receiving end using the extra vectors.

Frames are useful in a variety of applications, including signal processing. The reason they are useful and therefore studied is that, just as in this example, the restriction on their structure allows for speed of computation, while their flexibility in number and length of vectors allows them to be constructed to fit a variety of different application situations.

The concept that inspired this project was the notion of *frame potential*. The frame potential function takes a frame as input and returns its "frame potential value". The frame potential value reveals whether or not the input frame is a *tight frame*, a specific type of frame with properties that make it very useful. Since tight frames are so useful in applications, it is important to be able to find them. The frame potential function is a tool that helps to do just that.

During an independent study in frames, Dr. Aceska and I were inspired by this frame potential function and adapted it to create what we called the *relative frame potential function*. This function, instead of returning one frame's isolated frame potential value, takes a frame as input and returns its "relative frame potential" *with respect to another frame*. Our conjecture was that this function would be able to reveal important information about the relationship between the two frames. Our exploration showed that this was true, and we were able to prove our conjecture.

For this thesis project, Dr. Aceska and I wanted to see if our results about relative frame potential would extend to what are called *fusion frames*. While traditional frames are sequences of vectors, fusion frames can be thought of as sequences of traditional frames. This makes them useful in distributed processing applications, where in order to quickly process a large amount of data, the processing job is broken down into smaller jobs and carried out by different computers, with the results then being combined together. We adapted the existing

fusion frame potential function, which reveals whether or not a fusion frame is a tight frame, to create a *relative fusion frame potential function*, which, similarly to our function for traditional frames, takes a fusion frame as input and returns its relative fusion frame potential value with respect to another fusion frame. The goal of our research was to see whether or not this function would reveal important information about the relationship between two fusion frames, as the traditional relative frame potential function had done for traditional frames. Our end product was a paper detailing our results on both traditional and fusion frames that also serves as a first draft of a paper we will be submitting to a journal in the near future.

Just as the frame potential function can be used to quickly determine whether or not a frame is a tight frame, our relative frame potential function can be used to quickly determine the relationship between two frames. Without using such a tool, obtaining this information about the relationship between two frames requires an extensive calculation that cannot be carried out quickly by a computer. However, the computation of the relative frame potential value of one frame with respect to another involves only operations that are easy for a computer to carry out with considerable speed.

Research Process

The first part of the process of attempting to extend our relative potential results to fusion frames was for me to learn about fusion frames and how traditional frame potential had been extended to fusion frames. I read through and studied the paper where a notion of fusion frame potential had first been introduced.

The next step was to adapt the fusion frame potential function introduced in this paper to involve two fusion frames, so that we could create a relative fusion frame potential function. This adaptation was fairly straightforward.

After we had developed our relative fusion frame potential function, it was time to begin testing it to see if it would behave as we wanted to, that is, if it would reveal the desired information about the relationship between two frames. To test the function, we carried out a number of example computations, calculating the relative fusion frame potential value for a number of different frames. To speed this process up and avoid human mistakes in the calculations, I programmed a function in Mathematica, a software system with built-in mathematical libraries, to compute these values for us.

One important thing that we wanted the function to do was to assign the same relative potential value to every pair of fusion frames that were *canonically dual* to one another. This is a special relationship between frames that each frame shares with *only one other frame*. We computed the relative fusion frame potential of several pairs of fusion frames that were canonically dual to one another and unfortunately found that the function was not doing this for every pair.

However, we observed that every pair of canonically dual fusion frames of a particular type called *orthonormal fusion bases* (OFB's) were being assigned the same relative potential value. We shifted our goal to attempt to find out why this was happening and then prove that it would always happen when we had a pair of canonically dual OFB's.

After determining our new goal, we narrowed the range of our example computations to pairs of canonically dual OFB's. It now became important to carry out a good number of example computations by hand, so that we could try to discover a pattern showing up in the intermediate steps of the calculation process that would point to a reason why these pairs of fusion frames were all being assigned the same relative potential value. We eventually did notice one pattern that seemed promising and attempted to find a reason that would explain this pattern's occurrence as well as why it might cause the relative potential value to be the same. Unfortunately, we were unable to determine either.

After this dead end, we pursued a path of exploration offered up by another observed pattern in the calculation process. Specifically, we observed that whenever two OFB's were canonically dual to one another, they were in fact the same OFB. Using the properties of OFB's, we were able to prove that this was always the case and that it caused the relative potential value to always be the same.

What I learned about research

During this project, I gained valuable experience in a number of key pieces of the math research process. First, I learned about how a seemingly-completed research project can be expanded to new areas, allowing for the discovery of new research questions to explore as well as additional possible application settings. After we found that our initial conjecture was incorrect, I learned how to adjust that conjecture to fit the data we were seeing and still end up with a worthwhile result. Finally, I gained further experience in presenting research in a paper format, a skill that will certainly be crucial in my later studies.

Paper

This paper is written with an intended audience of a math undergraduate major or minor. It would certainly be helpful for the reader to have already had experience in a linear algebra class, but I have stated most of the linear algebra concepts and results that I have used in the paper so that the reader can learn or review them as they read. No prior knowledge of frames is necessary for the reader.

Paper Overview

The introduction introduces the concepts of frames and gives a brief overview of the project topic. The second section contains all of the background on tradi-

tional frames that the reader needs to know in order to understand the project topic and results. Section 2.2 details the important relationships between frames that our relative frame potential function helps to identify. Section 3 defines the original frame potential function that inspired this project, and Section 4 introduces our relative frame potential function and states and proves our results related to it. It also includes example computations as well as an explanation of the limitations of the function. Section 5 then gives all the background on fusion frames and fusion frame potential that the reader needs to know, and Section 6 discusses our relative fusion frame potential function, details our results related to it, and includes several example computations.

1 Introduction

Frames are of interest in a number of different applications because they are sequences of vectors that share a useful property with orthonormal bases while having flexibility in a few ways that orthonormal bases do not. If we have an orthonormal basis for a Hilbert space $\mathbb{H}$, then we can write any $f \in \mathbb{H}$ as a linear combination of the basis vectors. The coefficients for this linear combination can be computed by taking the inner products of f with each of the basis vectors. This property is very useful, since it allows for quick and easy computation of the coefficients. When we instead use frames to represent vectors in $\mathbb{H}$, the coefficients can still be computed easily, but the flexible structure of frames makes them much more suited to certain applications.

To illustrate this, we consider an example of an application of frames. Suppose we are sending a signal through a communication system, e.g. sending a text message. If we represent this signal as the sequence of coefficients of a linear combination of orthonormal basis vectors, we can compute the coefficients quickly using inner products. This allows for speed of communication. However, since bases contain only linearly independent vectors, losing one coefficient during transmission results in an entire dimension of information lost. If we instead represent the signal using a frame, the coefficients can still be computed using inner products, so the speed of communication is retained. In addition, frame vectors do not need to be linearly independent, so they may contain more vectors than bases. This property is called *redundancy*, and it allows for protection against information loss when sending the signal. If one coefficient is lost, the signal can still be pieced together from the remaining coefficients. A discussion of this can be found in the introduction of [1].

This project deals with frames for finite-dimensional subspaces. The topic which is the inspiration for this project is frame potential, which is a theoretical version of potential energy in physics. In physics, a system of objects acting under a force moves to minimize potential energy. For example, an extended spring has positive potential energy, and when it is released, it will contract to its natural position where potential energy is zero. The frame potential function measures the potential of a "system" of vectors and is minimized when the input frame is a tight frame: a type of frame with nice properties. In our project, we adapt the frame potential function to take the *relative frame potential* of a frame with respect to another frame, exploring when this function is minimized. We then explore the extension of relative potential to fusion frames, which are a generalization of traditional frames useful in distributed processing applications [2], using as a basis the notion of fusion frame potential discussed in [3].

2 Frames for Finite-Dimensional Spaces

We begin by defining traditional frames for Hilbert spaces (of finite or infinite dimension). The definitions and results in this section can be found in [1].

Definition 1. A sequence of vectors $\{f_i\}$ in a Hilbert space $\mathbb{H}$ is a *frame for* $\mathbb{H}$ if and only if there exist positive real constants $-\infty < A \leq B < +\infty$ such that for every $f \in \mathbb{H}$,

$$A\|f\|^2 \leq \sum_i |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

A is called the lower frame bound for $\{f_i\}$, and B is called the upper frame bound. A frame is called *tight* if $A = B$.

It turns out that in finite-dimensional spaces, frames are exactly the spanning sets of the space. In infinite-dimensional spaces, things are not this simple. However, our exploration of relative potential will be for finite-dimensional spaces, so the remainder of the paper will involve only finite-dimensional spaces. Recall that a spanning set of an n-dimensional space $\mathbb{H}^n$ is a set of vectors $\{v_i\}_{i=1}^k$ in $\mathbb{H}^n$ such that every vector $x \in \mathbb{H}^n$ can be written as a linear combination of the vectors v_i in the spanning set:

$$x = a_1v_1 + a_2v_2 + ... + a_kv_k.$$

Proposition 2. *A sequence of vectors $\{f_i\}_{i=1}^k$ in $\mathbb{H}^n$ is a frame for $\mathbb{H}^n$ if and only if $\{f_i\}_{i=1}^k$ spans $\mathbb{H}^n$.*

Note that since bases are spanning sets, every basis is a frame. However, frames that are not bases possess the useful property of redundancy ($k \geq n$) discussed in the introduction. We give a few examples of frames here:

Example 3.

$$F = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

is a frame for $\mathbb{R}^2$. Note that F is not a basis for $\mathbb{R}^2$, since its vectors are not linearly independent. In other words, it possesses redundancy. It is also important to note that since a frame is a sequence instead of a set of vectors, if we were to switch any of the vectors in F, we would have a different frame.

Example 4.

$$F = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

is a frame for $\mathbb{R}^4$. Note that there are several repeated vectors in this frame.

2.1 Frame Operators

Each frame has an associated analysis, synthesis, and frame operator.

Definition 5. Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{H}^n$. The analysis operator $\theta : \mathbb{H}^n \to \mathbb{C}^k$ is defined by

$$\theta f = \begin{bmatrix} \langle f, f_1 \rangle \\ \langle f, f_2 \rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, f_k \rangle \end{bmatrix} = \sum_{i=1}^{k} \langle f, f_i \rangle e_i,$$

where $\{e_i\}_{i=1}^k$ is the standard orthonormal basis for $\mathbb{C}^k$.

Definition 6. The *synthesis operator* of a frame $F = \{f_i\}_{i=1}^k$ is the adjoint operator of the analysis operator θ. Recall from linear algebra that the adjoint A^* of a matrix A is the transpose of the matrix containing the complex conjugates of the entries in A. If the entries in A are real numbers, then A^* is simply the transpose of A. In the case of the synthesis operator $\theta^* : \mathbb{C}^k \to \mathbb{H}^n$ for F, θ^* is defined by

$$\theta^* \left(\sum_{i=1}^{k} c_i e_i \right) = \sum_{i=1}^{k} c_i f_i.$$

Definition 7. The *frame operator* for a frame $F = \{f_i\}_{i=1}^k$ is $S : \mathbb{H}^n \to \mathbb{H}^n$ defined by

$$S = \theta^* \theta.$$

We write θ and θ^* as matrices with respect to the standard orthonormal basis for $\mathbb{C}^k$ and a fixed orthonormal basis for $\mathbb{H}^n$ as follows:

$$\theta = \begin{bmatrix} \leftarrow & f_1^* & \rightarrow \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \leftarrow & f_k^* & \rightarrow \end{bmatrix}, \theta^* = \begin{bmatrix} \uparrow & & & & \uparrow \\ f_1 & . & . & . & f_k \\ \downarrow & & & & \downarrow \end{bmatrix}.$$

Here f_i is the ith frame vector of F written as a column vector. The analysis operator is the matrix that has the frame vectors as its rows, and the synthesis operator is the matrix with the frame vectors as its columns.

2.2 Dual Frames

A crucial part of frame theory is the dual frame. As mentioned in the introduction, frames share a useful and convenient property with orthonormal bases: when writing a vector as a linear combination of frame vectors, the coefficients of the linear combination can be quickly computed using inner products. Recall that for an orthonormal basis $\{u_i\}_{i=1}^n$ for $\mathbb{H}^n$, any vector $f \in \mathbb{H}^n$ can be written as

$$f = \langle u_1, f \rangle u_1 + \langle u_2, f \rangle u_2 + ... + \langle u_n, f \rangle u_n.$$

The coefficients are the inner products of each of the basis vectors with f. In frame theory, dual frames are used to compute the necessary coefficients.

Definition 8. Let $\{f_i\}_{i=1}^k$ be a frame for $\mathbb{H}^n$. A *dual frame* for $\{f_i\}_{i=1}^k$ is a frame $\{g_i\}_{i=1}^k$ such that for every $f \in \mathbb{H}^n$,

$$f = \sum_{i=1}^{k} \langle f, g_i \rangle f_i.$$

Thus any vector $f \in \mathbb{H}^n$ can be written as a linear combination of the frame vectors f_i, where the coefficients are the inner products between f and the dual frame vectors g_i.

For a frame $F = \{f_i\}_{i=1}^k$, a frame $G = \{g_i\}_{i=1}^k$ satisfying the following equation is a dual frame for F:

$$\begin{bmatrix} \uparrow & & & \uparrow \\ f_1 & . & . & f_k \\ \downarrow & & & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & g_1^* & \rightarrow \\ & . & \\ & . & \\ & . & \\ \leftarrow & g_k^* & \rightarrow \end{bmatrix} = I_n.$$

Frames can have infinitely many dual frames. Some, however, have only one. Every frame does have at least one dual frame, though. This is the *canonical dual frame*. This dual frame is the only dual frame of a frame F that can be obtained by applying an invertible operator to the vectors in F, and the invertible operator that we use to do this is the inverse of the frame operator of F.

Definition 9. Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{H}^n$, and let S be the frame operator for F. The *canonical dual frame* for F is the frame $G = \{S^{-1} f_i\}_{i=1}^k$, where the ith vector in G is obtained by applying the inverse of S to the ith vector in F. The frame operator for G is S^{-1}.

2.3 Tight Frames

As was mentioned in Definition 1, a frame $F = \{f_i\}_{i=1}^k$ for $\mathbb{H}^n$ is called *tight* if its lower frame bound, A, and upper frame bound, B, are equal. One useful feature of tight frames is that the frame operator $S = \theta^* \theta$ of a tight frame is a scalar multiple of the identity operator. Thus, the canonical dual frame $\{S^{-1} f_i\}_{i=1}^k$ of a tight frame $F = \{f_i\}_{i=1}^k$, where $S = aI_n$ for a constant a, is simply $\{a f_i\}_{i=1}^k$, and so any vector f in $\mathbb{H}^n$ can be written as a linear combination of the vectors in F using only f, the vectors in F, and the constant a to compute the coefficients. We give an example of a tight frame here:

Example 10.

$$F = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix} \right\}$$

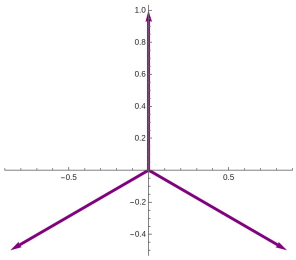

Figure 1: Mercedes-Benz frame (made in Mathematica)

is a tight frame for $\mathbb{R}^2$. It is called the Mercedes-Benz frame. The frame operator for F is

$$S = \begin{bmatrix} 0 & \sqrt{3}/2 & \sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} = \frac{3}{2} I_2.$$

3 Frame Potential

As discussed in the introduction, the frame potential of a finite sequence of vectors is a theoretical version of potential energy in physics developed by John Benedetto and Matthew Fickus [1]. In physics, a system of objects acting under a force will move to minimize its potential energy. The frame potential function takes a finite sequence of vectors in $\mathbb{H}^n$ as input and is minimized when the frame is a tight frame. The function is defined as follows:

Definition 11. [1] The *frame potential* of a frame $F = \{f_i\}_{i=1}^k$ for $\mathbb{H}^n$, P_F, is

$$P_F = \sum_{i=1}^{k} \sum_{j=1}^{k} |\langle f_i, f_j \rangle|^2.$$

The frame potential function can also be written as the trace of the square of the frame operator:

$$P_F = tr(S^2).$$

It can be shown:

Proposition 12. *[1] Let $F = \{f_i\}_{i=1}^k$ be a sequence of vectors in $\mathbb{H}^n$ with $\sum_{i=1}^k \|f_i\|^2 = L$. Then we have*

$$P_F \geq \frac{L^2}{n},$$

with equality if and only if F is a tight frame.

We now give a few examples of frame potential computations:

Example 13. Let F be the tight frame in Example 10:

$$F = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix} \right\}.$$

Then $L = 1+1+1 = 3$, and $P_F = 9/2 = L^2/n$, which is the expected minimum value, since F is tight.

Let's also look at an example where F is non-tight:

Example 14. Let F be the frame described in Example 3:

$$F = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Then $L = 1 + 2 + 2 = 5$. $P_F = 13 > 25/2 = L^2/n$. In this case, the frame potential is greater than the minimum value, since F is non-tight.

4 Relative Frame Potential

The research question that we posed, inspired by frame potential, was to adapt the frame potential function to return the frame potential of one frame *with respect to* another frame. We called this adapted function the "relative frame potential" and defined it as follows, adapting the original frame potential function to involve two frames in the obvious way:

Definition 15. Let $F = \{f_i\}_{i=1}^k$ and $G = \{g_i\}_{i=1}^k$ be frames for $\mathbb{H}^n$. Then the *relative frame potential* $P_F(G)$ of G with respect to F is defined by

$$P_F(G) = \sum_{i=1}^{k} \sum_{j=1}^{k} |\langle f_i, g_j \rangle|^2.$$

The hypothesis was that the relative frame potential would be minimized when the input frame G is the canonical dual frame for F. It turns out that for frames for $\mathbb{R}^n$, if the domain of P_F is limited to dual frames of F (the input frame G will be dual to F), then this is true, and the minimum value is n, the dimension of the space. We begin by showing that when G is the canonical dual of F, $P_F(G) = n$. This result and the corresponding proof are given once for tight frames for $\mathbb{R}^n$ and a second time for general frames for $\mathbb{R}^n$, since there is a method to prove the result for tight frames that is nicer than the proof of the general result.

Proposition 16. *Let F be a tight frame for $\mathbb{R}^n$, and let G be the canonical dual frame for F. Then the relative frame potential of G with respect to F is equal to n. That is,*

$$P_F(G) = n.$$

Proof. Let $F = \{f_i\}_{i=1}^k$ be a tight frame for $\mathbb{R}^n$, and let $G = \{g_i\}_{i=1}^k$ be its canonical dual. We will need two intermediate lemmas to prove this theorem.

Lemma 17. *Let F be a tight frame for $\mathbb{R}^n$. Then the set of column vectors of θ, the analysis operator for F, is a pairwise orthogonal collection of vectors all having the norm $\sqrt{\frac{L}{n}}$, where $L = \sum_{i=1}^k \|f_i\|^2$.*

A similar result for frames for $\mathbb{C}^n$ is stated and proved as Proposition 3.25(ii) in [1], and the proof given here of Lemma 17 is modeled after the one given there:

Proof. For any $f \in \mathbb{R}^n$, we have

$$\theta f = \begin{bmatrix} \leftarrow & f_1^* & \rightarrow \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \leftarrow & f_k^* & \rightarrow \end{bmatrix} f = \begin{bmatrix} \langle f_1, f \rangle \\ \langle f_2, f \rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f_k, f \rangle \end{bmatrix}.$$

Then $\|\theta f\|^2 = \sum_{i=1}^k |\langle f_i, f\rangle|^2 = \sum_{i=1}^k |\langle f, f_i\rangle|^2$ for all $f \in \mathbb{R}^n$. Recall that since F is a tight frame, its upper and lower frame bounds are equal ($A = B$). The value of the frame bounds will be $\frac{L}{n}$ (this fact can be found in [1]). Then by Definition 1,

$$\|\theta f\|^2 = \sum_{i=1}^k |\langle f, f_i\rangle|^2 = \frac{L}{n}\|f\|^2$$

for all $f \in \mathbb{R}^n$. Then

$$\|\theta f\| = \sqrt{\frac{L}{n}}\|f\|, \forall f \in \mathbb{R}^n,$$

and so

$$\|\frac{\sqrt{n}}{\sqrt{L}}\theta f\| = \frac{\sqrt{n}}{\sqrt{L}}\|\theta f\| = \|f\|, \forall f \in \mathbb{R}^n.$$

Recall from linear algebra that a linear operator $T : \mathbb{H}_1 \to \mathbb{H}_2$, where $\mathbb{H}_1$ and $\mathbb{H}_2$ are Hilbert spaces, is called an isometry if $\|Tx\| = \|x\|$ for all $x \in \mathbb{H}_1$. Thus we can see that $\frac{\sqrt{n}}{\sqrt{L}}\theta$ is an isometry, and so the matrix $\frac{\sqrt{n}}{\sqrt{L}}\theta$ has orthonormal columns (this is a property of isometries that can be found in [1]). Thus the columns of θ (i.e. the rows of θ^*) are pairwise orthogonal and all have the norm $\sqrt{\frac{L}{n}}$. □

Lemma 18. *Let S be the frame operator for a tight frame F for $\mathbb{R}^n$. Then $S = \frac{L}{n} I_n$, where I_n is the identity operator for $\mathbb{R}^n$.*

Proof. By definition, $S = \theta^*\theta$. By Lemma 17, the rows of θ^* (i.e. the columns of θ), are pairwise orthogonal. Then every entry of S off the main diagonal will be 0, since every entry off the main diagonal is a dot product between two distinct columns of θ. Every entry on the diagonal is a dot product between

a column of θ and itself, which is the norm of the column vector squared. By Lemma 17, the norm of each column vector of θ is $\sqrt{\frac{L}{n}}$, and so each diagonal entry of S will be $\frac{L}{n}$. Thus $S = \frac{L}{n} I_n$. □

Now the proposition can be proven. By Lemma 18, the frame operator S for F is $S = \frac{L}{n} I_n$. Then $S^{-1} = \frac{n}{L} I_n$. Then the canonical dual frame G is

$$G = \{g_i\}_{i=1}^k = \{S^{-1} f_i\}_{i=1}^k = \{\frac{n}{L} f_i\}_{i=1}^k.$$

Thus

$$P_F(G) = \sum_{i=1}^k \sum_{j=1}^k |\langle f_i, g_j \rangle|^2 = \sum_{i=1}^k \sum_{j=1}^k |\langle f_i, \frac{n}{L} f_j \rangle|^2 = \sum_{i=1}^k \frac{n^2}{L^2} \sum_{j=1}^k |\langle f_i, f_j \rangle|^2.$$

Then since F is a tight frame with frame bound $\frac{L}{n}$, we have by Definition 1,

$$P_F(G) = \sum_{i=1}^k \frac{n^2}{L^2} \sum_{j=1}^k |\langle f_i, f_j \rangle|^2 = \sum_{i=1}^k \frac{n^2}{L^2} (\frac{L}{n} \|f_i\|^2) = \frac{n}{L} \sum_{i=1}^k \|f_i\|^2 = \frac{n}{L}(L) = n.$$

□

Next the result is given for the general (not necessarily tight) case:

Proposition 19. *Let F be a frame for $\mathbb{R}^n$, and let G be the canonical dual frame for F. Then the relative frame potential of G with respect to F is equal to n. That is,*

$$P_F(G) = n.$$

Proof. Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{R}^n$, and let $G = \{g_i\}_{i=1}^k$ be its canonical dual. We will prove this result by analyzing the matrix product $\theta_F \theta_G^*$ between the analysis operator of F and the synthesis operator of G. To this end, we state and prove the following intermediate lemma:

Lemma 20. *Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{R}^n$, and let $G = \{g_i\}_{i=1}^k$ be its canonical dual. If θ_F is the analysis operator for F, and θ_G is the analysis operator for G, then the matrix product $\theta_F \theta_G^*$ is an orthogonal projection matrix.*

Proof. By definition, an orthogonal projection matrix is a matrix that is self-adjoint and whose square is itself. That is, we need to show that $(\theta_F \theta_G^*)^* = \theta_F \theta_G^*$, and that $(\theta_F \theta_G^*)^2 = \theta_F \theta_G^*$. (Recall that a matrix A is self-adjoint if $A^* = A$.) We first show that it is self-adjoint. By general matrix properties,

$$(\theta_F \theta_G^*)^* = (\theta_G^*)^* \theta_F^* = \theta_G \theta_F^*. \tag{1}$$

Since G is the canonical dual for F, $G = \{S^{-1} f_i\}_{i=1}^k$, where S is the frame operator for F. Then θ_G^* can be written as the product $S^{-1} \theta_F^*$, since θ_G^* has as

its columns the vectors in G, and the product $S^{-1}\theta_F^*$ results in a matrix with the vectors $S^{-1}f_i$ as its columns. Thus we have

$$\theta_G\theta_F^* = (S^{-1}\theta_F^*)^*\theta_F^* = ((\theta_F^*\theta_F)^{-1}\theta_F^*)^*\theta_F^* = (\theta_F^*)^*((\theta_F^*\theta_F)^{-1})^*\theta_F^*. \quad (2)$$

Again by matrix properties,

$$(\theta_F^*)^*((\theta_F^*\theta_F)^{-1})^*\theta_F^* = \theta_F((\theta_F^*\theta_F)^*)^{-1}\theta_F^*. \quad (3)$$

Then, combining equations 2 and 3, we get

$$\theta_G\theta_F^* = \theta_F((\theta_F^*\theta_F)^*)^{-1}\theta_F^* = \theta_F(\theta_F^*\theta_F)^{-1}\theta_F^* = \theta_F S^{-1}\theta_F^* = \theta_F\theta_G^*.$$

Putting this together with equation 1, we have $(\theta_F\theta_G^*)^* = \theta_F\theta_G^*$, as desired, and so we have shown that $\theta_F\theta_G^*$ is self-adjoint.

Now we show that $(\theta_F\theta_G^*)^2 = \theta_F\theta_G^*$. Since we know $\theta_F\theta_G^*$ is self-adjoint, we can write

$$(\theta_F\theta_G^*)^2 = (\theta_F\theta_G^*)(\theta_F\theta_G^*)^*. \quad (4)$$

Then we have

$$(\theta_F\theta_G^*)^2 = (\theta_F\theta_G^*)(\theta_G\theta_F^*) = \theta_F(\theta_G^*\theta_G)\theta_F^*. \quad (5)$$

Since S^{-1} is the frame operator for G, we have $S^{-1} = \theta_G^*\theta_G$, and so

$$(\theta_F\theta_G^*)^2 = \theta_F S^{-1}\theta_F^* = \theta_F\theta_G^*. \quad (6)$$

Since we have shown that $\theta_F\theta_G^*$ is self-adjoint and that $(\theta_F\theta_G^*)^2 = \theta_F\theta_G^*$, we have shown that $\theta_F\theta_G^*$ is an orthogonal projection matrix by definition. □

Now that we have proved this intermediate lemma, to complete the proof of 19 we need to define a particular kind of operator:

Definition 21. Any bounded linear operator A on a Hilbert space $\mathbb{H}$ is called a *Hilbert-Schmidt Operator* if for some basis $\{v_j\}$ of $\mathbb{H}$,

$$\sum_j \|Av_j\|^2 < \infty,$$

and for any Hilbert-Schmidt operator A,

$$\sqrt{Tr(A^*A)} = \left(\sum_j \|Av_j\|^2\right)^{1/2}.$$

The above information on Hilbert-Schmidt operators can be found in Appendix B of [4]. Since $\theta_F\theta_G^*$ is a linear operator on $\mathbb{R}^n$, it is bounded. If we take $\{v_j\}_{j=1}^n$ to be the standard orthonormal basis for $\mathbb{R}^n$, then for every $1 \leq j \leq n$:

$$\|\theta_F\theta_G^*v_j\|^2 = \sum_{i=1}^n |a_{ij}|^2,$$

where a_{ij} is the entry in the ith row and jth column of $\theta_F\theta_G^*$. Thus

$$\sum_{j=1}^{n}\|\theta_F\theta_G^* v_j\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 < \infty.$$

So $\theta_F\theta_G^*$ is a Hilbert-Schmidt operator. Note that the entries of $\theta_F\theta_G^*$ are $a_{ij} = \langle f_i, g_j\rangle$. Then since we have just shown that $\theta_F\theta_G^*$ is self-adjoint and that $(\theta_F\theta_G^*)^2 = \theta_F\theta_G^*$ in the proof of the previous lemma, we have

$$Tr(\theta_F\theta_G^*) = \sum_{i=1}^{k}\sum_{j=1}^{k}|\langle f_i, g_j\rangle|^2 = P_F(G).$$

Therefore since $\theta_F\theta_G^*$ is an orthogonal projection by Lemma 20, and the trace of an orthogonal projection is equal to its rank, the relative frame potential $P_F(G)$ is equal to the rank of $\theta_F\theta_G^*$. The rank of $\theta_F\theta_G^*$ is equal to the rank of θ_G^* (this is due to a property of rank: the rank of a matrix product AB is equal to the rank of A for any $m \times n$ matrix A and $n \times k$ matrix B of rank n). Since the columns of θ_G^* form a frame for $\mathbb{R}^n$, the columns span $\mathbb{R}^n$ by Proposition 2. Thus the rank of θ_G^* is equal to n, and so we have

$$P_F(G) = n.$$

□

So far we have shown that when the input frame G is the canonical dual, the value of the relative potential will be the dimension of the space. Now we want to show that when G is any other dual frame, the value of the relative potential will be strictly greater than the dimension. To prove this, we will need an intermediate result that we will not prove here. The statement of this result and a corresponding proof can be found in [1]. We state it here:

Lemma 22. *Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{R}^n$, let $G = \{g_i\}_{i=1}^k$ be the canonical dual frame for F, and let $H = \{h_i\}_{i=1}^k$ be a dual frame (not necessarily the canonical dual) for F. Then for every $f \in \mathbb{R}^n$,*

$$\sum_{i=1}^{k}|\langle f, g_i\rangle|^2 \leq \sum_{i=1}^{k}|\langle f, h_i\rangle|^2,$$

with equality if and only if H is also the canonical dual frame.

We are now ready to state and prove the main result of this section, using Lemma 22 in combination with Proposition 19:

Proposition 23. *Let F be a frame for $\mathbb{R}^n$, and let H be a dual frame for F. Then $P_F(H) \geq n$, with equality if and only if H is the canonical dual frame for F.*

Proof. Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{R}^n$, let $G = \{g_i\}_{i=1}^k$ be the canonical dual frame to F, and let $H = \{h_i\}_{i=1}^k$ be a dual frame for F. Then by Lemma 22, we have

$$P_F(G) = \sum_{i=1}^{k}\sum_{j=1}^{k} |\langle f_i, g_j\rangle|^2 \leq \sum_{i=1}^{k}\sum_{j=1}^{k} |\langle f_i, h_j\rangle|^2 = P_F(H),$$

with equality if and only if $G = H$.

Then applying Proposition 19, $n \leq P_F(H)$ with equality if and only if H is the canonical dual for F. □

Thus $P_F(G)$ is minimized exactly when G is the canonical dual frame, and the minimum value of the function is n.

Example 24. Let F be the frame described in Example 3:

$$F = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Then the frame operator for F is

$$S = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then the frame operator for the canonical dual of F is

$$S^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

Applying S^{-1} to each of the vectors in F, we find that our canonical dual is

$$G = \left\{ \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{bmatrix} \right\}.$$

Then $P_F(G) = 2$, which is the dimension of the space.

Now we take an alternate dual for F:

$$H = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} \right\}.$$

Then $P_F(H) = 4 > 2$.

It is important to note that this minimization result only holds when the domain of the relative potential function $P_F(G)$ is limited to dual frames for F. We give two examples of relative frame potential computations with non-dual frames that match or fall below the minimum value n:

Example 25. Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{R}^n$, and let $G = \{g_i\}_{i=1}^k$ be its canonical dual. Suppose $\langle f_i, g_1\rangle \neq 0$ for some f_i in F. Then consider the frame $H = \{\frac{g_1}{2}, g_2, ..., g_k\}$. Then we will have

$$\begin{aligned}
P_F(H) &= \sum_{i=1}^{k}\left(|\langle f_i, \frac{1}{2}g_1\rangle|^2 + \sum_{j=2}^{k}|\langle f_i, g_j\rangle|^2\right) \\
&= \frac{1}{4}\sum_{i=1}^{k}|\langle f_i, g_1\rangle|^2 + \sum_{i=1}^{k}\sum_{j=2}^{k}|\langle f_i, g_j\rangle|^2 \\
&< \sum_{i=1}^{k}|\langle f_i, g_1\rangle|^2 + \sum_{i=1}^{k}\sum_{j=2}^{k}|\langle f_i, g_j\rangle|^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k}|\langle f_i, g_j\rangle|^2 \\
&= P_F(G) \\
&= n.
\end{aligned}$$

Thus, in this example, the relative frame potential of H relative to F will be strictly less than n, the minimum value for relative frame potential between two dual frames. This tells us that H is not a dual frame for F, since if it was, then $P_F(H)$ would be greater than or equal to n by Proposition 23.

Example 26. Let $F = \{f_i\}_{i=1}^k$ be a frame for $\mathbb{R}^n$, and let $G = \{g_i\}_{i=1}^k$ be its canonical dual. Then consider the frame $H = \{-g_1, g_2, ..., g_k\}$. Then we have

$$\begin{aligned}
P_F(H) &= \sum_{i=1}^{k}\left(|\langle f_i, -g_1\rangle|^2 + \sum_{j=2}^{k}|\langle f_i, g_j\rangle|^2\right) \\
&= \sum_{i=1}^{k}\left(|-\langle f_i, g_1\rangle|^2 + \sum_{j=2}^{k}|\langle f_i, g_j\rangle|^2\right) \\
&= \sum_{i=1}^{k}\left(|\langle f_i, g_1\rangle|^2 + \sum_{j=2}^{k}|\langle f_i, g_j\rangle|^2\right) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k}|\langle f_i, g_j\rangle|^2 \\
&= P_F(G) \\
&= n.
\end{aligned}$$

Thus in this example, the relative frame potential of H relative to F is exactly n, the minimum value for relative frame potential of a dual frame. If g_1 is the zero vector, then H would be the same as the canonical dual frame G, and so

this relative potential value makes sense. However, if g_1 is not the zero vector, then H is not the same as the canonical dual frame. In this case, we know that H cannot be an alternate dual for F either, since if it was, $P_F(G)$ would be strictly greater than n. This example illustrates how when the domain of $P_F(G)$ is extended to include non-duals, the relative potential value n is no longer unique. From this example we can also see that, for any frame F, reversing the sign of any number of nonzero vectors of the canonical dual will never result in a dual frame for F.

Since we have successfully created a relative frame potential function P_F with respect to a frame F, adapted from the single-input frame potential function, that when limited to dual frames is minimized exactly when the input is the canonical dual, a natural furthering of these results would be to see if they can be extended to fusion frames, a generalization of traditional frames (the frames we have been dealing with up to this point). The next section discusses our exploration of this objective. We will begin by giving some necessary background on the topic of fusion frames.

5 Fusion Frames

As mentioned in the introduction, fusion frames are a generalization of traditional frames that are well-suited for distributed data processing: breaking up the processing job into chunks that different computers execute at the same time and then combining the results [2]. The reason for this is that fusion frames are sequences of subspaces instead of vectors [2]. Because of this, we can think of fusion frames as frames made up of frames for the subspaces. We define fusion frames for finite-dimensional spaces. The following definition for fusion frames can be found in [2]:

Definition 27. A sequence of closed subspaces $\{W_i\}_{i=1}^k$ of an n-dimensional Hilbert space $\mathbb{H}^n$ is a *fusion frame for* $\mathbb{H}^n$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{k} \|P_i f\|^2 \leq B\|f\|^2$$

for any $f \in \mathbb{H}^n$, where P_i is the orthogonal projection onto W_i. Similarly to traditional frames, a fusion frame is called *tight* if $A = B$.

Note that we can also view a fusion frame $\{W_i\}_{i=1}^k$ as the sequence $\{P_i\}_{i=1}^k$ of orthogonal projections onto the subspaces.

The fusion frame operator S of a frame $\{P_i\}_{i=1}^k$ is the sum of the projections:

$$S = \sum_{i=1}^{k} P_i.$$

One useful result about fusion frames is that for a sequence of subspaces $\{W_i\}_{i=1}^k$ of $\mathbb{H}^n$, if $\{f_i^j\}_{j\in J_i}$ for some index set J_i is a traditional frame for

subspace W_i, then the union of the vectors in every $\{f_i^j\}_{j\in J_i}$ is a traditional frame for $\mathbb{H}^n$ if and only if $\{W_i\}_{i=1}^k$ is a fusion frame for $\mathbb{H}^n$ [5].

Example 28. In $\mathbb{R}^3$, let W_1 be the plane $z = 0$, and let W_2 be the plane $y = z$. Then $P = \{W_1, W_2\}$ is a fusion frame for $\mathbb{R}^3$. We can also view P as the sequence $\{P_1, P_2\}$ of orthogonal projections onto W_1 and W_2, where

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The fusion frame operator for P is

$$S = P_1 + P_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Note also that we can take any frame F_1 for W_1 and any frame F_2 for W_2, and the union F of all the vectors in F_1 and F_2 will be a traditional frame for $\mathbb{R}^3$. For example, let

$$F_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}, F_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Then $F_1 \cup F_2$ is a traditional frame for $\mathbb{R}^3$.

5.1 Fusion Frame Potential

In order to explore the notion of relative fusion frame potential, we first need to understand fusion frame potential without respect to another frame. This is explored in [3] and defined there as follows:

Definition 29. The fusion frame potential $FFP(P)$ of a fusion frame $P = \{P_i\}_{i=1}^k$ is

$$FFP(P) = Tr(\sum_{i=1}^{k} P_i)^2.$$

Equivalently,

$$FFP(P) = \sum_{i=1}^{k}\sum_{j=1}^{k} Tr(P_i P_j).$$

Similarly to traditional frame potential, fusion frame potential is minimized exactly when the fusion frame is tight. The minimum value is as follows:

Proposition 30. *Let $P = \{P_i\}_{i=1}^k$ be a sequence of orthogonal projections $P_i : \mathbb{H}^n \to \mathbb{H}^n$ with $Tr(P_i) = L_i$. Then*

$$FFP(P) \geq \frac{1}{n}(\sum_{i=1}^{k} L_i)^2,$$

with equality if and only if $\{P_i\}_{i=1}^k$ *is a tight fusion frame.*

The result stated in Proposition 30 was published in [3]. Note that each L_i is also the rank of P_i since the trace of an orthogonal projection is equal to its rank, or in other words, each L_i is the dimension of the subspace W_i that P_i projects onto.

6 Relative Fusion Frame Potential

Using this notion of fusion frame potential, we propose to adapt the potential function to create a *relative fusion frame potential function*. Again, we do this in the obvious way:

Definition 31. Let $P = \{P_i\}_{i=1}^k$ and $Q = \{Q_i\}_{i=1}^k$ be fusion frames for $\mathbb{H}^n$, where P_i and Q_i are orthogonal projections. Then the relative fusion frame potential $RFFP_P(Q)$ of Q with respect to P is

$$RFFP_P(Q) := \sum_{i=1}^{k}\sum_{j=1}^{k} Tr(P_i Q_j).$$

Note that for frames P and Q, $RFFP_P(Q) = RFFP_Q(P)$, since trace of a matrix product is independent of the order of the matrix product. This is what we want, since it would be strange if the relative potential of Q with respect to P was not the same as the relative potential of P with respect to Q. Note also that we can write

$$RFFP_P(Q) = Tr(S_P S_Q),$$

where S_P is the fusion frame operator for P and S_Q is the fusion frame operator for Q. This is because $Tr(A+B) = Tr(A)+Tr(B)$ for any matrices A and B of the same dimensions, and the fusion frame operators of P and Q are the sums of their respective projection operators. First applying trace and sum properties,

$$RFFP_P(Q) = \sum_{i=1}^{k}\sum_{j=1}^{k} Tr(P_i Q_j) = Tr\left(\sum_{i=1}^{k}\sum_{j=1}^{k} P_i Q_j\right) = Tr\left(\sum_{i=1}^{k} P_i \left(\sum_{j=1}^{k} Q_j\right)\right)$$

Then:

$$Tr\left(\sum_{i=1} P_i S_Q\right) = Tr\left(\left(\sum_{i=1}^{k} P_i\right) S_Q\right) = Tr(S_P S_Q).$$

6.1 Computing the Canonical Dual Fusion Frame

As with traditional frames, the conjecture to test is if the relative potential $RFFP_P(Q)$ is minimized when Q is the canonical dual of P. To explore this, we need to know how to compute the canonical dual frame for a fusion frame.

Viewing a fusion frame P as a sequence of subspaces $\{W_i\}_{i=1}^k$, the canonical dual frame Q is $\{S^{-1}W_i\}_{i=1}^k$, where S is the fusion frame operator for P. This

makes sense, since it is the same process we followed to compute the canonical dual of a traditional frame. However, it is worth noting that if we view P as the sequence of orthogonal projections onto each subspace $\{P_i\}_{i=1}^k$, then $\{S^{-1}P_i\}_{i=1}^k$, that is, applying S^{-1} to each projection operator, will not give us a fusion frame, since the projections resulting from the products $S^{-1}P_i$ are not necessary orthogonal projections. They are projections onto the subspaces $S^{-1}W_i$ of the canonical dual, but they are not the orthogonal projections we need in order to fulfill the definition of fusion frame. In order to write Q as a sequence of orthogonal projections $\{Q_i\}_{i=1}^k$, it is necessary to first compute the subspaces $S^{-1}W_i$ of Q and then find the orthogonal projections Q_i for those $S^{-1}W_i$. While experimenting with the sequence $Q' = \{S^{-1}P_i\}_{i=1}^k$,however, we found that computing the relative frame potential using this sequence as the "canonical dual frame" gives the following value:

$$RFFP_P(Q') = \sum_{i=1}^{k} rank(P_i) = \sum_{i=1}^{k} dim(W_i).$$

Proof. Let S be the fusion frame operator for P. We begin by rewriting the sum:

$$RFFP_P(Q') = \sum_{i=1}^{k}\sum_{j=1}^{k} Tr(P_i S^{-1} P_j) = Tr\left(\sum_{i=1}^{k} P_i \sum_{j=1}^{k} S^{-1}P_j\right).$$

Then we rewrite using the definition of fusion frame operator as follows:

$$RFFP_P(Q') = Tr\left(SS^{-1}\sum_{j=1}^{k} P_j\right) = Tr\left(\sum_{j=1}^{k} P_j\right) = \sum_{j=1}^{k} Tr(P_j).$$

Since the trace of an orthogonal projection is equal to its rank, which is equal to the dimension of the subspace it projects onto, this completes the proof. Note also that $\sum_{j=1}^{k} Tr(P_j)$ is the same as the trace of the frame operator S. □

6.2 Relative Fusion Frame Potential of the Canonical Dual

Unfortunately, the relative potential $RFFP_P(Q)$ of the canonical dual Q of a general fusion frame $P = \{P_i\}_{i=1}^k$ with $\{L_i\}_{i=1}^k$ the ranks of the projections P_i is not consistent. We give an example to illustrate this:

Example 32. Let W_1 be the line $\langle t, t, 0\rangle$ in $\mathbb{R}^3$, and let W_2 be the plane $x = 0$. Then $P = \{W_1, W_2\}$ is a fusion frame for $\mathbb{R}^3$. Then the canonical dual fusion frame for P is the frame $Q = \{V_1, V_2\}$, where V_1 is the x-axis and V_2 is the plane $x + y = 0$. Then $RFFP_P(Q) = 2$.

Now let U_1 be the x-axis, and let U_2 be the plane $x = 0$ in $\mathbb{R}^3$. Then $R = \{U_1, U_2\}$ is a fusion frame for $\mathbb{R}^3$, and its canonical dual T is itself. Then $RFFP_R(T) = 3$.

Both P and R are fusion frames for $\mathbb{R}^3$ with 2 subspaces, the first having dimension 1 and the second having dimension 2. However, the relative potential of their canonical duals have different values.

However, there is a special case where we can predict the relative potential of the canonical dual fusion frame. This is when the frame is an *orthonormal fusion basis*. A fusion frame $P = \{W_i\}_{i=1}^k$ for $\mathbb{H}^n$ is called an orthonormal fusion basis when $\bigoplus_{i=1}^k W_i = \mathbb{H}^n$ [6]. When P is an orthonormal fusion basis, its canonical dual Q is itself. This means that the relative potential of Q will be equal to the fusion frame potential $FFP(P)$ of P, and since the subspaces W_i are all orthogonal to one another, we can predict this value. We state and prove this result here:

Proposition 33. *Let P be an orthonormal fusion basis for an n-dimensional Hilbert space $\mathbb{H}^n$. Then*

$$RFFP_P(Q) = FFP(P) = n,$$

where Q is the canonical dual of P.

Proof. We prove this using an intermediate lemma which we now state and prove:

Lemma 34. *Let P be an orthonormal fusion basis for $\mathbb{H}^n$. Then the canonical dual fusion frame of P is P.*

Proof. Let $P = \{W_i\}_{i=1}^k$ be an orthonormal fusion basis for $\mathbb{H}^n$ with projections $\{P_i\}_{i=1}^k$. For each W_i, let $F_i = \{f_j^i\}_{j \in J_i}$, where J_i is an index set, be a frame for W_i. Consider the product Sf_j^i, where S is the fusion frame operator for P.

$$Sf_j^i = \left(\sum_{i=1}^k P_i \right) f_j^i.$$

Note that for every $1 \leq m \leq k$ such that $m \neq i$, $P_m f_j^i$ is the zero vector. This is because $P_m = U_m^* U_m$, where U_m is a matrix whose rows are an orthonormal basis for W_m. f_j^i is orthogonal to every vector in this basis. Then

$$Sf_j^i = P_i f_j^i.$$

But the orthogonal projection of f_j^i onto W_i is just f_j^i, so

$$Sf_j^i = f_j^i.$$

Therefore,

$$f_j^i = S^{-1} f_j^i.$$

Thus $S^{-1} W_i = W_i$, and so the canonical dual of P is P. □

Using this lemma, we can now prove the proposition. Again let $P = \{W_i\}_{i=1}^k$ be an orthonormal fusion basis with projections $\{P_i\}_{i=1}^k$. Then by the previous lemma, the canonical dual of P is P. Therefore the relative potential of the canonical dual is

$$RFFP_P(P) = \sum_{i=1}^{k}\sum_{j=1}^{k} Tr(P_iP_j) = FFP(P).$$

Since $\bigoplus_{i=1}^k W_i = \mathbb{H}^n$, we know that $\sum_{i=1}^k L_i$, where L_i is the dimension of W_i, is equal to n. For each pair of projections P_i and P_j in P where $i \neq j$, the entries of P_iP_j are all zeroes, since W_i and W_j are orthogonal, $P_i = U_i^*U_i$, and $P_j = U_j^*U_j$, where U_i is a matrix whose rows are an orthonormal basis for W_i and U_j is a matrix whose rows are an orthonormal basis for W_j. Therefore

$$RFFP_P(P) = FFP(P) = \sum_{i=1}^{k}\sum_{j=1}^{k} tr(P_iP_j) = \sum_{i=1}^{k} tr(P_iP_i).$$

Since P_i are orthogonal projections, $P_iP_i = P_i$, so

$$RFFP_P(P) = FFP(P) = \sum_{i=1}^{k} tr(P_i) = \sum_{i=1}^{k} L_i = n.$$

This completes the proof. □

We give an example of computing the relative potential of Q with respect to P when P is an orthonormal fusion basis, and Q is the canonical dual:

Example 35. In $\mathbb{R}^3$, let W_1 be the xy-plane, and let W_2 be the z-axis. Then $P = \{W_1, W_2\}$ is an orthogonal fusion basis.

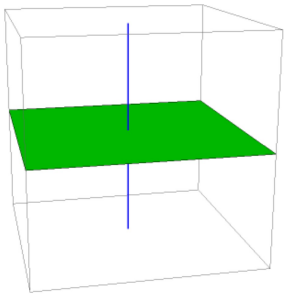

Figure 2: Orthonormal fusion basis P (made in Mathematica)

Note that in order to construct an orthonormal fusion basis, our only options are to use 3 perpendicular lines for our subspaces (for example, the x-, y-, and z-axes), or to use one plane and a line that is perpendicular to the plane. If we try to use more than 3 lines, one of them will have to not be orthogonal

to at least one of the other 3 lines. If we try to use more than one plane, the intersection of each pair of planes will have positive dimension. Finally, if we try to have one plane and multiple lines, the extra lines will have to be contained in the plane in order to be orthogonal to the first line.

Continuing with our example, our corresponding orthogonal projections are

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the fusion frame operator $S = P_1 + P_2 = I_3$ is the identity operator, and so S^{-1} is also the identity operator. Thus when we apply S^{-1} to the vectors in any frames F_1 for W_1 and F_2 for W_2, we will obtain the same frames. Therefore, our canonical dual fusion frame Q for P is the same as P, and so

$$RFFP_P(Q) = FFP(P) = 3.$$

Thus the relative potential of the canonical dual of P is equal to the dimension of the space, which illustrates the previous result.

We now give an example of computing the relative potential of Q with respect to P, where P is *not* an orthonormal fusion basis, and Q is the canonical dual.

Example 36. In $\mathbb{R}^3$, let W_1 be the xy-plane, and let W_2 be the plane given by the equation $x + y = 0$. Then $P = \{W_1, W_2\}$ is a fusion frame for $\mathbb{R}^3$. The corresponding orthogonal projections are

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The fusion frame operator for P is

$$S = P_1 + P_2 = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and so

$$S^{-1} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 3/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now create frames for each W_i:

$$F_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}, F_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We apply S^{-1} to each vector in F_1 and F_2 to get frames G_1 and G_2 for our canonical dual subspaces:

$$G_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2` \\ 0 \end{bmatrix} \right\}, G_2 = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

G_1 spans the xy-plane, and G_2 spans the plane $x+y=0$, so our canonical dual fusion frame Q for P is the same as P. Thus $RFFP_P(Q) = FFP_P(P) = 6$. Note that while the relative potential and regular potential share the same value in this example, that value is not the minimal value for $FFP(P)$, which is $\frac{1}{n}\left(\sum_{i=1}^{k} L_i\right)^2 = \frac{1}{3}(2+2)^2 = \frac{16}{3}$. This tells us that P is non-tight.

Finally, we give an example of computing the relative fusion frame potential of Q with respect to P, where P is non-tight and Q is the canonical dual, where $RFFP_P(Q)$ is not the same as $FFP(P)$, and in fact, $RFFP_P(Q) < \frac{1}{n}\left(\sum_{i=1}^{k} L_i\right)^2$:

Example 37. In $\mathbb{R}^3$, let W_1 be the xy-plane, and let W_2 be the plane $y = z$. Then $P = \{W_1, W_2\}$ is a fusion frame, and the corresponding orthogonal projections are:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Then our fusion frame operator and its inverse are:

$$S = P_1 + P_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}, S^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

We now choose bases for our subspaces:

$$F_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, F_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Then we apply S^{-1} to each of the vectors in our frames to obtain frames G_1 and G_2 for the subspaces of our canonical dual:

$$G_1 = \left\{ \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}, G_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

G_1 spans the plane $y+z=0$, and G_2 spans the plane $y=0$, so the orthogonal projections of our canonical dual frame Q are:

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $RFFP_P(Q) = 5$. Note that $FFP(P) = 7$, and the minimum value of $FFP(P)$ for a fusion frame for $\mathbb{R}^3$ with 2 subspaces of dimension 2 is 16/3. Thus the relative potential of Q with respect to P is not equal to the frame potential of P by itself, and the relative potential is less than the minimum frame potential value.

7 Further Exploration

Our future goals related to traditional frames include studying in further depth the behavior of the relative potential of traditional frames when the domain is expanded to include non-dual frames. Following this, we would like to explore the relative potential function when the domain is expanded to include finite sequences of vectors that are not frames.

With regards to fusion frames, our goal is to continue studying the relative potential of canonical duals with respect to frames that are not orthonormal fusion bases to determine if we can predict these values. In addition, we plan to study the relative fusion frame potential of alternate duals and then eventually explore what happens when the domain is expanded to include non-duals and finite sequences of subspaces that are not fusion frames.

8 Conclusion

The relative frame potential $P_F(G)$ of a traditional frame G, when the domain of P_F is limited to dual frames for F, is minimized exactly when G is the canonical dual frame for F. Furthermore, the value of the relative potential of the canonical dual is n, the dimension of the space. Computing the relative potential is a simple calculation that can easily determine whether or not a dual frame is the canonical dual. To see whether or not a given dual frame G is the canonical dual of a frame F without the relative potential function, we would need to compute the canonical dual of F and compare it to G. Computing the canonical dual requires inversion of an $n \times n$ matrix, a computationally taxing operation for a computer to complete in high dimensions. In contrast, computing the relative potential involves only simple operations that require little computation time by a computer. Once we have computed the relative potential of G, we can compare the potential value to the dimension of the space, n. G is the canonical dual if and only if its relative potential is n. In a similar fashion, computing the relative potential of a frame G with respect to a frame F can tell us that G is not a dual frame for F if the result is a value less than n.

Currently, we can predict the relative fusion frame potential of Q with respect to P when P is an orthonormal fusion basis and Q is its canonical dual. Further study is currently needed to fully understand the behavior of the relative fusion frame potential function $RFFP_P(Q)$ when P is a general fusion frame or when Q is not the canonical dual.

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