SEMI-GROUPS IN PROBABILITY

Senior Thesis

by

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Although the notion of a semi-group is more primitive than that of a group, it is of more recent origin. Mathematicians have used semi-groups extensively, but the concept was formulated and named in 1904. By the late twenties the algebraic theory of semi-groups got under way but progressed slowly. There was doubt concerning the value and possible implications of such a primitive notion.

In 1929, M. H. Stone indicated the usefulness of this theory in his discussions of linear transformations in Hilbert space (7). He soon developed important theorems dealing with the representation of one-parameter groups of unitary transformations in Hilbert space (8). This is the earliest encounter we have with the semi-group property of operators which will be defined later in this paper.

We will not be concerned with purely algebraic aspects of semi-group theory, except for the basic definitions which will now be set forth.

A groupoid is a collection of elements S, on which a binary operation $\circ$ is defined such that $a \circ b$ is uniquely defined as an element of S for every ordered pair of elements a, b of S.

A binary operation is associative if

$$(a \circ b) \circ c = a \circ (b \circ c).$$

A semi-group is an associative groupoid.

It is possible to think of the operation as a mapping of the product set S X S into S by a function F(a,b) which has a value in S if a and b are elements of S. Then the associative property is expressed by the functional equation

$$F(a,F(b,c)) = F(F(a,b),c). \tag{4}$$

Obviously, every group is a semi-group since the postulates stated above are those of a group except that a unit element and inverses are not assumed. Some semi-groups may contain a unit element, and some elements of such semi-groups may have inverses.

An elementary example of a semi-group is the set of positive integers under addition. Addition is well-defined, closed, and associative. The elements of this set also commute under addition; that is $a + b = b + a$. Such a semi-group is said to be abelian. This semi-group fails to be a group since it does not contain a unit element, and none of the elements have inverses.

We have not specified the structure of S. By doing so we arrive at interesting mathematical theories in a variety of areas which have shown the importance of this relatively simple notion.

For example S may be a subset of a ring or of a topological space, or of an algebra of operators. Then, according to the structure of S, we have algebraic, or topological or transformation semi-groups.

From the algebraic structure follow the usual notions of homomorphisms, isomorphisms, and automorphisms for semi-groups.

A semi-group U is a topological semi-group if U is a Hausdorff space and to every x, y in U and every neighborhood $N(x \circ y)$ of $x \circ y$, there are neighborhoods $N(x)$ of x and $N(y)$ of y such that $x \circ N(y) \subset N(x \circ y)$ and $N(x) \circ y \subset N(x \circ y)$. (3,p. 257) This states that U is a semi-group under an operation $\circ$ such that $x \circ y$ is a continuous function of x and y in the topology of the space.

This paper will be primarily concerned with transformation semi-groups in the theory of probability. A few preliminary definitions will illustrate how these arise.

A linear system $\mathfrak{X}$ is a Banach space if

1) with every element x, there is associated a real number $\|x\|$, called the norm of x with the properties

   N_1 $\|x\| \geq 0$ and $\|x\| = 0$ if and only if x is the zero element.

   N_2 $\|ax\| = |a|\|x\|$

   N_3 $\|x + y\| \leq \|x\| + \|y\|$

2) $d(x,y) = \|x - y\|$

3) $\mathfrak{X}$ is complete in the resulting topology. (3, p.16)

Let $\mathfrak{X}$ be a Banach space. A bounded linear transformation of $\mathfrak{X}$ into itself is a mapping $x \longrightarrow T(x)$ which assigns to each element x of $\mathfrak{X}$ a definite element $y = T(x)$ of $\mathfrak{X}$ which satisfies the following conditions:

1) $T(ax_1 + bx_2) = aT(x_1) + bT(x_2)$

2) $\|T(x)\| \leq M\|x\|$ for all x in $\mathcal{X}$.

B is a Banach algebra if B is an algebra as well as a space and if, in addition, $\|xy\| \leq \|x\|\|y\|$. It can be shown that the set of all linear transformations on a Banach space into itself forms a Banach algebra $B(\mathcal{X})$ which has the identity transformation as a unit element. (3)

Here the product is operator composition.

$$(T_1 \circ T_2)x = T_1T_2(x) = T_1[T_2(x)] \ .$$

If the elements of a semi-group S are of the form $T = T(a)$ where $T(a)$ is in $B(\mathcal{X})$ for each a, and if a belongs to an index set **A** which itself is a semi-group, then $B(\mathcal{X})$ is a parametrized semi-group. The relation between the composition in A and composition in S is

$$T(a \circ b) = T(a)T(b). \qquad (4)$$

From group theory we discover that operator composition is associative. Consider any x in $\mathcal{X}$. Then

$$(T_1 \circ T_2) \circ T_3(x) = (T_1 \circ T_2)T_3(x) = T_1T_2(T_3(x)) = T_1 [T_2(T_3(x))] .$$

$$T_1 \circ (T_2 \circ T_3)(x) = T_1 \circ (T_2T_3(x)) = T_1 \cdot [T_2(T_3(x))] = T_1T_2(T_3(x))$$

$$= T_1 [T_2(T_3(x))] \ .$$

Hence $(T_1 \circ T_2) \circ T_3 = T_1 \circ (T_2 \circ T_3)$ since x was any element of $\mathcal{X}$.

Therefore, to show that the set of transformations we define is a semi-group, we need only to show

$$T(a \circ b) = T(a)T(b).$$

From now on, this well be referred to as the semi-group property.

We will be concerned with one-parameter transformation semi-groups. For our purposes we will take $A = E^+$, the set of positive numbers, and $\circ$ is $+$ so that the elements satisfy the law

$$T(a + b) = T(a)T(b), \quad a > 0, \quad b > 0.$$

One-parameter transformation semi-groups arise in many problems of analysis. We will give some examples.

Example 1. Translations are the simplest of one-parameter transformation semi-groups. Consider $\mathcal{X}$ to be the set of differentiable functions on $[0,\infty]$ and define

$$T(a)[f](u) = f(u + a), \quad a \geq 0.$$

To verify the semi-group property we must show

$$T(a + b)[f](u) = T(a)T(b)[f](u).$$

$$T(a + b)[f](u) = f(u + a + b) = f(u + b + a)$$
$$= T(a)[f](u + b) = T(a)T(b)[f](u).$$

Example 2. Fractional integration provides us with another example. Let $\mathcal{X}$ be the set of continuous functions on $[0,1]$ and let

$$T(a)[f](u) = \frac{1}{\Gamma(a)} \int_0^u (u - t)^{a-1} f(t)\,dt \qquad a > 0.$$

The semi-group property has long been known. For complex values of a with $\mathrm{Re}(a) > 0$, this is the Riemann-Liouville integral. The semi-group property also holds in this case (4, p.61).

Example 3. The integral

$$T(y)[f](x) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u + x)}{u^2 + y^2}\,du$$

gives the value at the point $x + iy$, $y > 0$ of a function harmonic in the upper half-plane whose boundary values on the real axis are f(x). It is evident that the semi-group property holds.

$$T(y + w)[f](x) = \frac{y + w}{\pi}\int_{-\infty}^{\infty} \frac{f(u + x)}{u^2 + (y^2 + w^2)}\, du$$

$$= T(y)\left[\frac{w}{\pi}\int_{-\infty}^{\infty} \frac{f(u + x)}{u^2 + w^2}\, du\right] = T(y)T(w)[f](x).$$

Example 4. An elementary instance of application in the area of diffusion equations is given by heat conduction in an infinite rod. The temperature T(x,t;f) at a place x, after the time t, and initial temperature f(x) satisfies

$$\frac{1}{2}\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

and is given by

$$T(t)[f](x) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{\frac{-u^2}{2t}} f(x + u)\, du.$$

To show that the semi-group property is satisfied, we will use the Chapman-Kolmogorov equation, but will delay the proof of this equation until later in this paper.

$$T(s + t)[f](x) = \frac{1}{\sqrt{2\pi(s+t)}}\int_{-\infty}^{\infty} e^{\frac{-u^2}{2(s+t)}} f(x + u)\, du$$

By the Chapman-Kolmogorov equation, this is equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(0,0;t,w)p(t,w;t+s,u)\ f(x + u)\ dw\ du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{\frac{-w^2}{2t}}}{\sqrt{2\pi t}}\ \frac{e^{\frac{-(u-w)^2}{2s}}}{\sqrt{2\pi s}}\ f(x + u)\ dw\ du$$

Now let $\alpha = w$, $\beta = u - w$. Then the Jacobian equals 1, and the integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{\frac{-\alpha^2}{2t}}}{\sqrt{2\pi t}}\ \frac{e^{\frac{-\beta^2}{2s}}}{\sqrt{2\pi s}}\ f(x + \alpha + \beta)\ d\alpha\ d\beta$$

$$= \int_{-\infty}^{\infty} \frac{e^{\frac{-\beta^2}{2s}}}{\sqrt{2\pi s}} T(t)[f](x+\beta)\, d\beta = T(s)T(t)[f](x).$$

Before we look into our most important examples, those in probability theory, we will discuss some of the theory of one-parameter transformation semi-groups.

One important element of this theory is that of the infinitesimal generator of a semi-group. To discuss this, many preliminary definitions and remarks must be put forth.

The algebraic definition of generator states that a subset K of S is a generator of the set S if for all s in S, s is in K or s is a product of elements in K.

An operator function $V(\xi)$ on a subset G of the real number system to $U(\mathfrak{X})$ is continuous in the strong operator topology at $\xi = \xi_0$ if

$\lim_{\xi \to \xi_0} \| [V(\xi) - V(\xi_0)](x) \| = 0$ for each x in $\mathfrak{X}$. (3, p.59)

For our purposes, G will be the set $[0,T]$ of parametric values.

Let $\mathfrak{X}$ be the space of continuous functions on $[a,b]$. For x in $\mathfrak{X}$ let $\|x\| = \max_{a \le t \le b} |x(t)|$. It can be verified that this is a norm. $\mathfrak{X}$ is a linear system under ordinary addition of ordinates. $\mathfrak{X}$ is a complete metric space in the resulting topology (5).

To make use of these concepts, let us consider the following example. Let a transformation T_s of x be defined thus

$$T_s(x) = \int_0^s x(u)\,du, \quad \text{for } x \text{ in } \mathfrak{X}.$$

Since

$$T_s(ax + by) = \int_0^s ax(u)du + \int_0^s by(u)du,$$

T_s is a linear transformation. We will now compute the norm of T_s.

$$\begin{aligned} \|T_s\| &= \max\left[\|T_s(x)\| ; \|x\| \leq 1\right] \\ &= \max\left[\left\|\int_0^s x(u)du\right\|; \|x\| \leq 1\right] \\ &= \max\left[s\|x\| ; \|x\| \leq 1\right] = s. \end{aligned}$$

Since $\|T_s(x)\| = s\|x\| = \|T_s\| \|x\|$, for all x in $\mathcal{X}$, this is a bounded linear transformation.

We will now show that the operator T_s is continuous in the strong operator topology at any s in $[0,T]$. Let s_0 be in $[0,T]$. Let x be in $\mathcal{X}$.

$$\begin{aligned} \lim_{s\to s_0} \left\|\left[T_s - T_{s_0}\right](x)\right\| &= \lim_{s\to s_0} \left\| \int_0^s x(u)du - \int_0^{s_0} x(u)du \right\| \\ &= \lim_{s\to s_0} \left\| \int_s^{s_0} x(u)du \right\| \\ &\leq \lim_{s\to s_0} |s - s_0| M, \text{ where } M = \max_{a\leq t\leq b} |x(t)| \\ &= 0. \end{aligned}$$

We can look once more at example 1 of one-parameter transformations, $T(a)[f](u) = f(u + a)$. We verified above that it was linear. We will now investigate boundedness.

$$\begin{aligned} \|T(a)\| &= \max\left[\|T(a)[f]\|; \|f\| \leq 1\right] \\ &= \max\left[\|f(\cdot + a)\|; \|f\| \leq 1\right] \\ &= 1 \end{aligned}$$

Since $\|T(a)[f]\| = \|f\| = \|T(a)\| \|f\|$, for all x in $\mathcal{X}$, this is a bounded linear transformation.

To show the translation transformation is strongly continuous, consider

$$\lim_{\xi \to \xi_0} \| [T(\xi) - T(\xi_0)] [f](x) \| = \lim_{\xi \to \xi_0} \| f(x + \xi) - f(x + \xi_0) \|$$
$$= 0$$

provided we are in the space of continuous functions.

Consider a one-parameter semi-group $G = [T(s); s > 0]$ of linear bounded transformations on a complex Banach space $\mathcal{X}$ into itself with the property that

$T[s + t](x) = T(s)[T(t)x]$ for all $s > 0$, $t > 0$, and all x in $\mathcal{X}$.

We assume that $T(s)$ is strongly continuous for $s > o$.

We define $A_h = \frac{1}{h}[T(h) - I]$, $h > 0$

and $A_o(x) = \lim_{h \to 0^+} A_h(x)$ whenever the limit exists (3, p.302). The set of elements for which $\lim_{h \to 0^+} A_h(x)$ exists is the domain of A_o, denoted by $D(A_o)$. $D(A_o)$ is a linear subspace of $\mathcal{X}$. A_o is called the infinitesimal generator of G.

In order to state the important theorem dealing with generators, the following terminology from topology will be needed.

A subset E of a topological space U is nowhere dense if and only if every nonempty open set in U contains a nonempty open set which is disjoint from E. A result of this definition is that if E is nowhere dense in U, then U - E is dense in U.

A set A is of the second category in M if it is not the sum of a sequence of sets nowhere dense in M. This brings us to our theorem.

If $D(A_o)$ is of the second category in $\mathfrak{X}$, then $\lim_{h\to 0^+} \|T(h) - I\| = 0$, A_o is a bounded linear operator, and $T(s) = \exp(sA_o)$ (3, p.308).

Making the translation example slightly more interesting, we examine it in light of this theorem. Consider

$$T(\beta t)[u](x) = u(x - \beta t), \quad \beta,\ t \geq 0$$
$$= D(t)u(x)$$

Then
$$\frac{D(h) - 1}{h} u(x) = \frac{u(x - \beta h) - u(x)}{h}$$
$$= \beta \frac{u(x - \beta h) - u(x)}{\beta h},$$

and
$$\lim_{h\to 0^+} \frac{\beta u(x - \beta h) - \beta u(x)}{\beta h} = -\beta \frac{du(x)}{dx}.$$

Hence $D(t) = \exp(-\beta t \frac{d}{dx})$ (2, p.286).

We have shown that A_o is defined for all differentiable functions. Thus the preceding theorem legalizes the exponential formula since $D(A_o) = \mathfrak{X}$.

Dunford and Schwartz (1, p.614) approach this theorem in the following way.

A family $\{T(t);\ t \geq 0\}$ of unbounded linear operators in $\mathfrak{X}$ will be called a strongly continuous semi-group if

1) $T(s + t) = T(s)T(t) \quad t,\ s \geq 0$

2) $T(0) = I$

3) For each x in $\mathfrak{X}$, $T(t)x$ is continuous in t on $[0,\infty]$.

If, in addition, the map $t \to T(t)$ is continuous in the uniform operator topology, the family $\{T(t);\ t \geq 0\}$ is called a uniformly continuous semi-group in $B(\mathfrak{X})$.

They then state the theorem as follows. Let $\{T(t)\}$ be a uniformly continuous semi-group. Then there exists a bounded operator A such that $T(t) = \exp(tA)$ for $t \geq 0$. The operator A is given by the formula

$$A = \lim_{h \to 0^+} \frac{T(h) - I}{h} .$$

The interested reader can pursue his study of this topic in either the book by Hille and Phillips (3) or the one by Dunford and Schwartz (1).

Our discussion turns now to probability theory, and we will deal first with convolution semi-groups.

Let F and G be Lebesque measurable scalar functions defined on $(-\infty,\infty)$. We define function F*G by putting

$$(F*G)(t) = \int_{-\infty}^{\infty} F(t - s)G(s)ds$$

for all values of t for which the integral exists. The function F*G is called the convolution of F and G (1, p.633).

We will prove the following results.

1) If F and G are Lebesque measurable defined on $(-\infty,\infty)$, then $F*G = G*F$.

2) If F, G, and H belong to $L(-\infty,\infty)$, then $(F*G)*H = F*(G*H)$.

Proof of 1:

$$(F*G)(t) = \int_{-\infty}^{\infty} F(t - s)G(s)ds$$

Let $w = t - s$. Then

$$\int_{-\infty}^{\infty} F(t - s)G(S)ds = \int_{\infty}^{-\infty} F(w)G(t - w)dw(-1)$$

$$= \int_{-\infty}^{\infty} G(t - w)F(w)dw = G*F.$$

Proof of 2:

$$((F*G)*H)(r) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F(s)G(t - s)ds \right\} H(r - t)dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t - s)F(s)H(r - t)dsdt$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} G(t - s)H(r - t)dt \right\} F(s)ds \text{ by Fubini's Theorem}$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} G(t)H(r - s - t)dt \right\} F(s)ds$$

$$= (F*(G*H))(r) \quad \text{for almost all } r \quad (1, \text{p.}634),$$

Thus the set of convolutions is an abelian semi-group.

In connection with probability theory Feller (2, p.284) introduces convolution semi-groups as follows.

For $t > 0$, let Q_t be a probability distribution satisfying $Q_s*Q_t = Q_{s+t}$. (The normal and Poisson distributions satisfy this with t proportional to the variance.) Let $G(t)$ be the associated operator, that is

$$G(t)u(x) = \int_{-\infty}^{\infty} u(x - y)Q_t dy.$$

Then the semi-group property is $G(s + t) = G(s)G(t)$. A convolution semi-group $\{G(t),\ t > 0\}$ is a family of operators associated with the probability distribution and satisfying this semi-group property.

The convolution semi-group is said to be continuous if $\lim_{t \to 0} G(t) = I$, where I is the identity operator.

As before the generator of a convolution semi-group is A if $\lim_{h \to 0} \frac{G(h) - I}{h} = A$.

A semi-group with a generator is continuous. However, the interesting point is that all continuous convolution semi-groups possess generators. The proof of this statement may be found in Feller (2, p.293).

For our next example let us look at the Poisson transformation given by the formula

$$(T_t x)(s) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} x(s - ku).$$

We will verify the semi-group property.

$$(T_w(T_t x))(s) = e^{-\lambda w} \sum_{m=0}^{\infty} \frac{(\lambda w)^m}{m!} \left[e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} x(s - ku - mu) \right]$$

Let $p = m + k$. Then

$$(T_w(T_t x))(s) = e^{-\lambda(w+t)} \sum_{p=0}^{\infty} \frac{1}{p!} \left[p! \sum_{m=0}^{p} \frac{(\lambda w)^m}{m!} \frac{(\lambda t)^{p-m}}{(p-m)!} x(s - pu) \right]$$

which, by the binomial formula,

$$= e^{-\lambda(w+t)} \sum_{p=0}^{\infty} \frac{1}{p!} (\lambda w + \lambda t)^p \, x(s - pu)$$

$$= (T_{w+t} x)(s) \qquad (9, \text{ p.}237),$$

To calculate the generator, let $F^k = x(s - ku)$ (i.e. the convolution). Then

$$\frac{(T_h x) - x}{h} = \frac{(e^{-\lambda h} - 1)x}{h} + \lambda e^{-\lambda h} \left[Fx + \frac{\lambda h}{2!} F^2 x + \ldots \right]$$

$$= \frac{(\not{1} - \lambda h + \frac{\lambda^2 h^2}{2!} - \ldots - \not{1})x}{h}$$

$$+ \lambda e^{-\lambda h} \left[Fx + \frac{\lambda h}{2!} F^2 x + \ldots \right]$$

$$\rightarrow (-\lambda + F\lambda)x.$$

Hence $A = \lambda(F - I)$. Thus the compound Poisson semi-group is generated by $\lambda(F - I)$, and we can denote the elements of this semi-group by $\exp(\lambda(F - I)t)$ (2, p. 286) by our previous theorem.

Without delving too deeply into probability theory, our final example will deal with Markov processes.

According to Feller, "The theory of semi-groups leads to a unified theory of Markov processes not obtainable by other methods" (2, p.337).

We will deal only with stationary transition probabilities, that is, those which remain invariant under transitions in time.

Loève states, "The stationary Chapman-Kolmogorov equation for stationary transition probabilities and the semi-group property for corresponding stationary transition operators (Markov endomorphisms) are equivalent" (6, p.575).

The Chapman-Kolmogorov equation is

$$\int_{-\infty}^{\infty} p(0,x;t,y)p(t,y;t+s,z)dy = p(0,x;t+s,z).$$

We will prove this equation for

$$p(s,x;t,y) = \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(y-x)^2}{2(t-s)}}$$

Obviously, it is stationary since

$$p(s+h,x;t+h,y) = \frac{1}{\sqrt{2\pi[(t+h)-(s+h)]}}\, e^{-\frac{(y-x)^2}{2[(t+h)-(s+h)]}}$$

$$= \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(y-x)^2}{2(t-s)}}$$

$$= p(s,x;t,y).$$

Proof of equation:

$$p(0,x;t,y)p(t,y;t+s,z)dy = \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-x)^2}{2t}}}{\sqrt{2\pi t}} \frac{e^{-\frac{(z-y)^2}{2(t+s-t)}}}{\sqrt{2\pi(t+s-t)}}dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{(y^2-2xy+x^2)}{2t} - \frac{(z^2-2zy+y^2)}{2s}}}{\sqrt{(2\pi)^2 ts}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{\frac{-y^2s+2xys+2zyt-ty^2}{2ts} - \frac{x^2}{2t} - \frac{z^2}{2s}}}{\sqrt{2\pi ts}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\frac{-(t+s)y^2 + 2\frac{(xs+zt)}{t+s}y - \left(\frac{xs+zt}{t+s}\right)^2 + \left(\frac{xs+zt}{t+s}\right)^2}{\frac{2ts}{t+s}} - \frac{x^2}{2t} - \frac{z^2}{2s}}}{\sqrt{2\pi ts}} \, dy$$

$$= \frac{1}{\sqrt{2\pi(t+s)}} \int_{-\infty}^{\infty} \frac{e^{-\left[y - \frac{xs+zt}{\frac{2ts}{t+s}}\right]^2}}{\sqrt{\frac{2\pi ts}{t+s}}} \, dy \; e^{-\frac{x^2}{2t} - \frac{z^2}{2s} + \frac{\left(\frac{xs+zt}{t+s}\right)^2}{\frac{2ts}{t+s}}}$$

$$= \frac{1}{\sqrt{2\pi(t+s)}} \cdot 1 \cdot e^{-\frac{x^2}{2t} - \frac{z^2}{2s} + \frac{(xs+zt)^2}{2ts(t+s)}}$$

$$= \frac{1}{\sqrt{2\pi(t+s)}} \; e^{-\frac{x^2(t+s)s - z^2(t+s)t + (xs+zt)^2}{2ts(t+s)}}$$

$$= \frac{1}{\sqrt{2\pi(t+s)}} \; e^{-\frac{\cancel{x^2s^2} - x^2st - z^2ts - \cancel{z^2t^2} + \cancel{x^2s^2} + 2xszt + \cancel{z^2t^2}}{2ts(t+s)}}$$

$$= \frac{1}{\sqrt{2\pi(t+s)}} \; e^{-\frac{x^2\cancel{st} + 2x\cancel{s}z\cancel{t} - z^2\cancel{ts}}{2\cancel{ts}(t+s)}}$$

$$= \frac{1}{\sqrt{2\pi(t+s)}} \; e^{\frac{-(x-z)^2}{2(t+s)}}$$

$= p(0,x;t+s,z).$

We define the stationary transition operator or Markov endomorphism T_t on the space G of bounded Borel functions g on $\mathcal{X}$ by

$$(T_t g)(x) = \int_{-\infty}^{\infty} g(y)p(0,x;t,y)dy.$$

With the Chapman-Kolmogorov equation we can verify the semi-group property.

$$(T_{t+s}g)(x) = \int_{-\infty}^{\infty} g(z)p(0,x;t+s,z)dz$$

which, by the Chapman-Kolmogorov equation,

$$= \int_{-\infty}^{\infty} g(z)\left[\int_{-\infty}^{\infty} p(0,x;t,y)p(t,y;t+s,z)dy\right]dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(z)p(0,x;t,y)p(t,y;t+s,z)dzdy$$

$$= \int_{-\infty}^{\infty} p(0,x;t,y)\left[\int_{-\infty}^{\infty} g(z)p(t,y;t+s,z)dz\right]dy$$

$$= \int_{-\infty}^{\infty} p(0,x;t,y)(T_s g)(y)dy$$

$$= (T_t T_s g)(x).$$

We can also show that $(T_t g)(x)$ is bounded when $g(x)$ is bounded and calculate the norm.

$$\|(T_s g)x\| < M\|g(x)\|$$

Hence it is bounded when g is bounded, and

$$\|T_s\| = \max_{\text{all } g(x)}\left\{\left\|\frac{(T_s g)x}{g(x)}\right\| ; \quad \|g(x)\| \neq 0\right\}$$

$$= \max\left\{\left\|\frac{\int g(y)p(o,x;s,y)dy}{g(x)}\right\|\right\}$$

$$= \max\left\{\frac{\left\|\int g(y)p(0,x;s,y)dy\right\|}{\|g(x)\|}\right\}$$

$$\leq \frac{\left\|\int \max_{-\infty<y<\infty}|g(y)|\,p(0,x;s,y)dy\right\|}{\|g(x)\|}$$

$$= \frac{\|g(x)\|\,1}{\|g(x)\|}$$

$$= 1.$$

The purposes of this paper were to give the reader an introduction to semi-groups and to indicate some applications of semi-groups in probability theory. The author also hopes that this paper will stimulate the interested reader to further study in this area.

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