Magnetic Stimulation of Living Tissue

Joshua Hatheway

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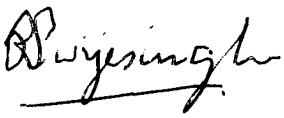
An Honors Thesis (HONRS 499)

By

Joshua Hatheway

Thesis Advisor:

Dr. Ranjith Wijesinghe



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Abstract

Magnetic stimulation appears to be a very promising medical diagnostic tool that is completely non-invasive. Non-invasive diagnostic tools are critical to the medical field because of their ability to give diagnostic data on all sorts of patients in a wide variety of conditions. This paper will give an overview of research efforts being made to construct an algorithm that will accurately model the optimal coil geometry to produce the desired stimulation of living tissues. The new model proposed is based upon the cylindrical coordinate system and utilizes modified Bessel functions to solve this problem. A successful model will hopefully lead to breakthroughs in the realm of diagnostic medical physics.

Acknowledgements:

-I would like to thank Dr. Ranjith Wijesinghe for his patience and advice throughout this project. He was extremely helpful and always made time to answer my endless questions in addition to being a great professor and mentor.

Introduction:

Magnetic stimulation is a tool that has been developed in recent years to assist in medical diagnostics. This tool can be used to stimulate nerves non-invasively so that tissue functionality can be determined. To date, the overwhelming amount of research done on this topic has been conducted using a spherical coordinate system, and has been primarily centered upon the brain and the head. Roth, Momen, and Turner published an algorithm that addresses this issue in the spherical coordinate system[1]. While extremely useful, the mainstream effort has somewhat neglected the cylindrical coordinate system and the body parts that more closely resemble the cylindrical shape. This tool appears to have much potential in medical diagnostics because of its non-invasive procedure.

Finding a non-invasive diagnostic method is absolutely critical in the field of medical physics. For instance, the well-known test, electroencephalography, or better known as the EEG, is a test that provides good data, but is extremely tedious to conduct. Some of these tests can take up to an hour to set up, even on the best patients. When also considering all those patients who are not as easy, and also those with illnesses that have difficulty being still, the process of connecting thirty-two electrodes becomes exponentially less appealing. One must also consider the purposes of the test. For example, one of the main purposes of electroencephalography is to determine whether organs are transferrable or not immediately before or after death[2]. In situations where time is clearly of the essence, a non-invasive technique would be extremely valuable. When perfected, magnetic stimulation

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[1] Roth, Bradley, Momen, Shokrollah, and Turner, Robert, (1994): "Algorithm for the design of magnetic stimulation coils," *Med. & Biol. Eng. & Comput.*, 32, 214-216.

[2] **electroencephalography**. (2009). In *Encyclopædia Britannica*. Retrieved December 08, 2009, from Encyclopædia Britannica Online: http://www.britannica.com/EBchecked/topic/183075/electroencephalography

will be one of the best non-invasive procedures available for nerve diagnostics and treatment. Because of its simple setup, essentially just a magnetic coil held near the body, nearly every patient will be more comfortable with the painless and non-invasive test rather than the previous method of using electrodes.

Theory:

From the classical study of electrodynamics and magnetism, there are two principal equations relating magnetic fields and currents. The first of these is the Biot-Savart Law, which relates current and magnetic field as follows:

$$\mathbf{B} = \int \frac{\mu_0}{4\pi} \frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}, \qquad (1)$$

and the second of these is Ampere's Law, which relates them the following way;

$$\oint_C \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_{\mathrm{enc}} \qquad (2)$$

The primary conclusion drawn from this equation is that a current induces a magnetic field, and this study is called magnetostatics[3]. More information on the derivations and applications of these laws and methods of finding magnetic fields can be found in Woosley, Roth, and Wikswo[4]. And while many great theories exist from this conclusion, it is also important to deduce that a magnetic field can in fact induce a current as well as a potential. It is this property that will be used in this paper to stimulate the desired axons and nerves

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[3] Griffiths, David, Introduction to Electrodynamics, (Upper Saddle River, NJ, 1999), 215.
[4] Woosley, James, Roth, Bradley, and Wikswo, John, (1985): "The Magnetic Field of a Single Axon: A Volume Conductor Model," *Mathematical Biosciences*, 76, pp 1-36.

to selected potentials or currents. From the results of Basser, Wijesinghe, and Roth, a potential of 90-100 mV will be desired for the stimulating potential[5].

The basic concept of this stimulation is that a current is passed through a coil of an arbitrary geometry. Once this current is passed through the coil, a magnetic field is induced through this equation:

$$\vec{B} = \frac{\mu_o N d\vec{I}}{dl} \qquad (3)$$

In this case, N stands for the number of turns, and dl is an infinitely small piece of the arbitrary coil shape in cylindrical coordinates. In order to produce desirable magnetic fields, careful consideration must be made towards the coil geometry. In previous papers, a "guess and check" method has been used to select coil geometry. This method has obvious limitations, as the perfect coil geometry needs to be "guessed," and a good guess could take years. A better but significantly more tedious solution is to solve this problem inversely, as also done by Roth, Momen, and Turner[6]. In other words, instead of first selecting a coil geometry and then experimentally determining the resulting field, this method uses the desired fields or stimulated potentials as a starting point, and then inversely solves for the coil geometry. When this coil geometry is found, it can then be constructed and implemented for diagnostic use. A model of this problem is shown below, with an infinitely long tissue cylinder assumed for calculations.

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[5] Basser, Peter, Wijesinghe, Ranjith, and Roth, Bradley, (1992): "The Activating Function for Magnetic Stimulation Derived From a Three-Dimensional Volume Conductor Model, *Transactions on Biomedical Engineering*, 39, pp 1207-1210.

[6] Roth, Bradley, Momen, Shokrollah, and Turner, Robert, (1994): "Algorithm for the design of magnetic stimulation coils," *Med. & Biol. Eng. & Comput.*, 32, 214-216.

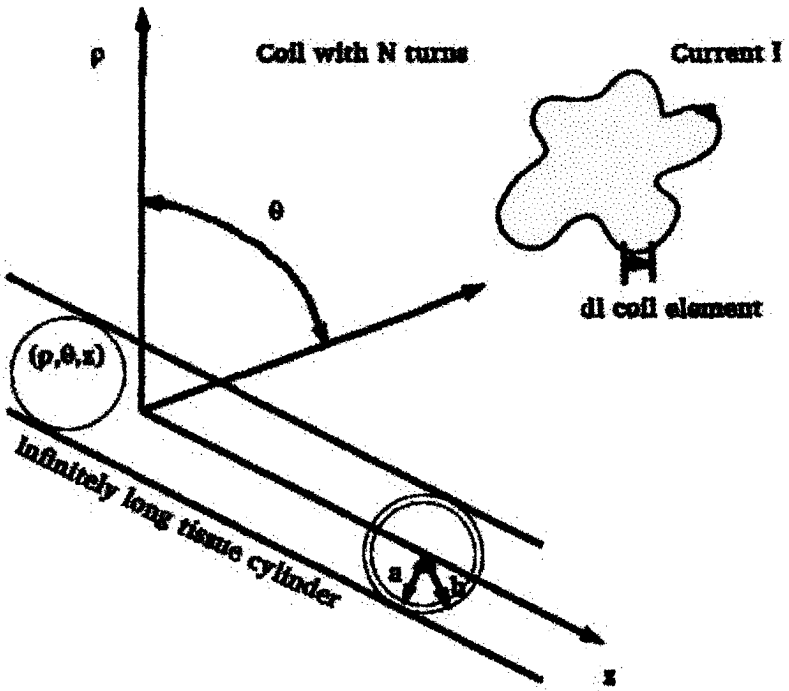

Figure 1. This drawing demonstrates the arbitrary coil shape of N turns and basic problem geometry of the cylindrical tissue.

Next, the parameters of the tissue must be explored further. The following cartoon gives a basic understanding of nerve structure;

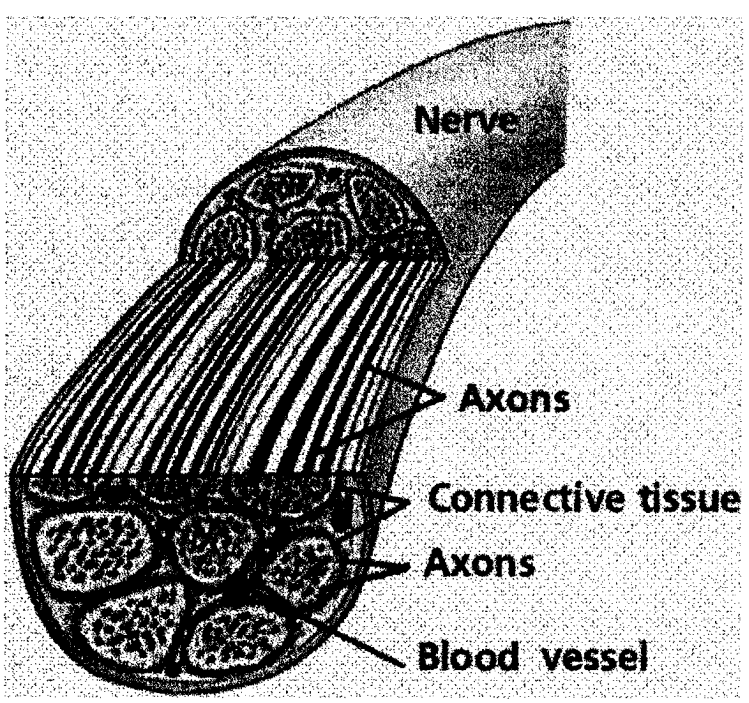

Figure 2. Schematic drawing of a nerve bundle. Note that inside the nerve are many axon bundles, which individually contain many single axons. This model will account for the axon bundle as well as the axons inside.

While this cartoon is not to scale and contains many inaccuracies, it is effective in demonstrating the model used in this paper[7]. Esselle and Stuchly created a model that approached this problem in cylindrical tissue, but it considered only the axon, and assumed all else as homogeneous surrounding tissue[8]. The model presented in this paper uses the same method as Esselle and Stuchly, but is expanded to account for the axon bundle. Then the nerve is assumed to be infinitely long, and all else outside the bundle sheath is assumed to be homogenous tissue. While this model is not exact, it is a more accurate approximation than has been used before. An even more accurate model could be achieved by accounting for every bundle in the nerve, but currently the time lost in the magnified complexity of those calculations is not worth the relatively trivial improvement in accuracy at this point in time.

To begin the calculations, several more parameters must be set. In this case, the bundle will be assumed to be uniform. Also, the transmembrane potential is assumed to be known, and the coordinate system will be based at the center of the bundle for mathematical simplification. Once these have been set, it is time to examine the electric fields present. The electric field that is most prominent is the quasi-static electric field that is induced by the coil, and it is,

$$d\vec{E}^{one} = -\frac{\mu_o N(dI/dt)d\vec{l}'}{4\pi R}. \qquad (4)$$

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[7] Nerve Structure [Online Image]. (n.d.). Retrieved December 8, 2009, from Cartage.org. http://www.cartage.org.lb/en/themes/sciences/lifescience/generalbiology/physiology/nervoussystem/Neuron/nervestruct.gif

[8] Esselle, K., and Stuchly, M., (1995): "Cylindrical tissue model for magnetic field stimulation of neurons: effects of coil geometry," *Transactions of Biomedical Engineering*, 42, pp 934- 941.

The second electric field is due to the surface charge present on the bundle and is defined by,

$$d\vec{E}^{two} = -\nabla\psi \quad (5).$$

By definition the Laplacian of the scalar potential is equal to zero as shown,

$$\nabla^2\psi = 0 \quad (6).$$

This has been assumed many times before and has yielded strong results[9]. Because of the quasi static condition, the normal of the electric field inside of the bundle is also zero and is indicated by

$$(d\vec{E}^{one} + d\vec{E}^{two}) \bullet \hat{r}\,|_{r=a} = 0 \quad (7).$$

However, knowing these electric fields is quintessential to this calculation, and must be done using computer code. These quasi static fields were calculated using equations found in a different paper from Esselle and Stuchly, and are quite complex, and would have been even worse to calculate without their much needed help[10].

From Wijesinghe, Gielen, and Wikswo, after applying a Fourier transformation, the potentials for the axon, the bundle, and surrounding tissue can be expressed as[11],

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[9] Nagarajan, Srikantan, and Durand, Dominique, (1996): "A Generalized Cable Equation for Magnetic Stimulation of Axons," *Transactions on Biomedical Engineering*, 43, pp 304-312.

[10] Esselle, Karu, and Stuchly, Maria, (1995): "Quasi-Static Electric Field in a Cylindrical Volume Conductor Induced by External Coils," *Transactions on Biomedical Engineering*, 41, pp 151-158.

[11] Wijesinghe, Ranjith, Gielen, Frans, and Wikswo, John, (1991): "A Model for Compound Action Potentials and Currents in a Nerve Bundle I: The Forward Calculation," *Annals of Biomedical Engineering*, 19, pp 43-72.

$$\phi_{axon}(\rho',\theta',k) = A_o(k)I_o(|k|\rho') + 2\sum_{m=1}^{\infty} A_m(k)I_m(|k|\rho')\cos m\theta', \quad (8)$$

$$\phi_{bundle}(\rho'^*,\theta',k) = B_o(k)I_o(|k|\rho'^*) + C_o(k)K_o(|k|\rho'^*) + 2\sum_{m=1}^{\infty}[B_m(k)I_m(|k|\rho'^*) + C_m(k)K_m(|k|\rho'^*)]\cos m\theta' \quad (9),$$

$$\phi_{outside}(\rho,\theta,k) = F_o(k)K_o(|k|\rho) + 2\sum_{m=1}^{\infty} F_m(k)K_m(|k|\rho)\cos m\theta. \quad (10)$$

I_m and K_m are modified Bessel Functions, and A_m, B_m, C_m, and F_m are all Fourier expansion coefficients. To solve for the Fourier coefficients, boundary conditions must be used. The first boundary condition is that the transmembrane potential is equivalent to the difference in potential between the internal and external membrane surfaces. The second boundary condition is that the radial current density across the axon membrane is continuous. For the third boundary condition, the radial current density at the outer surface of the nerve is continuous, and the final condition is that the potential at the outer surface of the nerve bundle is continuous. Then the Gaussian elimination method was used to find the Fourier coefficients. Once all these are solved and found, the optimal coil geometry for the desired parameters can be determined.

Results and Discussion:

To solve for the Fourier coefficients and the modified Bessel Functions, a computer code was written in Fortran and simulated on Ball State University's CCN cluster. This code also calculated the coil geometry and simulated the resulting fields and potentials. A copy of this code and its subroutines are attached to the end of this document. The simulation showed strong results, and further simulations will be run in the future to continue the verification of this method with other parameters and desired potentials. Esselle and

Stuchly's results are shown in graphical form below, and are similar to the simulation's results, although with less precision. [12]

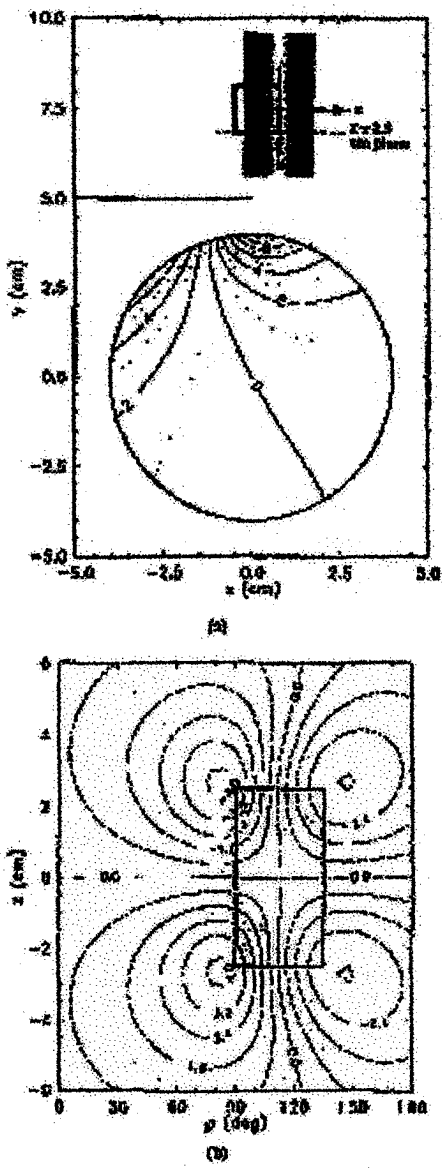

Figure 3. Contour plots for a square coil: (a) cross sectional plane; (b) cylindrical surface.

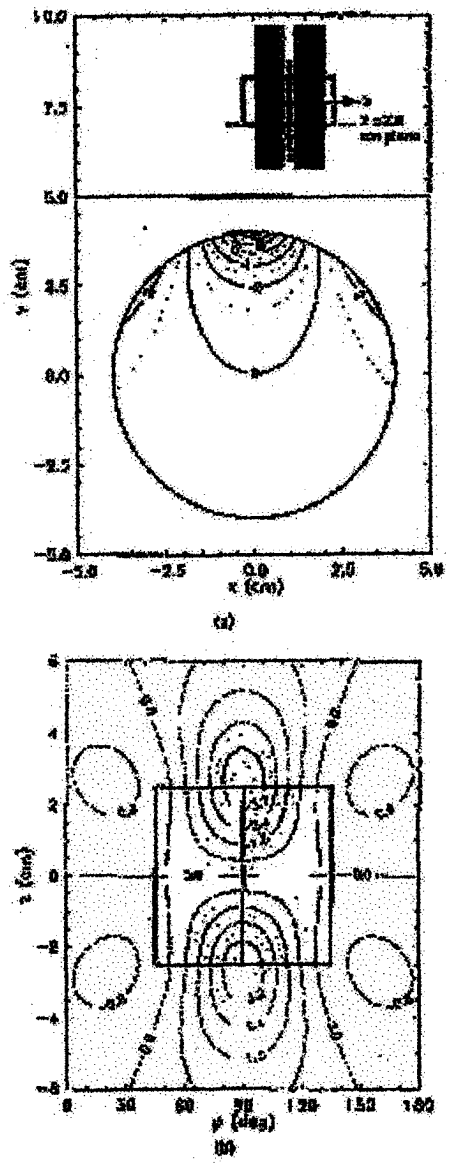

Figure 4. Contour plots for a planar DS coil: (a) cross sectional plane; (b) cylindrical surface.

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[12] Esselle, K., and Stuchly, M., (1995): "Cylindrical tissue model for magnetic field stimulation of neurons: effects of coil geometry," *Transactions of Biomedical Engineering*, 42, pp 934- 941.

The results also indicate that the assumptions made in this model were reasonable. However, it must be noted that this model does have significant limitations. Primarily, it is based upon the assumption that all the external tissue surrounding the nerve bundle is homogenous. While small areas of heterogeneous tissue may prove negligible, this model will not be sufficient for areas that do not satisfy this condition. A more complex model may be needed for these special cases in the future. Nevertheless, this model produced strong results and is worth further investigation. Simulations and code are attached.

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```
**************************************************
C
C
C       FILE: MAGFIELD.F
C       PROGRAMMER:
C
C       LIBRARY: CAPLIB
C
C       CALLS:  NTEST, FTCALL, CKN, XXX, PAIN, ICSCCU, ICSEUV
C               SQUISH, CKN, CSOLVE, CDCOMP
C
C               INPUT2=......MAGNETIC FIELD
C
C
        PROGRAM MAGFIELD
C
C
        COMPLEX INPUT(1025),OUT(1025),LAMBDA
        DOUBLE COMPLEX C(60,60),AA(60)
        COMPLEX AC(3600)
        COMPLEX AN(2049),B(60),WORK(60), COND
        COMPLEX ANISOP,EFFCON,GBZ,GBR,ZMEM,XAL,XBL,XTL,ALPHA
        DOUBLE PRECISION CR(60,60),CI(60,60),AAR(60),AAI(60)
        DOUBLE PRECISION CX(20,4),CDX(20,4),CXC(20,4),CDXC(20,4),CXDTO(3,4)
        REAL GMEN,GINT,TEMP2,CMEN,DTOR,TEMP4,TEMP3,GEXT,U0,CTOR
        REAL DBUN,BBUN,GS,PI,GINTER,XDTOR,DT,ST,XCTOR,TEMP1
        REAL TEMP5,U,E2,E,P,GG,AAA
        DOUBLE PRECISION CXB(20,4),CDXB(20,4),CXA(20,4),CDXA(20,4)
        REAL X,R,XA,XT,T,XC,CBUN,XB
        DOUBLE PRECISION CDXDT(3,4),CXCTO(3,4),CDXCT(3,4)
        DOUBLE PRECISION CXBL(20,4),CDXBL(20,4),CXAL(20,4)
        DOUBLE PRECISION CDXAL(20,4),CXTL(20,4),CDXTL(20,4)
        REAL BUF1(5020), BUF2(5020), BUF3(5020),BBB(5020),CCC(5020)
        REAL VALS(25),WTS(2),K,HDR(24),A,TLB(1000)
        REAL DOW,DOWN,BETA,SHI,VGIVEN
        REAL K1,K2,NFILT
        REAL REAL1, REAL2
        REAL RPARAM(5)
        DOUBLE PRECISION SUM1,SUM2,SUM,X1,Y1,RSUM,ESUM,X2,Y2,X3
        DOUBLE PRECISION Y3,X4,Y4,X11,Y11,RSUM1,ESUM1,RSUM2,ESUM2
        LOGICAL TOROID,NOISE
        INTEGER IPVT(60),TOKEN,IP(60)
        INTEGER IF,IDCN,NUM,IDSI,N2,N4,N1,IDCI,L,NDIM,IFOUT,N3
        INTEGER IER
        INTEGER IDSN,II1,JJ,N,II,I2,I,J,N5
```

```
      INTEGER IPARAM(6),IPATH,IROW(3600),JCOL(3600)
      character*1 ANSWER
C
      COMMON /BUF1/ BUF1
      COMMON /BUF2/ BUF2
      COMMON /BUF3/ BUF3
      COMMON /FDIF/ DZ, DK
      COMMON /HDR/ HDR
C
      NDIM=10
C     NDIM=4
C
C
c     OPEN DATA FILE
      OPEN (UNIT=10,FILE='VMEM.DAT', STATUS='NEW')
C
      PI=ACOS(-1.00)
      U0=4.0*PI*1.0E-7
C     print *, U0
C
C     INPUT CALCULATION PARAMETERS
C     I stopped here at 12:47 07/30/08
C
      N=256
      M=0
      CALL NTEST(N,M)
      A=0.01
      A=A/1000.0
      R=2.1
      R=R/1000.0
      BBUN=2.0
      BBUN=BBUN/1000.0
      DBUN=5.0
      DBUN=DBUN/1000000.0
C
      T=2.*BBUN/3.
      CBUN=BBUN+DBUN
      CMEM=0.0027
      CMEM=CMEM*0.1
C     READ *,GMEM
      GMEM=0.0000208
      GMEM=GMEM*10000.0
c
      P=0.89
      GS=GEXT
C     U=VALS(10)
```

```
        U=16.5
        IF(A.EQ.0.0) A=U/(4.1282*1000000.0)
        AAA=0.0
        AAA=AAA/1000000.0
        IF(AAA.EQ.0.0) AAA=A
        ST=0.482*(AAA)**2
        GINTER=0.882
CC
CC NOISE FILTER
CC      WE NEED TO ADD THIS IN THE FUTURE
        K1=5.0
        K2=100.
        K1=K1*1000.0
        K2=K2*1000.0
2600    CONTINUE
C
C
600     CONTINUE
C
C
C  PUTTING DATA INTO USEFUL FORM
C
C
        DO 700 I=1,N
700       INPUT(I)=CMPLX(BUF1(I),0.0)
C
        DT=1.0/70.0 ! this is the conversion factor from SFAS.DAT to
        DZ=U*DT/1000.0
        DK=(2.*PI)/(DZ*FLOAT(N))
C
C  SELECT OUTPUT FILE AND DATA SET
C
C
        TEMP1=AAA*(1./(GINTER*(1.-P)*ST)+1./(GINT*P*ST))
        TEMP2=(1.-P)/(1.+P)*GINTER
        TEMP3=AAA*SQRT(3.0)/2.
        TEMP4=GINTER*(1.-P)+GINT*P
        TEMP5=GINT*P/(GINTER*(1-P))
C
C  CALCULATE FFT
C
        CALL FTCALL(INPUT,M,0)
C
C         *** MAIN LOOP ***
C
C       FILTER CALCULATIONS IN TRANSFORM SPACE
```

```
C
      DO 800 L=1,N
C
      IF(L.EQ.(N/2+1)) GO TO 900
      K=(L-N/2-1)*DK
C
c     print *,'K=',K
C
      ZMEM=1./CMPLX(GMEM,K*U*CMEM)
      LAMBDA=CSQRT(CMPLX(TEMP1)/ZMEM)
      GBR=TEMP2+TEMP3/ZMEM
      GBZ=TEMP4/(1.0+TEMP5*(1.0/((LAMBDA/K)**2+1.0)))
      ANISOP=CSQRT(GBZ/GBR)
C     PRINT *,'ANISOP=',ANISOP,'LAMBDA=',LAMBDA,'ZMEM=',ZMEM
      EFFCON=CSQRT(GBZ*GBR)
C
C  CALCULATE ARGUMENTS OF BESSEL FUNCTIONS
C
C
      X=ABS(K)*R
      XA=ABS(K)*A
      XT=ABS(K)*T
      XCTOR=ABS(K)*CTOR !IF(TOROID) XCTOR=ABS(K)*CTOR
      XDTOR=ABS(K)*DTOR !IF(TOROID) XDTOR=ABS(K)*DTOR
      XC=ABS(K)*CBUN
      XB=ABS(K)*BBUN
      XAL=ABS(K)*A*ANISOP
      XTL=ABS(K)*T*ANISOP
      XBL=ABS(K)*BBUN*ANISOP
C
C  CHECK FOR TOO LARGE AN ARGUMENT
C
      IF(ABS(X-XA).GT.70.0) GO TO 900
      IF(ABS(XT).GT.70.) GO TO 900
      IF(ABS(XC).GT.70.) GO TO 900
      IF(ABS(XB).GT.70.) GO TO 900
      IF(CABS(XAL).GT.70.) GO TO 900
      IF(CABS(XTL).GT.70.) GO TO 900
      IF(CABS(XBL).GT.70.) GO TO 900
c     IF(TOROID.AND.(XCTOR.GT.70.)) GO TO 900
c     IF(TOROID.AND.(XDTOR.GT.70.)) GO TO 900
C
C  PREPARE NOISE FILTER
C
        NFILT=0.5*(1.0+COS((ABS(K)-ABS(K1))*3.1415/
     1          (ABS(K2)-ABS(K1))))
```

```
                IF(ABS(K).LT.ABS(K1)) NFILT=1.0
          IF(ABS(K).GT.ABS(K2)) NFILT=0.0
        IF(ABS(K).GT.ABS(K2)) GO TO 900
1000    CONTINUE
C
C
C  SETUP COEFFICIENT MATRIX
C
C
C       GET BESSEL FUNCTION VALUES
C
C
        DO 1100 I=1,20
        DO 1100 J=1,4
        CX(I,J)=0.0D0
        CDX(I,J)=0.0D0
        CXC(I,J)=0.0D0
        CDXC(I,J)=0.0D0
        CXB(I,J)=0.0D0
        CDXB(I,J)=0.0D0
        CXA(I,J)=0.0D0
        CDXA(I,J)=0.0D0
        CXBL(I,J)=0.0D0
        CDXBL(I,J)=0.0D0
        CXAL(I,J)=0.0D0
        CDXAL(I,J)=0.0D0
        CXTL(I,J)=0.0D0
        CDXTL(I,J)=0.0D0
1100    CONTINUE
C
C
C       print *,'XBL=',XBL,'XAL=',XAL,'XTL=',XTL
        CALL CKN(X,0.,CX,CDX,NUM+1)
        CALL CKN(XC,0.,CXC,CDXC,NUM+1)
        CALL CKN(XB,0.,CXB,CDXB,NUM+1)
        CALL CKN(XA,0.,CXA,CDXA,NUM+1)
C       CALL CKN(REAL(XCL),AIMAG(XCL),CXCL,CDXCL,10)
        CALL CKN(REAL(XBL),AIMAG(XBL),CXBL,CDXBL,NUM+1)
        CALL CKN(REAL(XAL),AIMAG(XAL),CXAL,CDXAL,NUM+1)
        CALL CKN(REAL(XTL),AIMAG(XTL),CXTL,CDXTL,10)
C
C
C
        DO 1200 II=1,60
        DO 1200 JJ=1,60
1200            C(II,JJ)=DCMPLX(0.0D0,0.0D0)
```

```
C
      DO 1300 II=1,NUM+1
            C(II,II)=DCMPLX(CXA(II,1),CXA(II,2))
            C(II,II+(NUM+1))=-(DCMPLX(CXAL(II,1),CXAL(II,2)))
            C(II,II+2*(NUM+1))=-(DCMPLX(CXAL(II,3),CXAL(II,4)))
            C(II+(NUM+1),II)=GINT/EFFCON*(DCMPLX(CDXA(II,1),
     1             CDXA(II,2)))
            C(II+(NUM+1),II+(NUM+1))=-DCMPLX(CDXAL(II,1),
     1             CDXAL(II,2))
            C(II+(NUM+1),II+2*(NUM+1))=-DCMPLX(CDXAL(II,3),
     1             CDXAL(II,4))
             C(II+2*(NUM+1),II+3*(NUM+1))=-DCMPLX(CXB(II,1),
     1             CXB(II,2))
            C(II+2*(NUM+1),II+4*(NUM+1))=-DCMPLX(CXB(II,3),
     1             CXB(II,4))
            C(II+3*(NUM+1),II+3*(NUM+1))=-GS/EFFCON*(DCMPLX
     1             (CDXB(II,1),CDXB(II,2)))
            C(II+3*(NUM+1),II+4*(NUM+1))=-GS/EFFCON*(DCMPLX
     1             (CDXB(II,3),CDXB(II,4)))
            C(II+4*(NUM+1),II+3*(NUM+1))=DCMPLX(CXC(II,1),
     1             CXC(II,2))
            C(II+4*(NUM+1),II+4*(NUM+1))=DCMPLX(CXC(II,3),
     1             CXC(II,4))
            C(II+4*(NUM+1),II+5*(NUM+1))=-DCMPLX(CXC(II,3),
     1             CXC(II,4))
            C(II+5*(NUM+1),II+3*(NUM+1))=DCMPLX(CDXC(II,1),
     1             CDXC(II,2))
            C(II+5*(NUM+1),II+4*(NUM+1))=DCMPLX(CDXC(II,3),
     1             CDXC(II,4))
             C(II+5*(NUM+1),II+5*(NUM+1))=-GEXT/GS*(DCMPLX
     1             (CDXC(II,3),CDXC(II,4)))
            C(II+2*(NUM+1),1+(NUM+1))=(-1.0)**(II-1)*
     1             DCMPLX(CXBL(II,1),CXBL(II,2))*DCMPLX(CXTL(II,1),
     2                    CXTL(II,2))
            C(II+2*(NUM+1),1+2*(NUM+1))=CMPLX(CXBL(II,3),
     1             CXBL(II,4))*DCMPLX(CXTL(II,1),CXTL(II,2))
        C(II+3*(NUM+1),1+(NUM+1))=(-1.0)**(II-1)
     1             *DCMPLX(CDXBL(II,1),CDXBL(II,2))
     2                    *DCMPLX(CXTL(II,1),CXTL(II,2))
            C(II+3*(NUM+1),1+2*(NUM+1))=DCMPLX(CDXBL(II,3),
     1             CDXBL(II,4))*DCMPLX(CXTL(II,1),CXTL(II,2))
1300        CONTINUE
C
C
      IF(NUM.LE.0) GO TO 1400
C
```

```
C
        DO 1500 II=1,NUM+1
          DO 1500 JJ=2,NUM+1
        C(II+2*(NUM+1),JJ+(NUM+1))=(-1)**(II-1)*(DCMPLX(CXTL((II+JJ-1),
     1          1),CXTL((II+JJ-1),2))+DCMPLX(CXTL((IABS(II-JJ)+1),
     2                  1),CXTL((IABS(II-JJ)+1),2)))*
     3                          DCMPLX(CXBL(II,1),CXBL(II,2))
        C(II+2*(NUM+1),JJ+2*(NUM+1))=(DCMPLX(CXTL((II+JJ-1),1)
     1          ,CXTL((II+JJ-1),2))+DCMPLX(CXTL(IABS(II-JJ)+1,1),
     2                  CXTL(IABS(II-JJ)+1,2)))*DCMPLX(CXBL(II,3),
     3                          CXBL(II,4))
        C(II+3*(NUM+1),JJ+(NUM+1))=(-1)**(II-1)*(DCMPLX(CXTL((II+JJ-1),
     1          1),CXTL((II+JJ-1),2))+CMPLX(CXTL((IABS(II-JJ)+1),1),
     2                  CXTL((IABS(II-JJ)+1),2)))*DCMPLX(
     3                          CDXBL(II,1),CDXBL(II,2))
        C(II+3*(NUM+1),JJ+2*(NUM+1))=(DCMPLX(CXTL((II+JJ-1),1),
     1          CXTL((II+JJ-1),2))+DCMPLX(CXTL((IABS(II-JJ)+1),1),CXTL
     2                  ((IABS(II-JJ)+1),2)))*DCMPLX(CDXBL(II,3)
     3                          ,CDXBL(II,4))
1500            CONTINUE
C
C
C SETUP VECTOR
C
C
1400    AA(1)=DCMPLX(1.0D0,0.0D0)
        DO 1600 II=2,6*(NUM+1)
1600          AA(II)=CMPLX(0.0D0,0.0D0)
C
        DO 1601 II=1,6*(NUM+1)
        AAR(II)=REAL(AA(II))
        AAI(II)=AIMAG(AA(II))
c       PRINT *,II,'AAR(I)=',AAR(II),'AAI(II)=',AAI(II)
1601    CONTINUE
C
        DO 1602 II=1,6*NDIM
        DO 1602 III=1,6*NDIM
        CR(II,III)=0.0D0
        CI(II,III)=0.0D0
1602    CONTINUE
        DO 1603 II=1,6*(NUM+1)
        DO 1603 III=1,6*(NUM+1)
        CR(II,III)=REAL(C(II,III))
        CI(II,III)=AIMAG(C(II,III))
c       PRINT *,II,III,C(II,III),'CR=',CR(II,III),'CI=',CI(II,III)
1603    CONTINUE
```

```
c       OPEN (UNIT=15,FILE='CMATRIX.DAT',STATUS='NEW')
c       DO 1604 II=1,6*(NUM+1)
c       DO 1604 III=1,6*(NUM+1)
c       WRITE(15,*),II,III,C(II,III),'CR=',CR(II,III),'CR=',CI(II,III)
c1604   CONTINUE
c       CLOSE (15)
c       print *,aar(1),aai(1),cr(4,4),ci(4,4)
C
C       STOP AT 3:01 ON 7/30/08
C
C               DO GAUSSIAN ELIMINATION.
C
C
C
        CALL DECC(6*(NUM+1),6*NDIM,CR,CI,IP,IER)
        CALL SOLC(6*(NUM+1),6*NDIM,CR,CI,AAR,AAI,IP)

1800    CONTINUE
C
C
C       .AXON TERM
C
C
C       RSUM=DBLE(GINT)*DBLE(A)*(DBLE(REAL(AA(1)))*DBLE(CXA(2,1))
C     1         -DBLE(AIMAG(AA(1)))*DBLE(CXA(2,2)))

        RSUM=DBLE(GINT)*DBLE(A)*(AAR(1)*CXA(2,1)
     1          -AAI(1)*CXA(2,2))
        ESUM=0.0D0
c       print *,'aar=',aar(1),'cxa=',cxa(2,1),'aai=',aai(1),'cxa=',cxa(2,2)
c       print *,'RUM=',RSUM,'ESUM=',ESUM
C
C       GO TO 2201
C
C        SHEATH TERM
C
C

        X1=0.0D0
        Y1=0.0D0
        X2=0.0D0
        Y2=0.0D0
        X3=0.0D0
        Y3=0.0D0
        X4=0.0D0
        Y4=0.0D0
        X1=DBLE(CBUN)*CXC(2,1)-DBLE(BBUN)*CXB(2,1)
```

```
      X2=DBLE(BBUN)*CXB(2,3)-DBLE(CBUN)*CXC(2,3)
C
      X3=AAR(1+3*(NUM+1))
      Y3=AAI(1+3*(NUM+1))
      X4=AAR(1+4*(NUM+1))
      Y4=AAI(1+4*(NUM+1))
C
      RSUM=RSUM+DBLE(GS)*(X3*X1+X4*X2)
      ESUM=ESUM+DBLE(GS)*(Y3*X1+Y4*X2)
C
C
C       BUNDLE TERM
C
C
      X1=0.0D0
      Y1=0.0D0
      X1=AAR(1+2*(NUM+1))
      Y1=AAI(1+2*(NUM+1))
      RSUM1=X1*CXTL(1,3)-Y1*CXTL(1,4)
      ESUM1=Y1*CXTL(1,3)+X1*CXTL(1,4)
C
C
       DO 1900 II=2,NUM+1
      X2=0.0D0
      Y2=0.0D0
      X2=AAR(II+2*(NUM+1))
      Y2=AAI(II+2*(NUM+1))
      RSUM1=RSUM1+2.0D0*(X2*CXTL(II,3)-Y2*CXTL(II,4))
      ESUM1=ESUM1+2.0D0*(Y2*CXTL(II,3)+X2*CXTL(II,4))
1900  CONTINUE
C
C
C
      RSUM=RSUM+DBLE(REAL(EFFCON))*DBLE(T)*(RSUM1*CXTL(2,1)
     1         -ESUM1*CXTL(2,2))-DBLE(AIMAG(EFFCON))*
     2                 DBLE(T)*(ESUM1*CXTL(2,1)
     3                         +RSUM1*CXTL(2,2))
C
C
      ESUM=ESUM+DBLE(REAL(EFFCON))*DBLE(T)*(ESUM1*CXTL(2,1)
     1         +RSUM1*CXTL(2,2))+DBLE(AIMAG(EFFCON))*DBLE(Y)*
     2                 DBLE(T)*(RSUM1*CXTL(2,1)
     3                         -ESUM1*CXTL(2,2))
C
C
C
```

```
C       BUNDLE, SECOND TERM
C
C
C
        X1=0.0D0
        Y1=0.0D0
        X2=0.0D0
        Y2=0.0D0
        X1=AAR(1+(NUM+1))
        Y1=AAI(1+(NUM+1))
        X2=AAR(1+2*(NUM+1))
        Y2=AAI(1+2*(NUM+1))
        RSUM1=0.0D0
        ESUM1=0.0D0
        RSUM2=0.0D0
        ESUM2=0.0D0
        RSUM1=X1*CXTL(1,1)-Y1*CXTL(1,2)
        ESUM1=X1*CXTL(1,2)+Y1*CXTL(1,1)
        RSUM2=X2*CXTL(1,1)-Y2*CXTL(1,2)
        ESUM2=X2*CXTL(1,2)+Y2*CXTL(1,1)
         DO 2000 II=2,NUM+1
        X1=0.0D0
        Y1=0.0D0
        X2=0.0D0
        Y2=0.0D0
        X1=AAR(II+(NUM+1))
        Y1=AAI(II+(NUM+1))
        X2=AAR(II+2*(NUM+1))
        Y2=AAI(II+2*(NUM+1))
        RSUM1=RSUM1+2.0D0*(X1*CXTL(II,1)-Y1*CXTL(II,2))
        ESUM1=ESUM1+2.0D0*(X1*CXTL(II,2)+Y1*CXTL(II,1))
        RSUM2=RSUM2+2.0D0*(X2*CXTL(II,1)-Y2*CXTL(II,2))
        ESUM2=ESUM2+2.0D0*(X2*CXTL(II,2)+Y2*CXTL(II,1))
2000    CONTINUE
2100    CONTINUE
        X1=0.0D0
        Y1=0.0D0
        X2=0.0D0
        Y2=0.0D0
        X1=DBLE(T)*CXTL(2,3)-DBLE(BBUN)*CXBL(2,3)
        Y1=DBLE(T)*CXTL(2,4)-DBLE(BBUN)*CXBL(2,4)
        X2=RSUM2*X1-ESUM2*Y1
        Y2=RSUM2*Y1+ESUM2*X1
        Y1=0.0D0
        X1=0.0D0
        X1=DBLE(BBUN)*(RSUM1*CXBL(2,1)-ESUM1*CXBL(2,2))+X2
```

```
      Y1=DBLE(BBUN)*(ESUM1*CXBL(2,1)+RSUM1*CXBL(2,2))+Y2
      RSUM=RSUM+DBLE(REAL(EFFCON))*X1-DBLE(AIMAG(EFFCON))*Y1
      ESUM=ESUM+DBLE(REAL(EFFCON))*Y1+DBLE(AIMAG(EFFCON))*X1
C
C
C     BUNDLE, THIRD TERM
C
C
      X1=0.0D0
      Y1=0.0D0
      X2=0.0D0
      Y2=0.0D0
      X11=DBLE(REAL(ANISOP))
      Y11=DBLE(AIMAG(ANISOP))
C
      X1=DBLE(REAL(ANISOP))/DBLE(ABS(K))/(X11**2+Y11**2)
     1          -DBLE(A)*CXAL(2,3)
      Y1=-DBLE(AIMAG(ANISOP))/DBLE(ABS(K))/(X11**2+Y11**2)
     1          -DBLE(A)*CXAL(2,4)
      X2=0.0D0
      Y2=0.0D0
      X2=AAR(1+2*(NUM+1))*X1-AAI(1+2*
     1          (NUM+1))*X1
      Y2=AAR(1+2*(NUM+1))*Y1+AAI(1+2*
     1          (NUM+1))*X1
C
C
      Y1=0.0D0
      X1=0.0D0
      X11=0.0D0
      Y11=0.0D0
      X11=AAR(1+(NUM+1))
      Y11=AAI(1+(NUM+1))
      X1=DBLE(A)*(X11*CXAL(2,1)-Y11*CXAL(2,2))
      Y1=DBLE(A)*(X11*CXAL(2,2)+Y11*CXAL(2,1))
      X2=X2+X1
      Y2=Y2+Y1
      RSUM=RSUM-(DBLE(REAL(EFFCON))*X2-DBLE(AIMAG(EFFCON))*Y2)
      ESUM=ESUM-(DBLE(REAL(EFFCON))*Y2+DBLE(AIMAG(EFFCON))*X2)
C
      DO 2200 I=1,3
      DO 2200 J=1,4
      CXCTO(I,J)=0.0D0
      CXDTO(I,J)=0.0D0
2200  CONTINUE
      RSUM=RSUM+DBLE(GEXT)*AAR(1+5*(NUM+1))*
```

```
     1          DBLE(R)*(DBLE(CBUN)*CXC(2,3)/DLOG(DBLE(DTOR
     2                  /CTOR))+((CXDTO(1,3))-(CXCTO(1,3))
     3                       /DBLE(ABS(K)))/((DBLE(DTOR)-DBLE(CTOR))))
C
C
C
        ESUM=ESUM+DBLE(GEXT)*AAI(1+5*(NUM+1))*
     1          DBLE(R)*(DBLE(CBUN)*CXC(2,3)/DLOG(DBLE(DTOR
     2                  /CTOR))+((CXDTO(1,3))-(CXCTO(1,3))
     3                      /DBLE(ABS(K)))/((DBLE(DTOR)-DBLE(CTOR))))
C
C
C
2201    continue
         AN(L)=CMPLX(SNGL(RSUM),SNGL(ESUM))*CMPLX(0.0,U0/R)*K/ABS(K)
C        IF(TOROID) AN(L)=AN(L)*SIN(K*E2)/(K*E2)
C        IF(NOISE)  AN(L)=AN(L)*NFILT
C
C
C
801     CONTINUE
         GO TO 800
C
C
900      AN(L)=CMPLX(0.0,0.0)
C
800      CONTINUE
C
C       **** END OF MAIN LOOP ****
C
C
C
C               CALCULATE INVERSE TRANSFORMS
C
C  BRANCH ACCORDING TO TOKEN VALUE
C
C       GO TO (2300,2400),TOKEN
C
C                       POTENTIAL
C
C
2300    CONTINUE
C

C
C       FOR EACH N, DO INVERSE FOURIER TRANSFORM
```

```
C
          DO 2500 L=1,N
                OUT(L)=(AN(L)*INPUT(L))
c       print *,L,'AN(L)=',AN(L),'INPUT(L)=',INPUT(L)
2500            CONTINUE
          CALL FTCALL(OUT,M,1)
C
        N5=N1+N3
C
        DO 500 L=1,N
          BUF1(L)=REAL(OUT(L))*1.0E9 ! Use 1000.0 for potential
c       PRINT *,L,BUF1(L)
        IF(N5.EQ.0) GO TO 500
        IF(L.GT.N5) BUF1(L)=0.0
500     CONTINUE
        OPEN(UNIT=20,FILE='POTENTIAL.DAT',STATUS='NEW')
        DO 510 I=1,N
        WRITE(20,*)float(I)*DT,BUF1(I)*32000/1000.! Now answer is in nT
510     CONTINUE
C
        CLOSE(UNIT=20)
C
C
C                MAGNETIC FIELD
C
2400    CONTINUE
C
        DO 2900 L=1,N
          OUT(L)=(AN(L)*INPUT(L)*1.0E9)
2900      CONTINUE
        CALL FTCALL(OUT,M,1)
C
C       OUTPUT
C
        DO 3000 L=1,N
        BUF2(L)=REAL(OUT(L))
        IF(N5.EQ.0) GO TO 3000
        IF(L.GT.N5) BUF2(L)=0.0
3000    CONTINUE
C
        OPEN(UNIT=21,FILE='BFIELD.DAT',STATUS='NEW')
        DO 520 I=1,N
        WRITE(22,*)I,BUF2(I)
520     CONTINUE

2800    CONTINUE
```

```
C       3100    CLOSE(33)
        STOP
        END
```

```
      SUBROUTINE SOLC (N, NDIM, AR, AI, BR, BI, IP)
C VERSION COMPLEX DOUBLE PRECISION
C I CHANGED THE IMPLICIT REAL*8 TO DOUBLE PRECISION
C
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        INTEGER N,NDIM,IP,NM1,K,KP1,M,I,KB,KM1
        DIMENSION AR(NDIM,N), AI(NDIM,N), BR(N), BI(N), IP(N)
C-----------------------------------------------------------------------
C  SOLUTION OF LINEAR SYSTEM, A*X = B .
C  INPUT..
C    N = ORDER OF MATRIX.
C    NDIM = DECLARED DIMENSION OF ARRAYS  AR AND AI.
C    (AR,AI) = TRIANGULARIZED MATRIX OBTAINED FROM DEC.
C    (BR,BI) = RIGHT HAND SIDE VECTOR.
C    IP = PIVOT VECTOR OBTAINED FROM DEC.
C  DO NOT USE IF DEC HAS SET IER .NE. 0.
C  OUTPUT..
C    (BR,BI) = SOLUTION VECTOR, X .
C-----------------------------------------------------------------------
        IF (N .EQ. 1) GO TO 50
        NM1 = N - 1
        DO 20 K = 1,NM1
                KP1 = K + 1
                M = IP(K)
                TR = BR(M)
                TI = BI(M)
                BR(M) = BR(K)
                BI(M) = BI(K)
                BR(K) = TR
        BI(K) = TI
        DO 10 I = KP1,N
                PRODR=AR(I,K)*TR-AI(I,K)*TI
                PRODI=AI(I,K)*TR+AR(I,K)*TI
                BR(I) = BR(I) + PRODR
                BI(I) = BI(I) + PRODI
10      CONTINUE
20      CONTINUE
        DO 40 KB = 1,NM1
        KM1 = N - KB
        K = KM1 + 1
        DEN=AR(K,K)*AR(K,K)+AI(K,K)*AI(K,K)
        PRODR=BR(K)*AR(K,K)+BI(K)*AI(K,K)
        PRODI=BI(K)*AR(K,K)-BR(K)*AI(K,K)
        BR(K)=PRODR/DEN
        BI(K)=PRODI/DEN
        TR = -BR(K)
```

```
        TI = -BI(K)
        DO 30 I = 1,KM1
                PRODR=AR(I,K)*TR-AI(I,K)*TI
                PRODI=AI(I,K)*TR+AR(I,K)*TI
                BR(I) = BR(I) + PRODR
                BI(I) = BI(I) + PRODI
30      CONTINUE
40      CONTINUE
50      CONTINUE
        DEN=AR(1,1)*AR(1,1)+AI(1,1)*AI(1,1)
        PRODR=BR(1)*AR(1,1)+BI(1)*AI(1,1)
        PRODI=BI(1)*AR(1,1)-BR(1)*AI(1,1)
        BR(1)=PRODR/DEN
        BI(1)=PRODI/DEN
        RETURN
C---------------------- END OF SUBROUTINE SOLC -----------------------
        END
```

```
C
      SUBROUTINE FFT(X,M,INVFLG)
      COMPLEX X(2048),U,W,TMP
      NPTS=2**M
      NM1=NPTS-1
      M1=M-1
      PI=3.1415926
C     TO DO INVERSE FFT COMPLEX CONJUGATE THE INPUT
      IF(INVFLG.EQ.0) GO TO 20
      DO 10 I=1,NPTS
10    X(I)=CONJG(X(I))
C
C     NUM IS THE NUMBER OF INPUTS TO THIS STAGE
C     N2 IS THE NUMBER OF BUTTERFLIES
C
20    continue
      DO 40 L=1,M1
      NUM=2**(M+1-L)
      N2=NUM/2
      W=(1.,0.0)
C     U IS THE INCREMENTAL KERNAL
      U=CEXP(CMPLX(0.,PI/N2))
      DO 40 J=1,N2
C
C     PERFORM THE BUTTERFLY
      DO 30 I=J,NPTS,NUM
      INDEX=I+N2
      TMP=X(I)+X(INDEX)
      X(INDEX)=(X(I)-X(INDEX))*W
30    X(I)=TMP
C     NOW INCREMENT W FOR THE NEXT BUTTERFLY
40    W=W*U
C     SPECIAL LOOP FOR LAST STAGE( WHERE W=1. FOR ALL)
      DO 50 J=1,NPTS,2
      INDEX=J+1
      TMP=X(J)+X(INDEX)
      X(INDEX)=X(J)-X(INDEX)
50    X(J)=TMP
C     FOR INVERSE FFT COMPLEX CONJUGATE THE RESULT AND DIVE BU NPTS
      IF(INVFLG.EQ.0) GO TO 70
      RN=1./NPTS
      DO 60 I=1,NPTS
60    X(I)=CONJG(X(I))*RN
70    CONTINUE
C     COULD EXIT HERE FOR CONVOLUTIONS. ETC
C************************************
```

```
C       NOW FO THE BIT-REVERSAL UNSHUFFLING
        N2=NPTS/2
        J=1
        DO 90 I=1,NM1
        IF(I.GE.J) GO TO 75
        TMP=X(J)
        X(J)=X(I)
        X(I)=TMP
75      K=N2
80      IF(K.GE.J) GO TO 90
        J=J-K
        K=K/2
        GO TO 80
90      J=J+K
        RETURN
        END
```

```
C
      SUBROUTINE DECC (N, NDIM, AR, AI, IP, IER)
C  VERSION COMPLEX DOUBLE PRECISION
C  I CHANGED IMPLICIT REAL*8 TO IMPLICIT DOUBLE PRECISION
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      INTEGER N,NDIM,IP,IER,NM1,K,KP1,M,I,J
      DIMENSION AR(NDIM,N), AI(NDIM,N), IP(N)
C-----------------------------------------------------------------------
C  MATRIX TRIANGULARIZATION BY GAUSSIAN ELIMINATION
C  ------ MODIFICATION FOR COMPLEX MATRICES --------
C  INPUT..
C     N = ORDER OF MATRIX.
C     NDIM = DECLARED DIMENSION OF ARRAYS  AR AND AI .
C     (AR, AI) = MATRIX TO BE TRIANGULARIZED.
C  OUTPUT..
C     AR(I,J), I.LE.J = UPPER TRIANGULAR FACTOR, U ; REAL PART.
C     AI(I,J), I.LE.J = UPPER TRIANGULAR FACTOR, U ; IMAGINARY PART.
C     AR(I,J), I.GT.J = MULTIPLIERS = LOWER TRIANGULAR FACTOR, I - L.
C                                                    REAL PART.
C     AI(I,J), I.GT.J = MULTIPLIERS = LOWER TRIANGULAR FACTOR, I - L.
C                                                    IMAGINARY PART.
C     IP(K), K.LT.N = INDEX OF K-TH PIVOT ROW.
C     IP(N) = (-1)**(NUMBER OF INTERCHANGES) OR 0 .
C     IER = 0 IF MATRIX A IS NONSINGULAR, OR K IF FOUND TO BE
C           SINGULAR AT STAGE K.
C  USE  SOL  TO OBTAIN SOLUTION OF LINEAR SYSTEM.
C  IF IP(N)=0, A IS SINGULAR, SOL WILL DIVIDE BY ZERO.
C
C  REFERENCE..
C     C. B. MOLER, ALGORITHM 423, LINEAR EQUATION SOLVER,
C     C.A.C.M. 15 (1972), P. 274.
C-----------------------------------------------------------------------
        IER = 0
        IP(N) = 1
        IF (N .EQ. 1) GO TO 70
        NM1 = N - 1
        DO 60 K = 1,NM1
        KP1 = K + 1
        M = K
        DO 10 I = KP1,N
              IF (DABS(AR(I,K))+DABS(AI(I,K)) .GT.
     &        DABS(AR(M,K))+DABS(AI(M,K))) M = I
10      CONTINUE
        IP(K) = M
        TR = AR(M,K)
```

```
      TI = AI(M,K)
      IF (M .EQ. K) GO TO 20
      IP(N) = -IP(N)
      AR(M,K) = AR(K,K)
      AI(M,K) = AI(K,K)
      AR(K,K) = TR
      AI(K,K) = TI
20    CONTINUE
      IF (DABS(TR)+DABS(TI) .EQ. 0.D0) GO TO 80
      DEN=TR*TR+TI*TI
      TR=TR/DEN
      TI=-TI/DEN
      DO 30 I = KP1,N
              PRODR=AR(I,K)*TR-AI(I,K)*TI
              PRODI=AI(I,K)*TR+AR(I,K)*TI
              AR(I,K)=-PRODR
              AI(I,K)=-PRODI
30    CONTINUE
      DO 50 J = KP1,N
              TR = AR(M,J)
              TI = AI(M,J)
              AR(M,J) = AR(K,J)
              AI(M,J) = AI(K,J)
              AR(K,J) = TR
              AI(K,J) = TI
      IF (DABS(TR)+DABS(TI) .EQ. 0.D0) GO TO 48
      IF (TI .EQ. 0.D0) THEN
      DO 40 I = KP1,N
              PRODR=AR(I,K)*TR
              PRODI=AI(I,K)*TR
              AR(I,J) = AR(I,J) + PRODR
              AI(I,J) = AI(I,J) + PRODI
40    CONTINUE
              GO TO 48
      END IF
      IF (TR .EQ. 0.D0) THEN
              DO 45 I = KP1,N
              PRODR=-AI(I,K)*TI
              PRODI=AR(I,K)*TI
              AR(I,J) = AR(I,J) + PRODR
              AI(I,J) = AI(I,J) + PRODI
45    CONTINUE
      GO TO 48
      END IF
      DO 47 I = KP1,N
              PRODR=AR(I,K)*TR-AI(I,K)*TI
```

```
                PRODI=AI(I,K)*TR+AR(I,K)*TI
                AR(I,J) = AR(I,J) + PRODR
                AI(I,J) = AI(I,J) + PRODI
47        CONTINUE
48        CONTINUE
50        CONTINUE
60        CONTINUE
70        K = N
          IF (DABS(AR(N,N))+DABS(AI(N,N)) .EQ. 0.D0) GO TO 80
          RETURN
80        IER = K
          IP(N) = 0
          RETURN
C----------------------- END OF SUBROUTINE DECC -------------------------
          END
C
```

```
C
      SUBROUTINE CKN(X,Y,SCK,SCI,NUM)
C
C     CK(#) HAS THE COMPLEX BESSEL FUNCTIONS
C     CI(#) HAS THE FIRST DERIVATIVES OF THE CK(#)
C
      DOUBLE PRECISION DD,RIO,EIO,RI1,EI1,RKO,EKO,RK1,EK1,XD,YD
      DOUBLE PRECISION AA,CK(20,4),CI(20,4)
      INTEGER NUM,K
      REAL X,Y,FAC,SS
      DOUBLE PRECISION SCI(20,4),SCK(20,4)
      COMPLEX SUM
C
      CALL XXX(X,Y,RIO,EIO,RI1,EI1,RKO,EKO,RK1,EK1)
C
c     print *,x,y
c     IF(Y.NE.0.0)    print *,x,y,RIO,EIO,RI1,EI1,RKO,EKO,RK1,EK1
      XD=DBLE(X)
      YD=DBLE(Y)
      CK(1,1)=RIO
      CK(1,2)=EIO
      CK(1,3)=RKO
      CK(1,4)=EKO
      CK(2,1)=RI1
      CK(2,2)=EI1
      CK(2,3)=RK1
      CK(2,4)=EK1
C
      CI(1,1)=RI1
      CI(1,2)=EI1
      CI(1,3)=-RK1
      CI(1,4)=-EK1
      DO 80 J=1,4
      SCI(1,J)=SNGL(CI(1,J))
      SCK(1,J)=SNGL(CK(1,J))
80    CONTINUE
      IF(NUM. GE. 1) GO TO 10
      RETURN
10    CONTINUE
      DD=2.0D0/(XD**2+YD**2)
      AA=(XD**2+YD**2)
C     WRITE(6,*)AA
      IF(NUM.EQ.1) GO TO 40
      DO 20 I=3,NUM+2
C
      IF(DSQRT(XD**2D0+YD**2D0).LE.0.1D0) GO TO 500
C
      CK(I,1)=CK(I-2,1)-DD*(XD*CK(I-1,1)+YD*CK(I-1,2))
     1 *DFLOAT(I-2)
      CK(I,2)=CK(I-2,2)-DD*(XD*CK(I-1,2)-YD*CK(I-1,1))
     1 *DFLOAT(I-2)
      GO TO 510
500   CONTINUE
      SUM=CMPLX(0.0,0.0)
      SS=1.
      DO 550 K=1,10
      FAC=0.
      SS=SS*FLOAT(K-1)
      IF(K.EQ.1) SS=1.
      CALL PACTOR((I+K-1),FAC)
      SUM=SUM+((CMPLX(X,Y)/2.)**(I-3+2*K))/FAC/SS
550   CONTINUE
      CK(I,1)=DBLE(REAL(SUM))
      CK(I,2)=DBLE(AIMAG(SUM))
C
510   CONTINUE
      CK(I,3)=CK(I-2,3)+DD*(XD*CK(I-1,3)+YD*CK(I-1,4))
     1 *DFLOAT(I-2)
      CK(I,4)=CK(I-2,4)+DD*(XD*CK(I-1,4)-YD*CK(I-1,3))
     1 *DFLOAT(I-2)
20    CONTINUE
C
40    CONTINUE
      DO 30 I=2,NUM+1
C     CI(I,1)=CK(I-1,1)-DFLOAT(I-1)/AA*(XD*CK(I,1)+YD*CK(I,2))
      CI(I,1)=.5D0*(CK((I-1),1)+CK((I+1),1))
C     CI(I,2)=CK(I-1,2)-DFLOAT(I-1)/AA*(XD*CK(I,2)-YD*CK(I,1))
```

```
        CI(I,2)=.5D0*(CK((I-1),2)+CK((I+1),2))
C       CI(I,3)=-CK(I-1,3)-DFLOAT(I-1)/AA*(XD*CK(I,3)+YD*
        CI(I,3)=-0.5D0*(CK((I-1),3)+CK((I+1),3))
C     1 CK(I,4))
C       CI(I,4)=-CK(I-1,4)-DFLOAT(I-1)/AA*(XD*CK(I,4)-YD*
        CI(I,4)=-0.5D0*(CK((I-1),4)+CK((I+1),4))
C     1 CK(I,3))
30      CONTINUE
C
        DO 100 I=1,NUM+1
        DO 100 J=1,4
        SCI(I,J)=CI(I,J)
        SCK(I,J)=CK(I,J)
c       print *,I,J,SCI(I,J),SCK(I,J)
100     CONTINUE

        RETURN
        END
```

```
**************************************************
C
C       FILE: BESSEL.F
C       PROGRAMMER:
C
C       LIBRARY: CAPLIB
C
C       PURPOSE: CALCULATE THE ZEROTH AND FIRST ORDER COMPLEX MODIFIED
C       BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND
C
C
C
        SUBROUTINE XXX(X,Y,RIO,EIO,RI1,EI1,RKO,EKO,RK1,EK1)
        DOUBLE PRECISION COEF(8,8),ARGU,ARGUT,CC,PI
        DOUBLE PRECISION TETA,A,B,AA,BB,RIO,EIO,RI1,EI1,RKO,EKO
        DOUBLE PRECISION RK1,EK1,R,AK,AL,XD,YD
        REAL X,Y
C
C       THE FOLLOWING DATA WERE TAKEN FROM THE PAGES
C
C                       378 AND 379 OF
C
C               "HANDBOOK OF MATHEMATICAL FUNCTIONS
C                               WITH
C       FORMULAS, GRAPHS, AND MATHEMATICAL TABLES"
C
C                       EDITED BY
C                               M. ABROMOWITZ
C                                       AND
C                               I. A. STEGAN
C
C       THE FORMULAS IN THE SAME PAGES WERE TRANSFERRED FROM REAL
C       VARIABLES TO COMPLEX VARIABLES AND REFORMULATE ALL THE
C       EQUATIONS.
C
        DATA((COEF(I,J),J=1,8),I=1,8)
     1    /3.5156229, 3.0899424, 1.2067492, 0.2659732,
     2    0.0360768, 0.0045813, 0.0000000, 0.0000000,
     3    0.01328592, 0.00225319, -0.00157565, 0.00916281,
     4    -0.02057706, 0.02635537, -0.01647633, 0.00392377,
     5    0.87890594, 0.51498869, 0.15084934, 0.02658733,
     6    0.00301532, 0.00032411, 0.00000000, 0.00000000,
     7    -0.03988024, -0.00362018, 0.00163801, -0.01031555,
     8    0.02282967, -0.02895312, 0.01787654, -0.00420059,
     1    0.42278420, 0.23069756, 0.03488590, 0.00262698,
```

```
     2    0.00010750, 0.00000740, 0.00000000, 0.00000000,
     3    -0.07832358, 0.02189568, -0.01062446, 0.00587872,
     4    -0.00251540, 0.00053208, 0.000000000, 0.00000000,
     5    0.15443144, -0.67278579, -0.18156897, -0.01919402,
     6    -0.00110404, -0.00004686, 0.00000000, 0.00000000,
     7    0.23498619, -0.03655620, 0.01504268, -0.00780353,
     8    0.00325614, -0.00068245, 0.00000000, 0.000000000/
C
C
      DATA PI/3.14159266535897932384626643/
C
C
      XD=DBLE(X)
      YD=DBLE(Y)
C
C     CHECK FOR ARGUMENT TOO BIG FOR FORTRAN
C     COMPLEX EXPONENTIAL FUNCTION
C
      IF(XD.EQ.0.0D0.AND.YD.GT.0.0D0) GO TO 100
      IF(XD.EQ.0.0D0.AND.YD.LT.0.0D0) GO TO 200
      ARGUT=YD/XD
      TETA=DATAN(ARGUT)
cc    print *,teta
      GO TO 300
100   TETA=PI/2.0D0
      GO TO 300
200   TETA=3.0D0*PI/2.0D0
300   CONTINUE
      ARGU=XD**2D0+YD**2D0
      R=DSQRT(ARGU)
      IF(ABS(R).GT.(88.0)) GO TO 999
C
C
C   I. CALCULATE BESSEL FUNCTION OF THE FIRST KIND (IO,I1)
C
      IF(R.GT.3.75D0) GO TO 500
C
      A=0.0D0
      B=0.0D0
      DO 400 I=1,6
      CC=2D0*DFLOAT(I)
      A=A+COEF(1,I)*((R/3.75D0)**(CC))*DCOS(CC*TETA)
400   B=B+COEF(1,I)*(R/3.75D0)**(CC)*DSIN(CC*TETA)
      RIO=A+1.0D0
      EIO=B
      CC=DSIN(CC*TETA)
```

```
C
C
      GO TO 700
C
C
500   CONTINUE
      A=0.0D0
      B=0.0D0
      DO 600 I=1,8
      A=A+COEF(2,I)*(3.75D0/R)**I*DCOS(DFLOAT(I)*TETA)
600   B=B-COEF(2,I)*(3.75D0/R)**i*DSIN(DFLOAT(I)*TETA)

      RIO=DEXP(R*DCOS(TETA))/DSQRT(R)*(DCOS(R*DSIN(TETA)-TETA/2.0D0)
     1         *(A+.39894228D0)-B*DSIN(R*DSIN(TETA)-TETA/2.0D0))
      EIO=DEXP(R*DCOS(TETA))/DSQRT(R)*((A+.398894228D0)*DSIN(R*
     1         DSIN(TETA)-TETA/2.0D0)+B*DCOS(R*DSIN(TETA)-TETA/2.0D0))
C
      IF(R.GT.3.75D0) GO TO 900
C
700   CONTINUE
C
      A=0.0D0
      B=0.0D0
      DO 800 I=1,6
      A=A+COEF(3,I)*(R/3.75D0)**(2*I)*DCOS(2.0D0*DFLOAT(I)*TETA)
800   B=B+COEF(3,I)*(R/3.75D0)**(2*I)*DSIN(2.0D0*DFLOAT(I)*TETA)
      RI1=(A+.5D0)*R*DCOS(TETA)-R*B*DSIN(TETA)
      EI1=B*R*DCOS(TETA)+(A+.5D0)*R*DSIN(TETA)
C
C
      GO TO 1100
C
C
900   CONTINUE
      A=0.0D0
      B=0.0D0
      DO 1000 I=1,8
      A=A+COEF(4,I)*(3.75D0/R)**I*DCOS(DFLOAT(I)*TETA)
1000  B=B-COEF(4,I)*(3.75D0/R)**I*DSIN(DFLOAT(I)*TETA)
      RI1=DEXP(R*DCOS(TETA))/DSQRT(R)*(DCOS(R*DSIN(TETA)-TETA/2.0D0)*
     1         (A+.39894228D0)-B*DSIN(R*DSIN(TETA)-TETA/2.0D0))
      EI1=DEXP(R*DCOS(TETA))/DSQRT(R)*(DSIN(R*DSIN(TETA)-TETA/2.0D0)*
     1         (A+.39894228D0)+B*DCOS(R*DSIN(TETA)-TETA/2.0D0))
C
1100  CONTINUE
C
```

```
C   II THIS PART OF THE PROGRAM CALCULATES THE FUNCTIONS K0 AND K1
C
        IF(R.GT.2.0D0) GO TO 1300
C
        AA=0.0D0
        BB=0.0D0
        DO 1200 I=1,6
        AA=COEF(5,I)*(R/2.0D0)**(2*I)*
     1                  DCOS(2.0D0*DFLOAT(I)*TETA)+AA
1200    BB=COEF(5,I)*(R/2.0D0)**(2*I)
     1          *DSIN(2.0D0*DFLOAT(I)*TETA)+BB
C
        RKO=AA-RIO*DLOG(R/2.0D0)+TETA*EIO-.57721566
        EKO=BB-RIO*TETA-EIO*DLOG(R/2.0D0)
C
        GO TO 1500
1300    CONTINUE
        AA=0.0D0
        BB=0.0D0

        DO 1400 I=1,6
        AA=COEF(6,I)*(2.0D0/R)**I*DCOS(DFLOAT(I)*TETA)+AA
1400    BB=BB-COEF(6,I)*(2.0D0/R)**I*DSIN(DFLOAT(I)*TETA)
        RKO=DEXP(-R*DCOS(TETA))/DSQRT(R)*(DCOS(R*DSIN(TETA)+TETA/2.0D0
     1          )*(AA+1.25331414D0)+BB*DSIN(R*DSIN(TETA)+TETA/2.0D0))
        EKO=DEXP(-R*DCOS(TETA))/DSQRT(R)*(-(AA+1.25331414D0)*DSIN(R*
     1          DSIN(TETA)+TETA/2.0D0)+BB*DCOS(R*DSIN(TETA)+TETA/2.0D0))
C
C
        IF(R.GT.2.0D0) GO TO 1700
C
1500    CONTINUE
C
C
        AK=0.0D0
        AL=0.0D0
        AA=0.0D0
        BB=0.0D0
        AK=DCOS(TETA)*R*(RI1*DLOG(R/2.0D0)-EI1*TETA)-DSIN(TETA)*R*
     1          (TETA*RI1+EI1*DLOG(R/2.0D0))
        AL=DSIN(TETA)*R*(RI1*DLOG(R/2.0D0)-EI1*TETA)+R*DCOS(TETA)*
     1          (TETA*RI1+EI1*DLOG(R/2.0D0))
        DO 1600 I=1,6
        AA=AA+COEF(7,I)*(R/2.0D0)**(2*I)*DCOS(2.0D0*DFLOAT(I)*TETA)
1600    BB=BB+COEF(7,I)*(R/2.0D0)**(2*I)*DSIN(2.0D0*DFLOAT(I)*TETA)
        RK1=(AA+AK+1.0D0)/R*DCOS(TETA)+(BB+AL)/R*DSIN(TETA)
```

```
      EK1=(BB+AL)/R*DCOS(TETA)-(AA+AK+1.0D0)/R*DSIN(TETA)
C
      RETURN
C
C
1700  CONTINUE
C
C
      AA=0.0D0
      BB=0.0D0
      DO 1800 I=1,6
      AA=AA+COEF(8,I)*(2.0D0/R)**I*DCOS(DFLOAT(I)*TETA)
1800  BB=BB-COEF(8,I)*(2.0D0/R)**I*DSIN(DFLOAT(I)*TETA)
      RK1=DEXP(-R*DCOS(TETA))/DSQRT(R)*((AA+1.25331414D0)*DCOS(R*
     1         DSIN(TETA)+TETA/2.0D0)+BB*DSIN(R*DSIN(TETA)+TETA/2.0D0))
      EK1=DEXP(-R*DCOS(TETA))/DSQRT(R)*(-(AA+1.25331414D0)*DSIN(R*
     1         DSIN(TETA)+TETA/2.0D0)+BB*DCOS(R*DSIN(TETA)+TETA/2.0D0))
      RETURN
C
C
C     CHECK FOR ERRORS
C
C
999   CONTINUE
      WRITE(5,10)
10    FORMAT(' ARGUMENT TOO LARGE...EROOR!')
      RETURN
      END
```