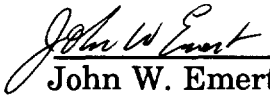


Graph Fractals

An Honors Thesis (Honrs 499)

by
Ben J. Kelly

Thesis Advisor


John W. Emert

Ball State University
Muncie, Indiana

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I. Introduction

Fractals have become in the last few years a popular topic of contemporary mathematics. Graph fractals generalize these classical fractals and share many of the same traits, such as non-integer dimension and self-similarity. Extending one's perspective to include Graph fractals allows one to model even more through the use of fractal notation.

Very little work on graph fractals currently exists in the literature. This project will provide a list of the available existing work, identify the relationship between graph fractals and classical fractals, create a set of examples of graph fractals, and analyze these example adapting the terminology of classical fractals.

II. Hausdorff and Topological Dimension

Benoit Mandelbrot formally defined a fractal as "...a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension" [2, 15]. For clarification, a definition of both the Hausdorff Dimension and the topological dimension will follow.

The form of the **Hausdorff dimension** to follow is described by Gerald A. Edgar [1, 105]. He states that a ratio list, (r_1, r_2, \dots, r_n) , is a finite list of the reducing ratios, where each ratio, r_i , corresponds to one of the n separate pieces that compose the following iteration. Associated with a ratio list, there exists a unique nonnegative number s such that

$$r_1^s + r_2^s + \dots + r_n^s = \sum_{i=1}^n (r_i^s) = 1.$$

If all the reducing ratios, r_i , are equal to r , the previous summation can be reduced to

$$nr^s = 1,$$

and therefore

$$S = \log n / \log(1/r).$$

The **topological dimension** of an object can be best described by establishing its "degrees of freedom", a nonnegative integer. A point, or set of distinct points, has a dimension equal to zero because its movement is completely restricted. To describe the dimension of a line, or curve in any form, consider the possible movement of a point on that line. Its movement is restricted to back and

forth along the line, a combination of one direction. Therefore, the line has a topological dimension of one. Next, consider the movement a point confined to a surface in the x-y plane. Its movement is restricted to back and forth along the x-axis and also back and forth along the y-axis, or any combination thereof. Therefore, the plane has a topological dimension of two. A similar analysis can also be applied to any object of higher order dimension.

Two examples of classical fractals will now be constructed and their dimensions computed.

III. Classical Fractals

Example 1: The **Cantor Set** is constructed from a single line segment of length one. The first iteration of the Cantor Set replaces the entire line segment with two copies of the original line of length one-third at either end. The second and all following iterations consist of two copies of the prior iteration reduced by one-third being placed at opposite ends of the previous line segment. **Figure 1** shows the first few iterations of the Cantor Set.

The Cantor Set has Hausdorff dimension

$$H_c = (\log 2) / (\log 3) \approx 0.6309$$

and Topological dimension

$$C_c = 0, \text{ so } H_c > C_c.$$

Example 2: The **Sierpinski Gasket** is constructed from an equilateral triangle including its interior. To begin the second iteration, divide the original figure into four equilateral, equal area triangles. To finish the second iteration, remove the middle, or inverted triangle, which leaves three copies of the original triangle in each of the corners. The next, and all following iterations repeat this process of subdividing the remaining solid triangles and removing the inverted triangles from them. **Figure 2** shows the first few iterations of the Sierpinski Gasket.

The Sierpinski Gasket has Hausdorff dimension

$$H_s = (\log 3) / (\log 2) \approx 1.5850$$

and Topological dimension

$$C_s = 1, \text{ so } H_s > C_s.$$

IV. Graph Fractals

A **Graph fractal** is described by a set of functions, (f_1, f_2, \dots, f_n) , a ratio list, (r_1, r_2, \dots, r_n) , with the results redirected back into the set of functions to be computed again. Graph fractals share many of the same traits as these classical fractals. These common traits include self-similarity and a non-integer Hausdorff (similarity) dimension. To be self-similar, a fractal must be composed of smaller copies of itself. The self-similarity attribute in graph fractals provides a way of creating fractals that are not globally self-similar.

For this paper, a formal definition of the similarity dimension will not be

necessary. Each example used herein to illustrate graph fractals will have a similarity dimension equal to its corresponding Hausdorff dimension.

Several examples of graph fractals will now be constructed and then analyzed using classical fractal comparisons.

V. Examples

Example 3: Begin with an equilateral triangle having each side length three and a regular hexagon having each side length one. The second iteration places a full size copy of the hexagon completely inside of the original triangle. To finish this stage, the figure created is reduced by a factor of $r = 1/3$, and then six copies are placed inside of the original hexagon. All following iterations place a full size copy of the hexagon figure of the previous iteration inside the triangle, and then taking six reduced copies of this triangle figure and placing them inside of the hexagon. **Figure 3** illustrates the first few iterations of this graph fractal.

$$H = S = (\log 6) / (\log 3) \approx 1.6309.$$

$$T = 1, \text{ so } H > T.$$

Example 4: The next example begins with a right triangle having a hypotenuse of length one and a square with sides of length one. The second iteration places a copy of the square reduced by a factor of $r = 1/3$ inside the original triangle along the middle 1/3 of the hypotenuse. To finish this iteration, four full size copies of the triangle figure just created are placed inside the original square. Each iteration to follow takes the square created in the previous iteration,

reduces it by $r = 1/3$, places it along the hypotenuse inside the triangle, and then four of these triangles are positioned back in the square. **Figure 4** depicts the first few iterations of this graph fractal.

$$H = S = (\log 4) / (\log 3) \approx 1.2619.$$

$$T = 1, \text{ so } H > T.$$

Example 5: The next example begins with a regular octagon having a side length of one and a square having a side length of $(1 + \sqrt{2})$. The second iteration places a single copy the square reduced by a factor a $r_i = (\sqrt{2 + \sqrt{2}}) / (1 + \sqrt{2})$ inside the original octagon with each corner of the square meeting every other corner of the octagon. Then, four copies of the octagon figure just produced are reduced by $r_j = 1/2$ and placed in each quadrant of the original square. Finally, to finish off the second iteration, the original square is reduced by $r_k = 1/(2 + \sqrt{2})$ and placed in the center of the larger square. Each subsequent iteration takes the square figure produced at the end of the previous iteration, reduces it by r_i , places it in the octagon, reduces this figure by r_j , and positions four of these and the square reduced by r_k into the original size square. **Figure 5** illustrates the first two iterations of this graph fractal.

$$H = S \leftrightarrow \sum (r^s) = 1 \leftrightarrow S \approx 1.540.$$

$$T = 1, \text{ so } H > T.$$

Example 6: The final example will present two different approaches to achieve the same graph fractal. The first will begin with a regular hexagon with

sides of length one and a rhombus with side length two and acute angles equal to 60 degrees. The second iteration places three copies of the rhombus reduced by a factor of $r = 1/2$ inside the hexagon, and then places this figure into the previous rhombus making sure to match two of its opposite corners with the obtuse angled corners of the rhombus. All iterations that follow place three reduced copies of the created rhombus figure into the hexagon, and then positioning this figure back into the rhombus. **Figure 6** illustrates the first few iterations of this graph fractal. To demonstrate a different direction to the same graph fractal, begin with the same two figures. This time place a full size hexagon inside the rhombus first, and then take three copies of this figure reduced by a factor of $r = 1/2$ and align them inside the original hexagon. Each iteration that follows will continue this pattern of a full size copy of the new hexagon figure being placed inside the rhombus followed by three reduced copies of this rhombus back into the hexagon. **Figure 7** illustrates the first few iterations of this graph fractal. Both **Figure 6** and **Figure 7** have been highlighted to show that the final graph fractal created is the same, although the framework behind them may appear different.

$$H = S = (\log 3) / (\log 2) \approx 1.5850.$$

$$T = 1, \text{ so } H > T.$$

VI. Conclusion

Although fractals have become a popular topic in contemporary mathematics, little attention has yet been given to graph fractals in the literature. The most common format at this time would consist of references in textbooks. Thus, the size of the list of literature exclusively devoted to graph fractals is not very extensive. Although little formal work has been done with graph fractals, they are very useful. By sharing self-similarity and non-integer dimension with classical fractals, graph fractals provide a way of creating fractals that are not globally self-similar. They create a way of modeling through the use of fractal notation.

The Cantor Set

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Figure 1



The Sierpinski Gasket

Figure 2

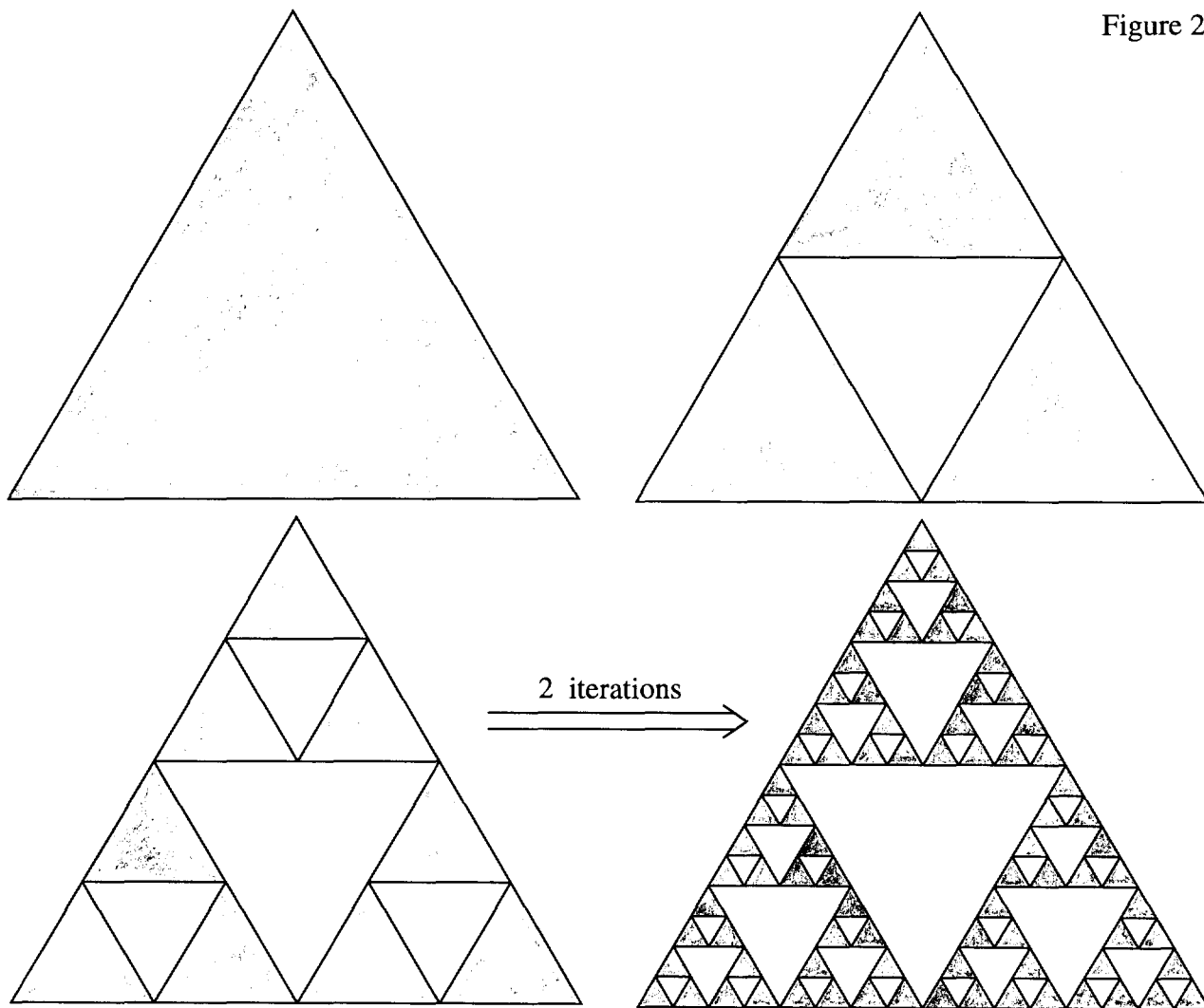


Figure 3

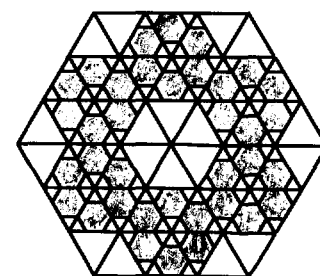
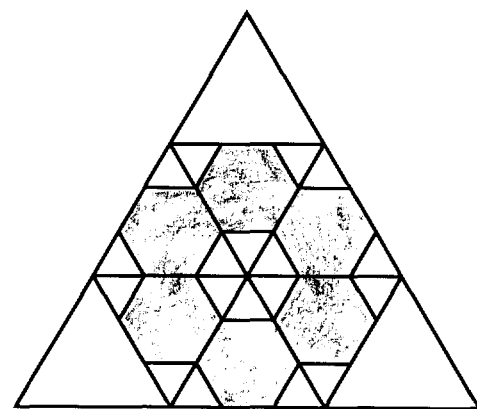
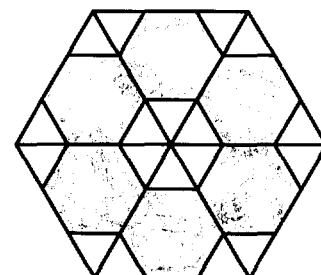
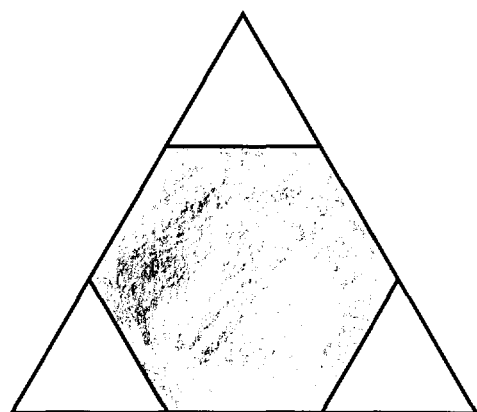
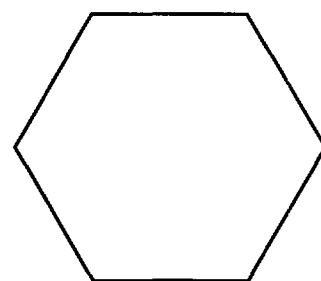
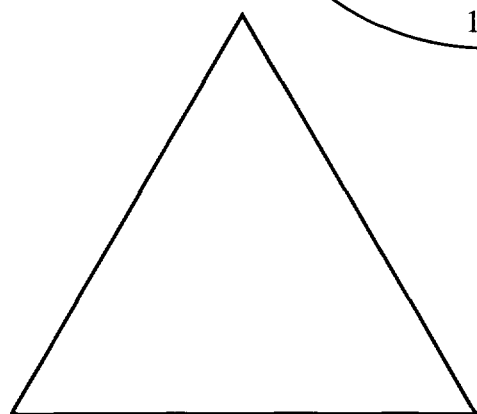
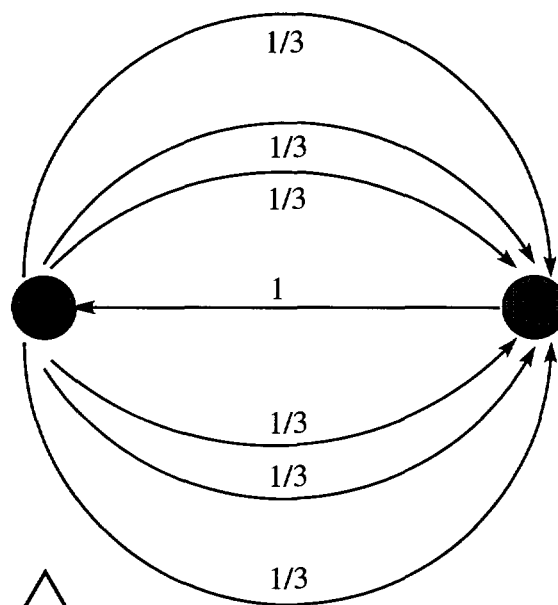


Figure 4

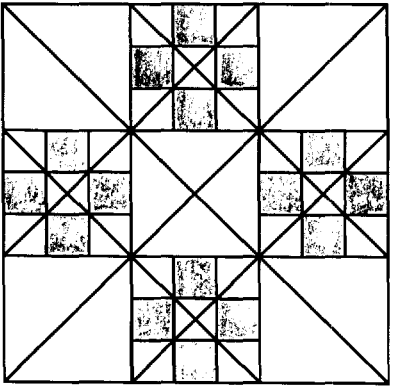
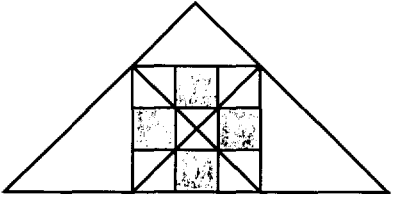
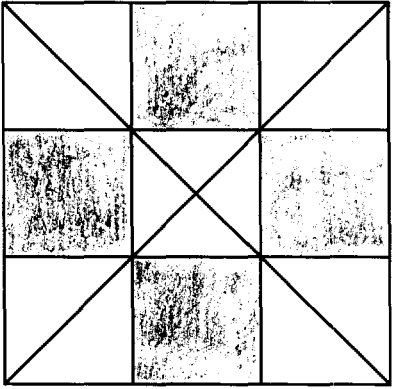
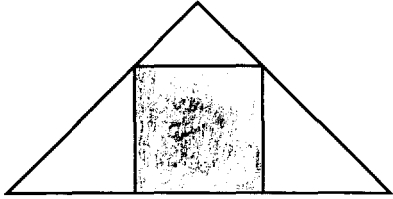
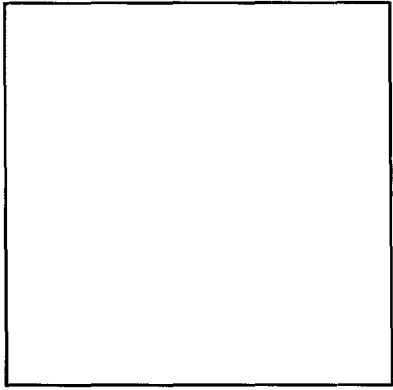
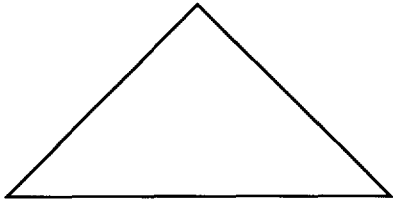
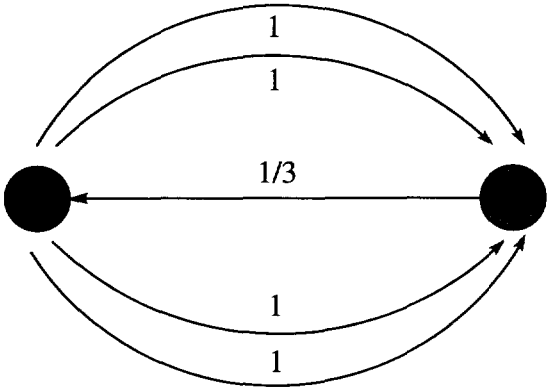


Figure 5

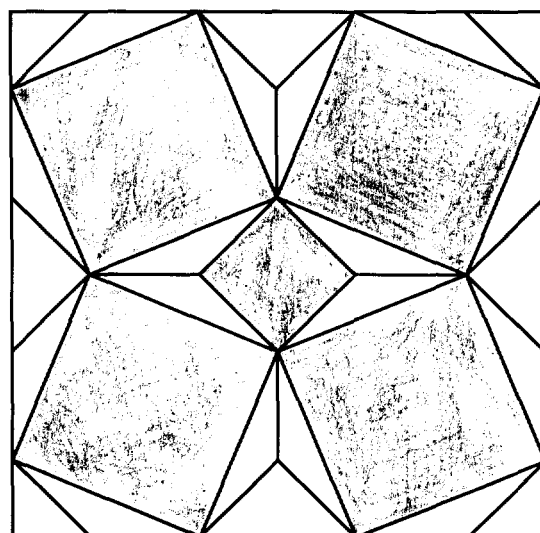
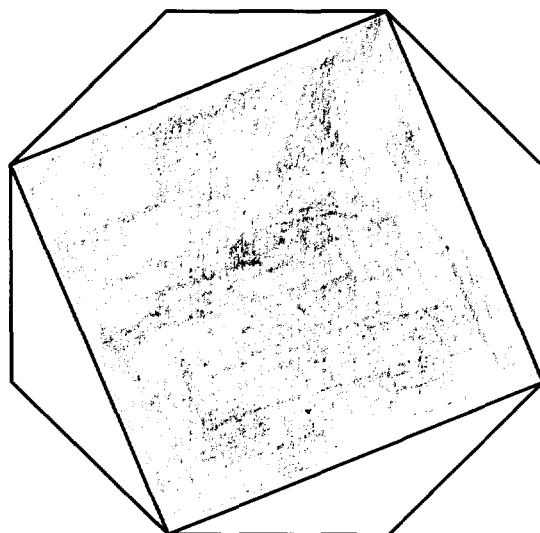
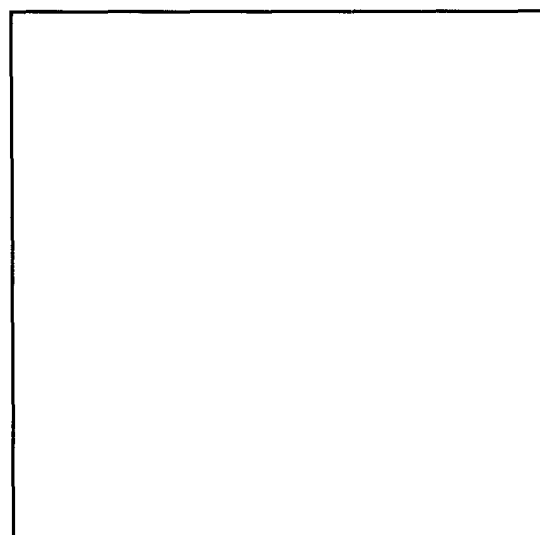
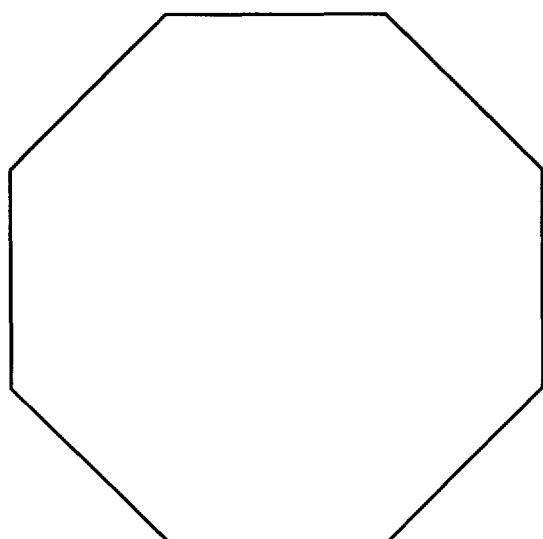
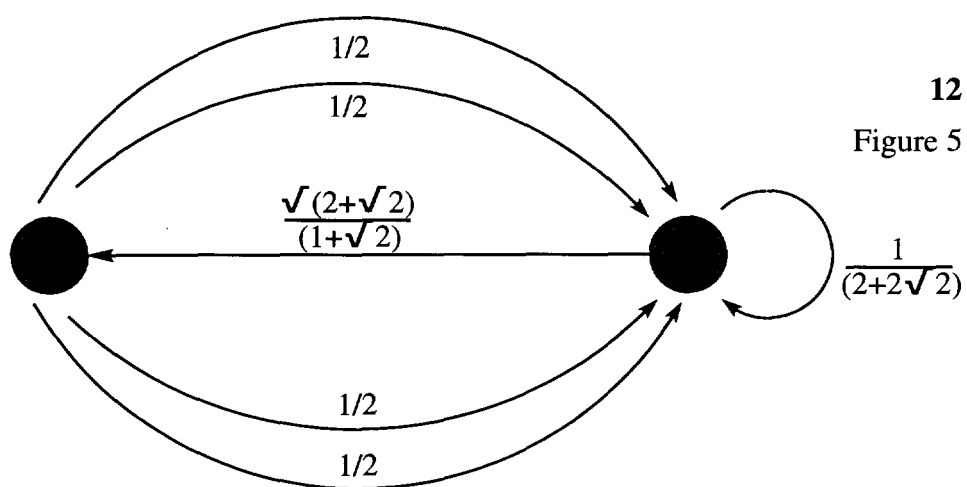


Figure 6

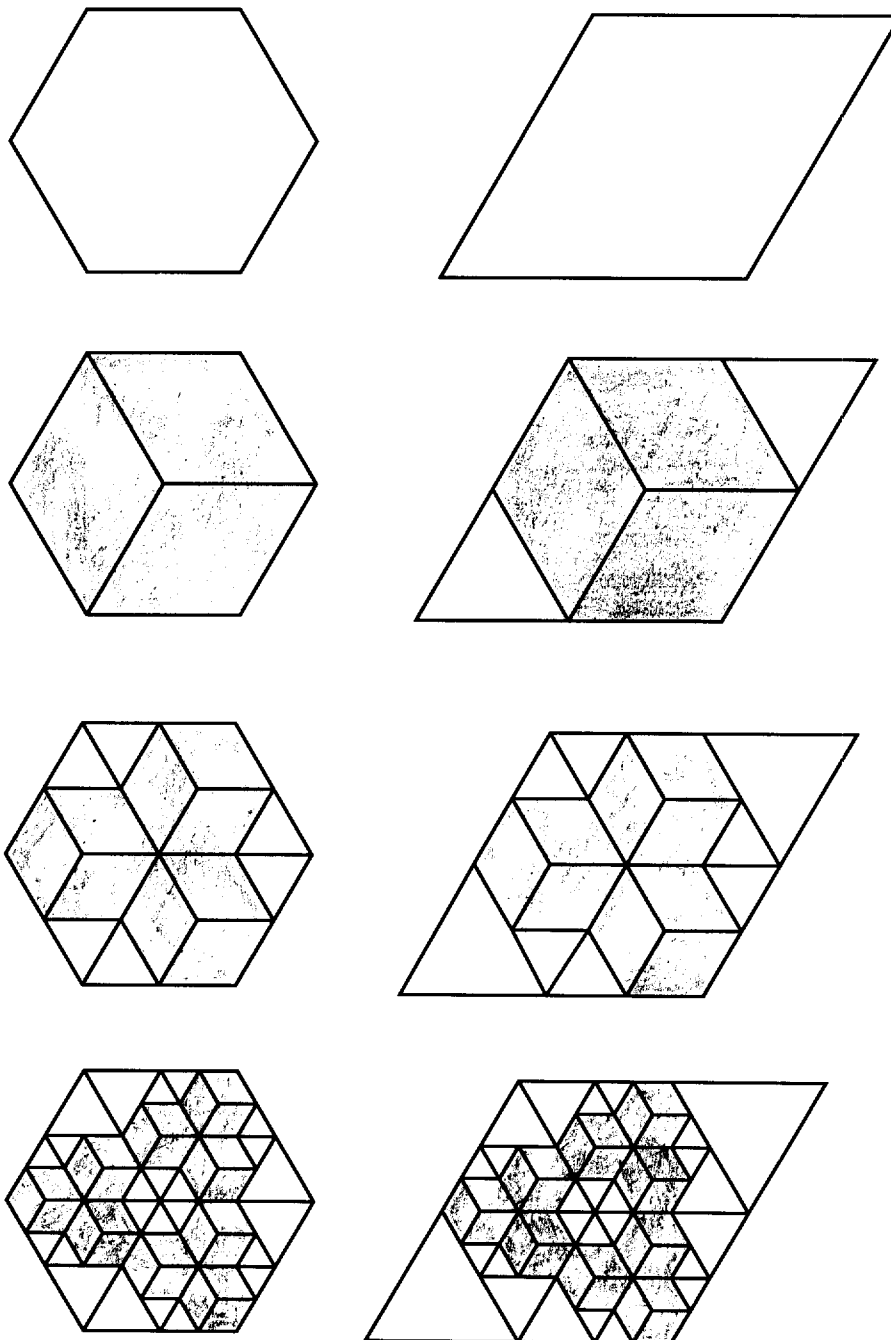
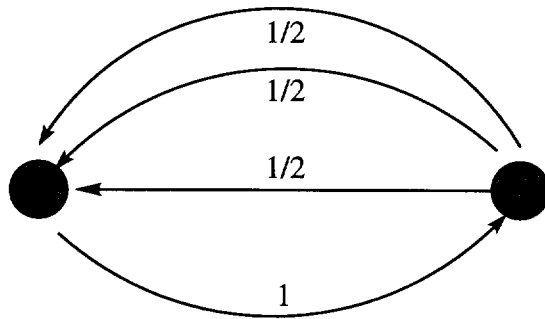
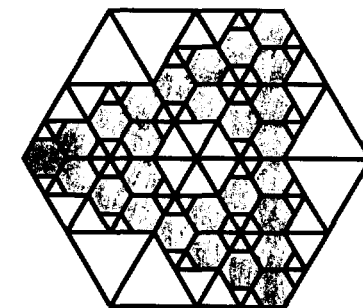
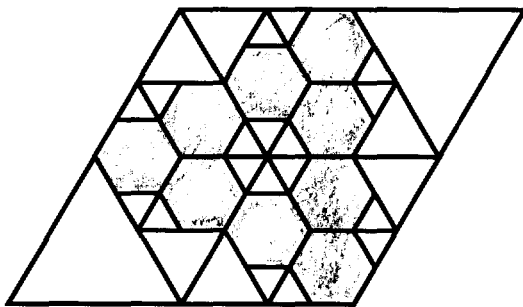
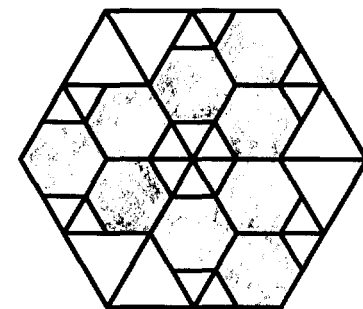
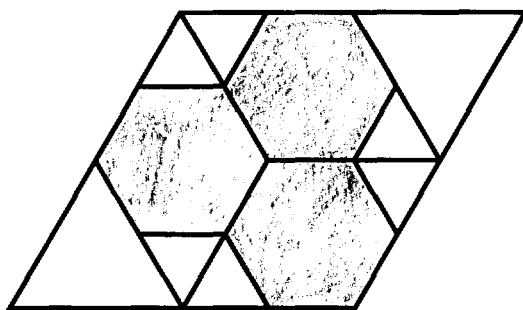
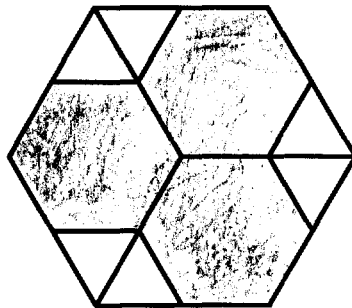
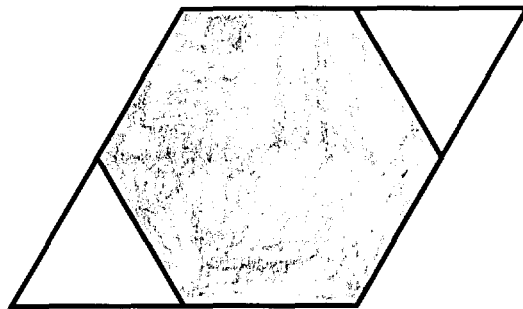
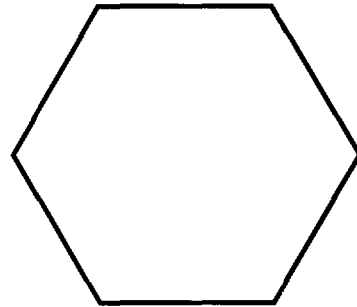
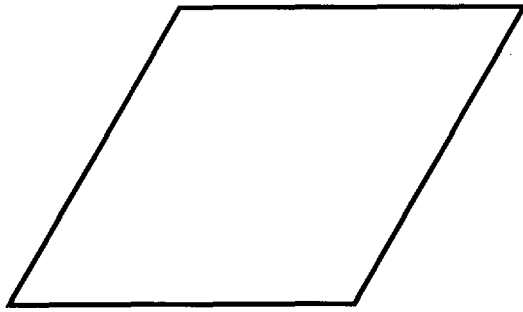
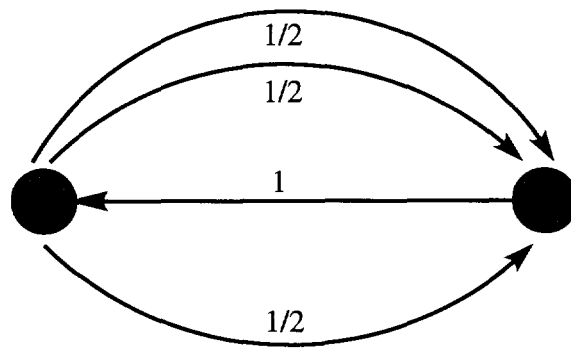


Figure 7



VIII. Existing Work

Below is a list of the existing work that pertains to graph fractals.

1. Edgar, G.A. and Mauldin, R. Daniel. "Multifractal Decompositions of Digraph Recursive Fractals." Proceedings of the London Mathematical Society. 1992. Vol 65(3):604-628.
2. Hu, Tian You and Lau, Ka Sing. "Fractal Dimensions and Singularities of the Weierstrass Type Functions." Transactions of the American Mathematical Society. American Mathematical Society, 1993. Vol 335(2):649-665.
3. Mauldin, R. Daniel and Williams, S. C. "Hausdorff Dimension in Graph Directed Constructions." Transactions of the American Mathematical Society. American Mathematical Society, Oct. 1988. Vol 309(2):811-829.
4. Spear, Donald W. "Measures and Self Similarity." Advances In Mathematics. Academic Press, Inc., 1992. Vol 91(2):143-157.
5. Von Haeseler. "Pascal's Triangle, Dynamic Systems and Attractors." Ergodic Theory and Dynamical Systems. University Press, 1992. Vol 12(3):479-486.

IX. References

[1] Edgar, Gerald A. Measure, Topology, and Fractal Geometry. Springer-Verlag, New York, 1990.

[2] Mandelbrot, Benoit B. The Fractal Geometry of Nature. W.H. Freeman and Co., New York, 1983.