

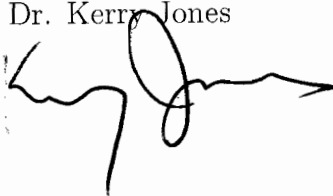
# Negatively Curved Groups and Their Applications

An Honors Thesis (Honors 499)

by

William Malone

Thesis Advisor  
Dr. Kerry Jones

A handwritten signature in black ink, appearing to read "Kerry Jones", with a large loop in the middle and a horizontal line extending to the right.

Ball State University  
Muncie, Indiana

May 2005

Expected Date of Graduation: May 7, 2005

2480  
.24  
2005  
A335

## Acknowledgements

I would like to first thank Dr. Kerry Jones for helping me with this thesis. He has proved an invaluable tool in helping me understand and comprehend the material. Also I would like to thank all the other professors that I have had throughout the years that have made my college career a success. Lastly, I would like to thank my roommates and my family for keeping me motivated to do homework and complete this thesis.

## Abstract

The goal of this thesis is to explore geometric group theory which is a fairly recent area of mathematics in which geometric techniques (specifically the theory of geodesic metric spaces) are applied to algebraic objects (specifically infinite finitely presented groups).

In the early 1980's Mikhail Gromov, James Cannon, and others developed methods analogous to the curvature methods in differential geometry, but applicable to a broader category of metric spaces. These techniques have subsequently proven valuable in understanding certain previously difficult groups by studying their Cayley Graphs (a geometric object derived from the group). We will define and describe these techniques and give some examples of their use.

## Introduction

The notion of a negatively curved group is at first highly non-intuitive because it links two areas of mathematics that are not usually associated with one another. Curvature is generally a property that we associate with curves and surfaces sitting inside  $\mathbb{R}^3$  while a group is an algebraic structure normally associated with abstract concepts like the integers or matrices. However, there is a way to define a group operation on paths sitting inside a geometric object called a manifold and from the curvature properties of these paths that make up the manifold we can determine whether or not the group itself should be negatively curved.

It should be mentioned that negatively curved groups are very interesting because of their algebraic properties and their applications in both computer science and art.

The main reason why these groups are interesting to mathematicians is that they make up the vast majority of all fundamental groups of three-manifolds. Now the reason that three manifolds are of interest is that current mathematical knowledge already knows as much as it ever will about manifolds of dimension five or greater, and everything worth knowing about manifolds of dimension two. This means that three and four manifolds are the only ones of considerable interest left. Another reason that mathematicians and others care about studying three manifolds is that this is the study of spaces that locally act like the world in which we live. In particular, it might be possible very soon to discern what the actual geometric shape of the universe is.

In addition to being able to help us study the world in which we live, negatively curved groups occur in everyday life in the fields of science and art. The first example of negatively curved groups shows up in art and specifically in terms of the perspective drawings done by Escher. Escher's Circle IV is a illustration

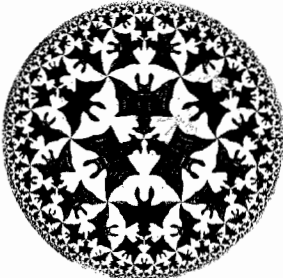


Figure 1: Escher's Circle IV  
look like.

of a Cayley graph of a negatively curved group. This is specifically a tiling of the hyperbolic plane with triangles when you consider the tips of the wings and where the feet come together to be the three vertices of the triangle and the lines connecting all these to be the edges.

This perspective that Escher uses gives us an idea of what the structure of a negatively curved group should

In addition to this use of perspective, negatively curved groups also show up in an

area of computer science studying finite state automata and their properties. These objects will be defined later in detail and we shall show how they relate to some of the examples of negatively curved groups that we come up with.

## Negatively Curved Groups and Their Applications

We would like to start the paper by building up the theory of negatively curved groups. To do this we will first carefully develop an intuitive notion of what a negatively curved group should be and then we will try to generalize this to a criterion that we can use to distinguish negatively curved groups from groups with non-negative curvature. Finally we will explain why these groups are of interest and what properties they have in detail. So to start we will review the concept of a group and negative curvature to see how they fit together in regards to three manifolds.

In order to start out we will define explicitly what a group is, as well as some other fundamental concepts that we will be using when we are making the rigorous definition of a negatively curved group to provide a common language that can be used throughout the paper.

The first series of definitions that we need will be a reminder of what exactly a group is and what properties it is required to have. A *set* is a collection of objects that needs not fulfill any further requirement. For example, {Bob, Fred, Sue} and {1,2,3,4,5} are both sets. Now we can define a binary operation on a set as follows:

**Definition 1.** *Given a set  $X$ , a binary operation on  $X$  is a function  $F$  from  $X \times X$  into  $X$ .*

Some examples of such a binary operation could be multiplication or addition if the integers (denoted  $\mathbb{Z}$ ) are our set, or function composition if the set we are working

with is made up entirely of functions. An example of an arithmetic operation that is not a binary operation on the integers would be division. Using this notion, we can form the definition of a group as follows:

**Definition 2.** A group  $\langle G, * \rangle$  is a set  $G$ , with a binary operation  $*$ , such that the following axioms are satisfied:

1. For all  $a, b, c \in G$ , we have  $(a * b) * c = a * (b * c)$
2. There is an element  $e$  in  $G$  such that for all  $x \in G$ ,  $e * x = x * e = x$ .
3. There is an element  $a'$  in  $G$  such that  $a * a' = a' * a = e$

[Fra67]

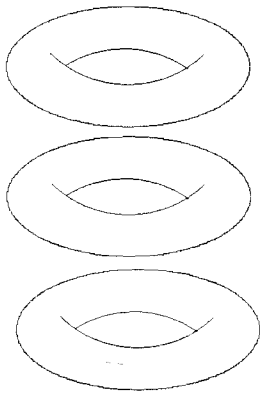
In addition, it is worth noting that not all groups are commutative ( $a * b = b * a$  for  $a, b \in G$ ), but those that are commutative are called *abelian* groups. Some common examples of groups are the integers under the binary operation of addition, and the set of all polynomials with rational coefficients under polynomial addition. However, the positive integers under addition is not a group because axioms 2 and 3 do not hold. It is important to note that for finite groups, a great deal is known due to a well defined machinery that has been built up, but for infinite groups we have much less machinery at our disposal which makes working with these groups much harder.

A specific type of group that is of particular interest in the study of negatively curved groups is the fundamental group of a  $n$ -manifold, which is a collection of points such that locally (within some  $\epsilon$ -neighborhood of the point) each neighborhood of every point acts like Euclidean  $n$ -space. So the definition of the fundamental group is:

**Definition 3.** The fundamental group of a path connected space  $X$  (space in which any two points lying in the space are connected by a path lying entirely in  $X$ ) at

a basepoint  $p$  is the group  $(G, *)$  where the set  $G$  is the set formed by the set of all equivalence classes of homotopically equivalent loops (two loops are homotopically equivalent if each can be continuously deformed into the other) in  $X$  starting and ending at  $p$  under the operation  $*$  of path concatenation, first running around one path and then around the other.

In the case of manifolds (which are a path connected space) the fundamental group at each basepoint will be the same throughout the whole manifold since we can connect any two points with a path and thus we can talk about the fundamental group of the manifold without confusion. Some examples of manifolds and their fundamental groups are the sphere which has trivial fundamental group (all loops can be contracted), and the torus (doughnut) which has fundamental group



$\mathbb{Z} \oplus \mathbb{Z}$ . Now the reason that the torus has this fundamental group is that any loop in the torus can be deformed into a loop that winds around the top of the doughnut  $m$  times denoted  $a^m$ , and a loop that winds through the doughnut hole  $n$  times denoted  $b^n$ . Now the path  $a$  and  $b$  will generate the fundamental group and they commute since the path  $aba^{-1}b^{-1}$  is contractible.

Thus if we multiply by  $b$  on the right and then by  $a$  on the right we get  $ab = ba$ . And since the group is infinite, abelian, and generated by two elements we can conclude that the fundamental group for the torus is  $\mathbb{Z} \oplus \mathbb{Z}$ .

In order to continue to develop our intuitive notion of a negatively curved group it is necessary to define what a surface is and how specifically to detect curvature in

a surface. In order to do this we will start out with the definition of the notion of a distance between points in a space.

**Definition 4.** *A set  $X$  is a metric space if with any two points  $p$  and  $q$  of  $X$  there is associated a number  $d(p,q)$  called the distance from  $p$  to  $q$  such that:*

- (1)  $d(p,q) > 0$  if  $p \neq q$  and  $d(p,p) = 0$
- (2)  $d(p,q) = d(q,p)$
- (3)  $d(p,q) \leq d(p,r) + d(r,q)$  for any  $r \in X$  [Rud76]

The distance function in a metric space is called a metric on the space it is acting on. A particular metric that we will be working with quite often is the path metric:

**Definition 5.** *The path metric  $d'$  on a metric space  $(X, d)$  is defined by setting  $d'(a, b)$  equal to the infimum (least upper bound) of lengths of rectifiable paths from  $a$  to  $b$ . If  $d = d'$  then we say that  $X$  is a path connected space.*

To go along with the definition of a metric space we can also define what it means for a certain set to be open in the metric space.

**Definition 6.** *Given a metric space  $X$ , a subset  $Y$  of  $X$  is considered open in  $X$  if for all  $p \in Y$  there exists an  $\epsilon > 0$  such that whenever  $d(p,q) < \epsilon$  then  $q \in Y$ . [Rud76]*

For example an open interval is open in  $\mathbb{R}$  because if you take any point in the interval you can explicitly find an  $\epsilon$  that will work. Specifically in this case choose  $\epsilon$  such that it is the minimum of the two distances from your point  $p$  to the two endpoints of the interval. A closed interval is not open in  $\mathbb{R}$  since for any  $\epsilon$  you pick at the endpoint  $a$  of the interval  $[a, b]$  you will always have a point  $p$  outside the interval such that  $d(a,p) < \epsilon$ . Now in order to create a surface we need to consider a function that takes an open set into Euclidean 3-space.



**Definition 7.** A coordinate patch  $x: D \rightarrow \mathbb{R}^3$  is a one-to-one regular mapping of an open set  $D$  of  $\mathbb{R}^2$  into  $\mathbb{R}^3$ . Such a patch is said to be a proper patch if the inverse function  $x^{-1}: x(D) \rightarrow D$  is continuous. [O’N66]

Now if we take several of these coordinate patches and make sure that their intersections overlap smoothly then we can create our surface.

**Definition 8.** A surface in  $\mathbb{R}^3$  is a subset  $M$  of  $\mathbb{R}^3$  such that for each point  $p$  of  $M$  there exists a proper patch in  $M$  whose image contains a neighborhood of  $p$  in  $M$ . [O’N66]

Given a point living in the surface we would like to be able to tell what kind of curvature the surface has at that point. In order to motivate such a notion of curvature we need to remember some concepts involving planar curves and the extension of these concepts into  $\mathbb{R}^3$ .

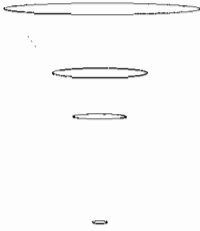
Given a planar curve, one way of thinking about the curvature at a point  $p$  on the curve is looking at the osculating circle at  $p$  where the osculating circle is the circle that best approximates the curve at the point  $p$ . One way to think about this is that three points will uniquely determine a circle in the plane thus take three points on the planar curve call them  $a, b$ , and  $c$ . The osculating circle will be the circle formed by letting  $a, b$ , and  $c$  limit to the point  $p$  along the planar curve. The curvature at  $p$  will be  $\frac{1}{r}$  where  $r$  is the radius of the osculating circle. Thus if the planar curve is a line the osculating circle has infinite radius and thus the curvature is zero which is consistent with the intuitive notion of curvature.

If we want to look at the curvature of a surface at a point  $p$  we can look at lines passing through  $p$  and try to apply a similar notion to the one that we used in the

plane to define curvature. If we are given a surface sitting inside  $\mathbb{R}^3$  (from calculus) we can find a normal vector to the surface at the point  $p$ . If we look at all possible planes that contain the normal vector and intersect these planes with the surface itself what we will get is a bunch of curves sitting inside the surface that pass through the point  $p$  and also live inside a plane. If we compute the curvature of all of these curves in their respective planes using the osculating circle (giving the curvature of those curves that have an osculating circle that opens in the direction that the normal vector is pointing a positive sign and the curvature of those curves in which the osculating circle opens away from the direction in which the normal vector is pointing a negative sign) then the Gaussian curvature of the point  $p$  will be the product  $a \times b$  where  $a$  is the minimum of the curvature of all the curves,  $b$  is the maximum of the curvature of all the curves, and the operation  $\times$  is normal multiplication of real numbers.

Using this as the basis for the definition of Gaussian curvature we can look at some notions that will help us visualize what such surfaces should look like. The first visualization of a surface  $S$  in  $\mathbb{R}^3$  that is negative is if a neighborhood of that point looks like a saddle. However, if the Gaussian curvature at a point sitting in  $S$  is positive then locally it looks like a dome, and if the Gaussian curvature is zero at a point in  $S$  then locally near that point the  $S$  looks flat like a table top. Another way to visualize the curvature of a surface involves making the surface shiny and considering the reflection that can be seen. If the reflection reverses both left and right, and up and down then the surface is positively curved. A good example of such a shiny surface with a point of positive curvature would be a spoon with the point sitting at the bottom of the bowl of the spoon. If the reflection does not

reverse any direction then it is negatively curved. A good example of this is if you were to look at a point sitting on the bell of a tuba, somewhat like the figure pictured to the left. Both of these visualizations should be able to help in imagining what a point of negative curvature looks like as opposed to non-negative curvature.



Combining the notion of curvature in  $\mathbb{R}^3$  along with the notion of the fundamental group an example of a group that should be negatively curved is the fundamental group of a set such that every point in that set has negative Gaussian curvature. It is important to note that such a set can not be formed in  $\mathbb{R}^3$ , but can be formed in  $\mathbb{R}^4$ .

In order to define what a negatively curved group is we need to escape from the restriction of the fundamental group and look at how a group could be negatively curved without associating it with a manifold of some sort. In order to do this we will construct a geometric object called the Cayley graph from the group presentation and then define a path metric on the Cayley graph so that we can look at specific curves lying in the graph and their properties within the path metric.

To start out it is necessary to develop what exactly it means to have a group presentation. The first step is to define what a generating set for a group is.

**Definition 9.** A set of elements  $A = \{a_i \mid i \in 1, 2, \dots, n \text{ for some integer } n\}$  is a generating set for a group  $G$  if the subgroup  $\{a_1^{j_1} * a_2^{j_2} * \dots * a_n^{j_m} \mid \forall j_k \in \mathbb{Z}\} = G$ .

For example, the set  $\{1, -1\}$  is a generating set for the integers. Now a particular kind of group in which the generators have infinite order (the number  $n$  such that

$a^n = I$  where  $I$  is the identity element in the group and  $a$  is a generator for the group) is called a free group:

**Definition 10.** *A free group  $G$  on a set of generators  $\{a_i\}$  is a group where the set consists of reduced words (words that never contain a generator and its inverse adjacent to each other) that can be formed using the generators and their inverses and the operation  $*$  consists of concatenation, followed by reduction (the process of repeatedly eliminating adjacent inverse generators).*

For example if the set of generators is  $a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}$  then some of the elements of the free group would be  $aabbaab^{-1}d^{-1}a^{-1}c^{-1}ad$ ,  $aaaaaaaaa$ , and  $d^{-1}c^{-1}dac^{-1}db^{-1}a^{-1}$ . It is important to note that not all groups are free. Some words in a group are trivial for reasons other than reduction. An example of this was the fundamental group of the torus which had the word  $aba^{-1}b^{-1} = e$ . Specifying these gives a nice shorthand for a group.

**Definition 11.** *A group with presentation, or a group presentation, is a free group on a set of generators along with a number of relations that specify certain sequences of letters are equivalent to the identity.*

For example, if we look at the free group on four generators together with the relations  $aa = 1, bb = 1$ , and  $db^{-1} = 1$  then the three words that we looked at earlier can be reduced as follows:

$$\begin{aligned} aabbaab^{-1}d^{-1}a^{-1}c^{-1}ad &= b^{-1}d^{-1}a^{-1}c^{-1}ad = b^{-1}b^{-1}bd^{-1}a^{-1}c^{-1}ad = a^{-1}c^{-1}ad \\ aaaaaaaaa &= a \\ d^{-1}c^{-1}dac^{-1}db^{-1}a^{-1} &= d^{-1}c^{-1}dac^{-1}a^{-1} \end{aligned}$$

Another example of a group presentation would be  $\{a, b \mid aba^{-1}b^{-1} = e\}$  which is a

presentation for  $\mathbb{Z} \oplus \mathbb{Z}$  which as we noted earlier was the fundamental group for the torus.

For the study of negatively curved groups we are interested in only those groups that have a finite presentation. Thus if a group has an infinite number of relations or generators we are not particularly interested in it. Now armed with the machinery of the group presentation we can construct a visual representation of what the group looks like via the Cayley graph.

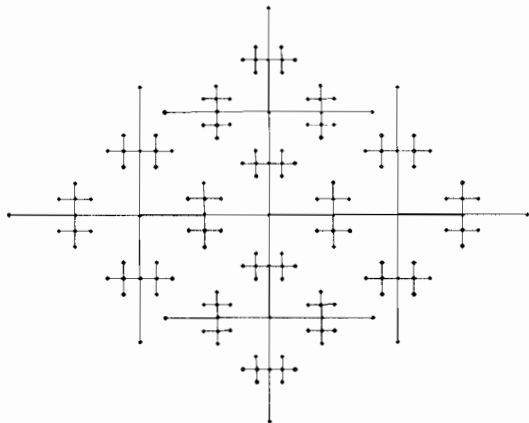
**Definition 12.** *Given a group presentation we can create a directed graph called a Cayley graph in the following manner. For a group  $G$  with a generating set  $S$  each element of the group is represented by a vertex in the graph and each directed edge from one group element to another in the group represents multiplication of the group element by a generator in the group presentation.*

In the Cayley graph there will be one edge coming out of a given vertex for each generator. Each of the relations in the group presentation can be seen as a closed loop in the Cayley graph. Some examples of Cayley graphs can be seen to the right.

The top two graphs are visual presentations for the group  $\mathbb{Z}$  with generating sets  $\{-1,1\}$  and  $\{-3,-2,2,3\}$  respectively. This shows that neither the Cayley graph, the generating set,

nor the group presentation is determined by the group. The third graph is a small part of the presentation for the free group on two generators in which each vertex has four neighbors. From this graph we can clearly see that there the only relations in the group come from backtracking along

an entire path because there are no closed loops present in the graph.



Now that we have created a Cayley graph from a given group presentation we would like to define a specific metric on the graph so that we can talk about distances between group elements and

lengths of paths in the Cayley graph.

**Definition 13.** *Let  $G$  be a group and  $S = \{g_i\}$  be a generating set for  $G$ . Then, we may define a metric on  $G$  by setting  $d(p, q) =$  the minimum length of a word in the generators of  $G$  and their inverses which represents  $pq^{-1}$ . The metric is called the word metric on  $G$  with respect to  $S$ .*

Using the word metric we can define a metric on the Cayley graph as follows: We will define the distance between two points in the Cayley graph to be the minimum number of segments needed to get from one point to the other. Inside the Cayley graph there can be many different paths from one point to another, but the paths of shortest length will be of particular interest in defining negative curvature.

**Definition 14.** *A curve sitting inside a path connected space  $X$  from a point  $p$  to a point  $q$  is considered to be a geodesic curve if it is a curve of the shortest possible length between two points.*

It is important to note that not all metric spaces have geodesics. The ones that do are called geodesic metric spaces and these are the metric spaces that are of particular interest in the study of negatively curved groups. We are interested in geodesic curves that live inside of a specific kind of metric space called a geometry. The definition of a geometry is somewhat technical, but this condition is satisfied by a wide variety of metric spaces including manifolds and Cayley graphs of finitely presented groups.

**Definition 15.** *A geometry is a metric space  $M$  with path metric  $m$  such that metric balls have compact closure. A group  $G$  acts geometrically on a geometry  $(M, m)$  if it acts by isometry (if the group acts in a one to one and onto fashion while at the same time preserving distance) and if its action is cocompact and properly discontinuous.*

*(1.) Proper discontinuity: for each compact set  $K$  in  $M$ , the set  $\{g \in G \mid \emptyset \neq K \cap gK\}$  is finite*

*(2.) Cocompactness: the orbit space  $M/G$  of the action is compact [Can91]*

A vastly simplified version of the definitions of proper discontinuity and cocompactness is that there is not too much group acting on the space and that there is not too much space for the group to act on, respectively. It is important to note that one of the motivations for this definition is that a finitely presented group acts geometrically on any of its Cayley graphs. These concepts allow us to fit certain groups to geometries in a way that important properties will be preserved.

Another important concept that we will need that works with the idea of a group acting geometrically on a geometry is the idea of two spaces being quasi-isometric. Generally speaking two groups will be quasi-isometric if they are similar in the large.

**Definition 16.** *Let  $X$  and  $Y$  be two metric spaces with metrics  $d$  and  $d'$  respectively.*

A map  $f: X \rightarrow Y$  is a quasi-isometry if there exist constants  $A, B, C$ , and  $D$  such that

$$A d(x, y) + B \leq d'(f(x), f(y)) \leq C d(x, y) + D$$

for all  $x, y \in X$ . The spaces  $X$  and  $Y$  are quasi-isometric if there exists a quasi-isometry  $f: X \rightarrow Y$  and a constant  $E \geq 0$  such that  $d'(f(X), y') \leq E$  for all  $y' \in Y$ .

[GH91]

A specific example of two spaces that are quasi-isometric are any metric space with finite diameter (maximum distance between any two points in the space) and a point. Given that the diameter of the space  $X$  is 15, then you can choose the constants  $A = 1, B = -16, C = 1, D = 0, E = 1$  with the result that  $d(x, y) - 16 \leq d'(f(x), f(y)) \leq d(x, y)$  and  $d'(f(X), y') = d'(y', y') = 0 \leq E = 1$ . Now we can relate the important concepts of a group acting geometrically on a geometry and two metric spaces being quasi-isometric in the following manner.

**Theorem 17.** *If a group  $G$  acts geometrically on two geometries  $X$  and  $Y$ , then  $X$  and  $Y$  are quasi-isometric. [Can91]*

In particular this theorem motivates the claim made earlier that a group  $G$  acts geometrically on any of its Cayley graphs. Now we will state the definition of negative curvature and see how it relates to groups and Cayley graphs.

**Definition 18.** *A geometry  $(M, m)$  has thin triangles if each geodesic triangle (a triangle where each of the sides is a geodesic in the space) in  $M$  satisfies the following condition. Given that the triangle has sides  $s_1, s_2$ , and  $s_3$  each point of  $s_1$  lies within*

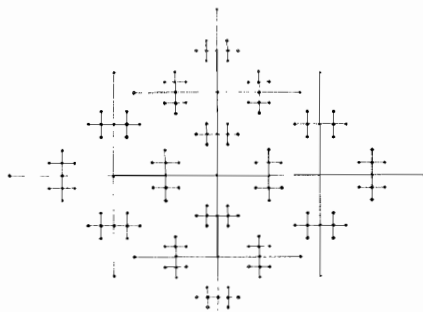


a distance  $C$  of the union  $s_2 \cup s_3$ . A group  $G$  is negatively curved if it has a finite generating set such that the associated Cayley graph has thin triangles. [Can91]

**Theorem.** Let  $X, X'$  be two geodesic metric spaces which are quasi-isometric. The space  $X$  is hyperbolic (negatively curved) if and only if  $X'$  is hyperbolic. [GH91]

An interesting problem arises from the definition of a group presentation and all the subsequent definitions that we made that relied on the group presentation. Can it be possible for the Cayley graph that arises from one group presentation to have thin triangles if another Cayley graph for the group does not? The answer to the question is no and the reason follows directly from the above theorem by the following reasoning. Since  $G$  acts geometrically on each of its Cayley graphs they are all quasi-isometric to each other and thus if one Cayley graph is negatively curved then all the Cayley graphs are negatively curved. Likewise, if one of the Cayley graphs is not negatively curved then all the Cayley graphs are not negatively curved.

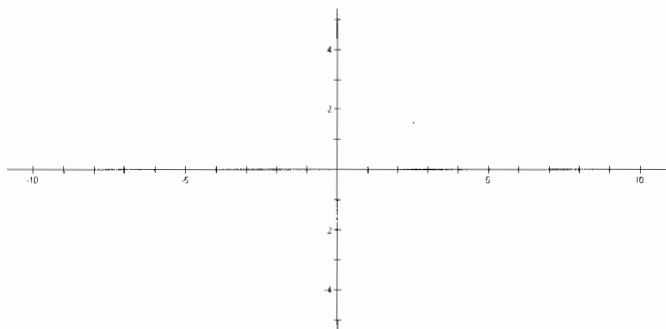
Utilizing the definition for negative curvature let us look at several examples to see some explicit examples of negatively curved groups as well as groups that are not negatively curved.



The first example that we will look at is the free group on two generators. The first thing that we need to do is construct the Cayley graph which has the presentation  $\{a, b \mid \text{no rela-}$

tions}. Every element  $q$  of the group will have four neighbors each can be written explicitly as  $qa, qb, qb^{-1}, qa^{-1}$ . A picture can be seen to the left.

It is important to note that with this example any path that does not contain trivial words elements such as  $aa^{-1}$  is a geodesic because it is the unique path from one vertex to another. Thus if we look at any geodesic triangle in the Cayley graph with sides  $s_1, s_2$ , and  $s_3$ , note that  $s_1 = s_2 \cup s_3$  and thus since they are equal the constant  $C$  by which the distance from  $s_1$  and  $s_2 \cup s_3$  is  $C = 0$  (it could be possible that we would have to relabel the sides first). And hence since we can bound the distance the free group on two generators is negatively curved.

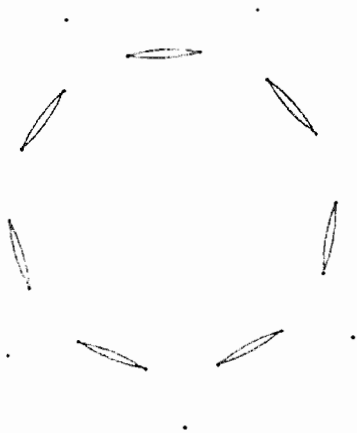


The second group that we will look at is the abelian group on two generators (fundamental group of the torus) which can be denoted by the group presentation  $\{x, y, x^{-1}, y^{-1} \mid xy =$

$yx\}$ . The Cayley graph for this group presentation is the familiar grid that is used to graph functions in the plane. Say that there exists a constant  $C$  such that it bounds all geodesic triangles in the Cayley graph. Create a geodesic triangle in this space where the geodesics are exactly the stairstep paths that best approximate the straight lines between two points such that the distance between  $S_1$  and  $S_1 = S_2 \cup S_3$  is less than  $C$ . It is important to note that geodesics are not unique in this group. For example, you can go from  $e$  to  $a^2b^2$  by six different geodesic paths:  $aabb, abab, baab, abba, baba,$

and  $baaa$ . In general the number of geodesic paths from  $e$  to  $a^m b^n$  is  $C(|m| + |n|, |n|)$  ( $C(x, y)$  denotes the number of ways to choose  $y$  number of things from  $x$  things), so  $C(4, 2) = 6$  in this case. One such geodesic triangle could be found by letting  $S_1$  be the straight line path connecting  $(2C, 2C)$  and  $(2C, -2C)$ ,  $S_2$  be the path from  $(2C, 2C)$  to the origin in which you first go along a path of length one in the  $y$ -direction toward the point  $(2C, 2C)$  and then go along a path of length one in the  $x$ -direction toward the point  $(2C, 2C)$ , and let  $S_3$  be a similarly defined path from the origin to the point  $(2C, -2C)$ . In this triangle the maximum distance from  $S_1$  to  $S_2 \cup S_3$  is greater than the maximum Euclidean distance between the best lines approximating the two sets which is  $\frac{2C}{\sqrt{2}}$  which is greater than  $C$ . Thus the abelian group on two generators is not negatively curved.

The final example that we will look at is the 2-3-7 triangle group. One group presentation for this group is  $\{x, y | x^2 = 1, y^3 = 1, \text{ and } (xy)^7 = 1\}$ . One circle of the Cayley graph for this group is pictured to the left where the dotted lines represent multiplication by  $y$  and the solid lines represent multiplication by  $x$ .



A geodesic in this group is a reduced word that contains no instances of the trivial words  $xx$ ,  $yyy$ ,  $yy$ ,  $y^{-1}y^{-1}$ ,  $x^{-1}x^{-1}$ ,  $y^{-1}y^{-1}y^{-1}$ , or the non-trivial words  $(xy)^4$  and  $(x^{-1}y^{-1})^4$ . The reason for the inclusion of

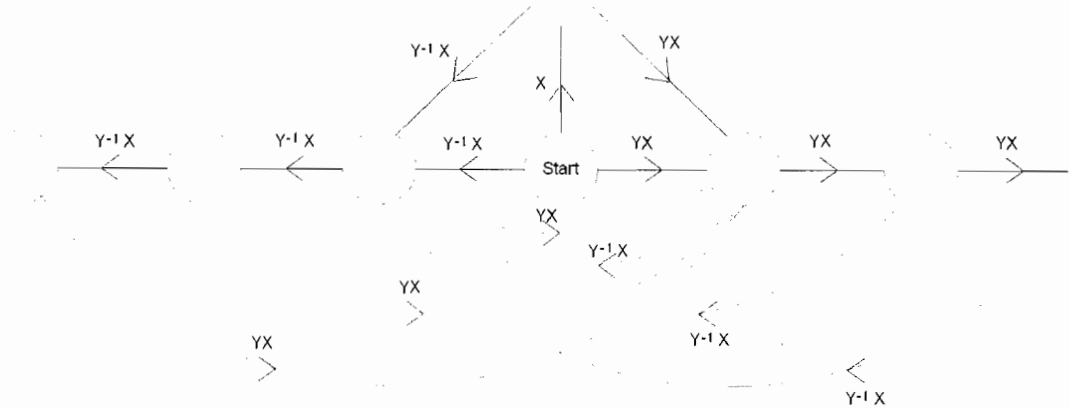
the non-trivial words is that if that was the path taken by the supposed geodesic curve contained one of the non-trivial words that represents going more than halfway around a circle like the one pictured to the left then going around the other side of the circle would have resulted in a shorter path. Now we would like to show that the 2-3-7 triangle group is negatively curved, thus we need to check if the Cayley graph has thin triangles. So take two geodesic paths in the Cayley graph represented by words  $A$  and  $B$  such that  $A$  connects the group elements  $a$  and  $b$ , and  $B$  connects the group elements  $b$  and  $c$ . It is possible for  $AB$  to not be a geodesic path because words that were not previously allowed could be introduced into  $AB$  where the end of  $A$  is connected to the beginning of  $B$ . Reduce any trivial words found in the connection. Also reduce any non-trivial words that meet the criteria mentioned above. This process will terminate at some point leaving a word that shall be denoted  $C$ . Let  $c_1$  denote the word that has the following two properties. First  $c_1$  starts at the first place where  $A$  and  $C$  are not identical. Second  $c_1$  ends at the first place when the ends of the words  $B$  and  $C$  differ. The maximum length of  $c_1$  is six because the largest replacement that we could have been forced to make was replacing the word  $(xy)^4$  or  $(x^{-1}y^{-1})^4$  by  $(y^{-1}x^{-1})^3$  or  $(yx)^3$  respectively which each have length 6. Now the distance between  $C$  and  $AB$  is zero for all points along the path  $C$  which are not in  $c_1$  and for all points inside  $c_1$  we are a maximum of three away from a part of  $C$  that is identical to  $AB$ . Thus for any geodesic triangle in the Cayley graph the distance between the union of any two sides and the third is bounded above by the constant three. And thus the 2-3-7 triangle group is negatively curved because the Cayley graph has thin triangles.

Now that we have seen some examples of negatively curved groups the question arises of what exactly they are good for, and why are they so important. Part of the answer is the applications that these groups have as the fundamental group of three-manifolds that are negatively curved. What was not mentioned is the prevalence of both negatively curved manifolds and negatively curved groups. According to Gromov, a finitely presented group is negatively curved with probability one, and out of the three manifolds the negatively curved ones are much more prevalent than non-negatively curved groups [Gro87]. Thus in learning more about negatively curved groups and their properties, we expand our knowledge about some of the areas of mathematics that were previously not accessible just from the topological point of view.

In groups that are given by presentations, each element is represented by at least one word in the generators. However, many words may represent the same element. It would be nice to be able to decide algorithmically precisely when two words represent the same element in a group given by a presentation. Another way to state this is that the *word problem* consists of finding an algorithm that given a word in the group can answer definitively yes or no, if the word represents the identity in the group. It is important to note that not all groups have this property. In order to construct the definition of an automatic group we must first construct some simple algorithms called finite state automata.

**Definition 19.** *A finite state automaton is a quintuple  $(S, A, \mu, Y, s_0)$  where  $S$  is a finite set, sometimes called the set of states,  $A$  is a finite set called the alphabet,  $\mu : S \times A \rightarrow S$  is a function called the transition function,  $Y$  is a subset of  $S$  called*

the subset of accept states, and  $s_0 \in S$  is called the initial state. [ECH<sup>+</sup> 92]



An example of this kind of automaton that will either accept a word or reject it for the 2-3-7 triangle group  $\{X, Y \mid XX = 1, YYY = 1, \text{ and } (XY)^7 = 1\}$  is pictured above. If the word ends in any circle or in the middle of a line then it is in an accept state and hence the word is accepted. The words that the automaton is excluding are any word containing the words,  $X^{-1}$ ,  $XX$ ,  $YY$ ,  $YY^{-1}$ ,  $Y^{-1}Y$ ,  $Y^{-1}Y^{-1}$ ,  $Y^{-1}XY^{-1}XY^{-1}XY^{-1}$ , and  $YXYXYXY$ . Another way to say this is that this automaton accepts the words described earlier as geodesics, after they have had all  $X^{-1}$  replaced by  $X$ , and rejects all others. Now the question arises of what can be accomplished with this automaton that accepts words or rejects them. One thing that can be accomplished with this automaton is to create an automatic structure on the group  $G$ .

**Definition 20.** Let  $G$  be a group. An automatic structure on  $G$  consists of a set  $A$  of semigroup generators of  $G$ , a finite state automaton  $W$  over  $A$ , and finite state automata  $M_x$  over  $(A, A)$ , for  $x \in A \cup \{\epsilon\}$ , satisfying the following conditions:

- (1.) The map  $\pi : L(W) \rightarrow G$  is surjective.
- (2.) For  $x \in A \cup \{\epsilon\}$ , we have  $(w_1, w_2) \in L(M_x)$  if and only if  $\overline{w_1x} = \overline{w_2}$  and both

$w_1$  and  $w_2$  are elements of  $L(W)$ . [ECH<sup>+</sup>92]

In this definition  $\epsilon$  is a padding symbol added to the end of a word to make it longer without altering the group element it represents, and  $\bar{w}$  means the group element represented by the word  $w$ .

Using the automaton that we already created in conjunction with five new automaton we can create an automatic structure on the 2-3-7 triangle group. The first automaton takes as its input two words and answers yes if the words are the same and no if the words are different (denoted  $M_\epsilon$ ). The other four automata  $M_x$ ,  $M_y$ ,  $M_{x^{-1}}$ , and  $M_{y^{-1}}$  all will answer yes if the second word represents the element in the group obtained by multiplying the first word by the subscript generator. For example, the  $M_x$  and  $M_{x^{-1}}$  automata are identical and answer yes if the two words differ only by either one having one extra  $x$  on the end. We know that the 2-3-7 triangle group is negatively curved and we have created an automatic structure on the group, a natural question to ask is whether all negatively curved groups have an automatic structure that can be defined on them.

**Theorem.** *If  $G$  is a negatively curved group, and if  $\Gamma$  is the Cayley graph for  $G$  based on a finite generating set for  $G$ , then the shortest-representative finite state automaton (an automaton in which all the words are the shortest representatives of group elements) defines an automatic structure on  $G$ .*

[Can91]

There are several nice things about automatic groups. The first is that any group that is automatic has a solvable word problem as we have already observed. The second is that all negatively curved groups are automatic groups as seen in the above

theorem. The third and real advantage of an automatic group is that “the entire global structure can be captured in a small family of finite state automata” [Can91]. This means that from an algebraic point of view negatively curved groups are especially nice because it gives us a way of attacking certain topological problems in regards to fundamental groups of negatively curved manifolds previously unassailable from the topological point of view by using algebraic methods to tackle these problems and make them vastly more manageable.

The initial reason that we began looking at negatively curved groups was to discover if we could discern a criteria that linked the notion of curvature from differential geometry with the algebraic concept of a group. We first developed an intuitive idea of what an example of such a group should be by looking at the fundamental group of a totally negatively curved manifold. Just looking at the fundamental groups of negatively curved manifolds does not provide the generality needed to adequately define negative curvature in groups. To do this it was necessary to define a structure that could be obtained from the group that was quasi-isometric to the group itself in order to preserve the negative curvature of the structure. The structure that worked for our purposes is a structure that is called a Cayley graph.

The Cayley graph was perfect for our purposes because a group acts geometrically on its Cayley graph, and thus under any two group presentations the resulting Cayley graphs are quasi-isometric and hence curvature is independent of group presentation for a finitely presented group. Finally we were able to define a finitely presented group to be negatively curved if its Cayley graph has thin triangles, and we have a condition that equates the notion of a group with the notion of being negatively



curved in the full generality that we desired.

Utilizing this definition we can find whole classes of groups that are negatively curved, and observe that they have some particularly nice algebraic properties, such as being automatic and hence having a solvable word problem. The reason that these groups are interesting to study is their application in regards to three-manifolds, which are of considerable interest to topologists. The hope is that more information can be gleaned about negatively curved manifolds, which are far and away the most prevalent, by looking at the algebraic properties associated with the fundamental groups of these manifolds, which are negatively curved. This will hopefully provide an adequate mechanism for discovering many of the interesting properties about three and four manifolds which are very hard to discern any other way.

## References

- [Can91] James W. Cannon, *The theory of negatively curved spaces and groups*, Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989), Oxford Sci. Publ., Oxford Univ. Press, New York, 1991, pp. 315–369. MR MR1130181
- [ECH<sup>+</sup>92] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. MR MR1161694 (93i:20036)
- [Fra67] John B. Fraleigh, *A first course in abstract algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1967. MR MR0225619 (37 #1212)
- [Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR MR919829 (89e:20070)
- [O’N66] Barrett O’Neill, *Elementary differential geometry*, Academic Press, New York, 1966. MR MR0203595 (34 #3444)
- [Rud76] Walter Rudin, *Principles of mathematical analysis*, third ed., McGraw-Hill Book Co., New York, 1976, International Series in Pure and Applied Mathematics. MR MR0385023 (52 #5893)

- [GH91] Etienne Ghys and Pierre de la Harpe, *Infinite groups as geometric objects (after Gromov)*, Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989), Oxford Sci. Publ., Oxford Univ. Press, New York, 1991, pp. 299–314.