Guitars Are Non-Linear!

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by

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Abstract:

Relatively recently in the world of mathematics education, there has been a push to connect concepts students are learning to areas outside of mathematics. It seems sensible, however, that these areas of application should be topics in which students would find interest. One such topic area that turns out to lend itself well to discussions of several aspects of the mathematical idea of non-linearity is guitars. This paper discusses the ways in which the guitar can be used to examine those applications of non-linearity. Ideas on how to implement those applications into a secondary mathematics classroom are also included.

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Guitars Are Non-Linear!

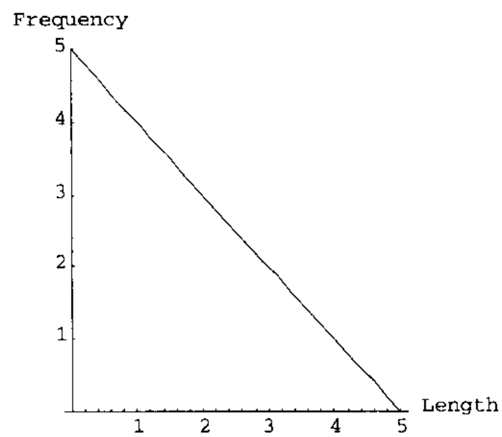
The Ideas

Non-linear functions are functions whose graphs are not straight lines. These functions include circles, parabolas, hyperbolas, quadratics, exponentials, trigonometric functions, and many other functions. Non-linear functions tend to be a bit more complex in general than linear equations, which makes them more interesting as well. Their complexity points to another fact about non-linear functions; the real world is non-linear. The straight line is a very specialized case of a function, and very few real life applications are truly linear. So, although it is easier to start a discussion of functions and relationships between variables by examining linear cases, it is of utmost importance that much emphasis be placed on non-linear functions and that all different sorts of non-linear functions be examined in-depth.

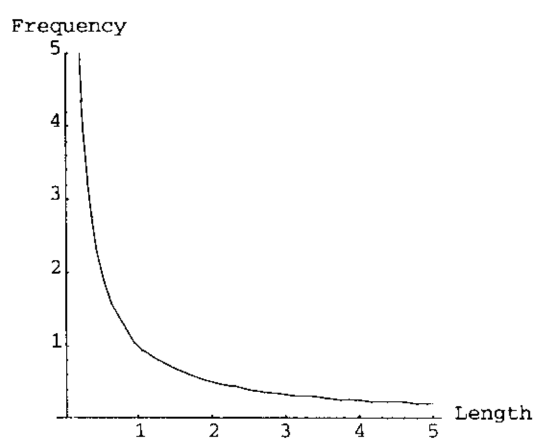
Applications of non-linearity are found in almost all aspects of the real world. They even show up in subjects to which one would not typically think that mathematics could apply. For instance, there are several interesting applications of non-linearity to be found in examining the standard acoustic guitar. Though most think of music and mathematics as totally separate realms, there are some interesting ideas about guitars, specifically their shape and the way they produce sounds, that automatically lead to discussions involving non-linearity.

One interesting example of an application of non-linearity found in discussing the working of guitars is the relationship between the vibration frequency of a taut string (at a constant tension) and the length of that string. In the case of a guitar, the length of the string can be considered the length of the portion of the string that vibrates when plucked. For instance, this would be the entire length of the string when the string is plucked 'open', or without being held down on any fret by the guitarist. The length of the string is essentially shortened when it is held down on a fret, because the string only vibrates between the lower bridge and the fret where it is held down. It is easy to recognize that there is a relationship between the frequency of the string and the length of the string, because that is how a guitar is played. Vibration frequency is directly related to the pitch heard. When the frequency at which a string is vibrating goes up, so does the pitch. When the frequency decreases, so does the pitch. When playing a guitar, a guitarist changes his hand position and the location of his fingers, which changes the lengths of the different strings, thereby creating different frequencies (and thus different pitches).

But how exactly can the relationship between frequency and length be expressed? It would probably seem at to a layman at first glance that the relationship would be linear, because that is the easiest way to view relationships between variables. This would mean, however, that if a string were shortened by any proportion, the frequency would increase by the same proportion. The relationship between the length and frequency would then be expressed mathematically by the linear equation $F = n \times L$ where F is the frequency, L is the length, and n is some constant of proportionality relating the two variables. (In this case, n would necessarily be negative because of the simple fact that as string length goes down pitch goes up.) The graph of a linear relationship of this sort would have the same form as the following.



This linear relationship does not hold, however. It is actually true in the case of strings that there is a non-linear function that provides a much better model for the relationship between frequency and length. This can best be seen through a graphical representation.



This graph depicting the actual relationship between frequency and string length shows a definitive non-linear relationship, but the picture itself still does not tell exactly what that relationship is. One who has had experience with differing types of non-linear functions and their corresponding graphs might be able to postulate a fairly educated hypothesis about the nature of the graph, and therefore the function relating the two variables. One specific bit of data that should be noted is that as the length of the string is halved, the frequency doubles, which means the pitch goes up by an entire octave. This alludes to an inverse relationship between the variables. This dependence can be expressed mathematically as

$$Frequency = \frac{K}{StringLength}$$

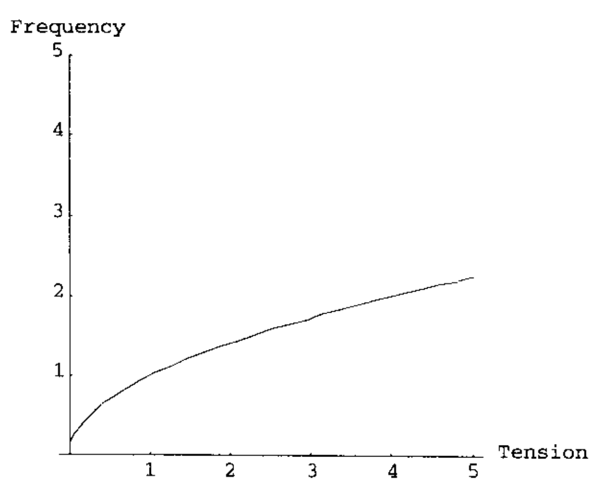
where K is a constant equal to the frequency of the plucked open string multiplied by the length of the open string. For instance, if an open "A" string (frequency equal to 440 Hz) is 2 feet long, and the "E" a fifth above the "A" was played (app.660 Hz) on the same string, the new length would have to be 1.33 feet, which is 2/3 as long as the open string.

Another noteworthy thing about the length-frequency relation is that the function has a restricted domain, which is to say that neither the length nor the frequency can exceed certain limits. What type of limits would be expected to be found intrinsic to this problem if looked at from the real-world case? If the guitar were actually held and played, what could be inferred about how high or low the frequencies could be, or how long or short the string could be allowed to be? It becomes apparent after playing with a guitar for some small time that for a given string, there is a lowest note and a longest length. A string cannot be made any longer (at least through normal conventions) than it is when it is played open. The frequency will also be the lowest it can be when the string is played open. However, what is the highest possible note and the shortest possible string? In theory, the string could get infinitely shorter (and the frequency infinitely higher) by the guitarist's finger falling closer and closer to the lower bridge. However, there are some facts that exist in the real case that cannot be ignored. Firstly, the short length of a guitar's fret board relative to the length of the strings does not allow for all the possible frequencies a string could play. If the fret board is ignored though (which is reasonable since it is possible to still play notes beyond the fret board), there are still a few major hitches. One is that a guitarist's fingers would not really be able to approach the bridge in infinitely small increments because the guitarist would not have that precision and his fingers would be too large. Using the previously addressed fact that as string length is halved the pitch goes up an octave, it becomes apparent that as the guitarist's fingers approach the bridge and higher and higher frequencies, entire octaves lie within a span no wider than a fingertip. Another reason the guitarist's fingers cannot approach the bridge infinitely is because there needs to be enough room between the guitarist's finger and the bridge to pluck the string. Yet another problem is that the range of frequencies to which the human ear is sensitive is not infinite. If the ear is the device by which the frequencies are detected, there is a point at which frequencies can no longer be detected. Most guitarists, unless playing some twentieth century piece by the likes of John Cage, would hardly want to play notes that no one could even hear.

The relationship between frequency and length is only one of the relationships related to guitar playing that is non-linear in nature. It can be found that there also exists a relationship between the tension of a string (of a given length) and its frequency. As a string is tightened, the pitch goes up. That is a basic fact that can be observed easily, and that is also used every time someone tunes up a guitar. How much does the pitch change for a certain change in tension, though? And is the amount of pitch change dependent on how tight the string is to begin with? As anyone who has ever strung and then tuned a guitar would have observed, a loose string when first being tightened changes pitch rapidly to approach the range the string is meant to be played in. But as the pitch gets close to the pitch it is meant for, the pitch changes much more slowly. This relationship can easily be observed to be non-linear by plucking and tightening the string simultaneously. If the relationship were linear, the rate at which the pitch would increase would stay constant as the rate of increase in tension remained constant. However, in reality, as the tension increases at a constant rate, the pitch changes much more dramatically at low tensions than at high. The actual non-linear function that describes the relationship between frequency and tension, assuming constant length and string mass per unit length, is

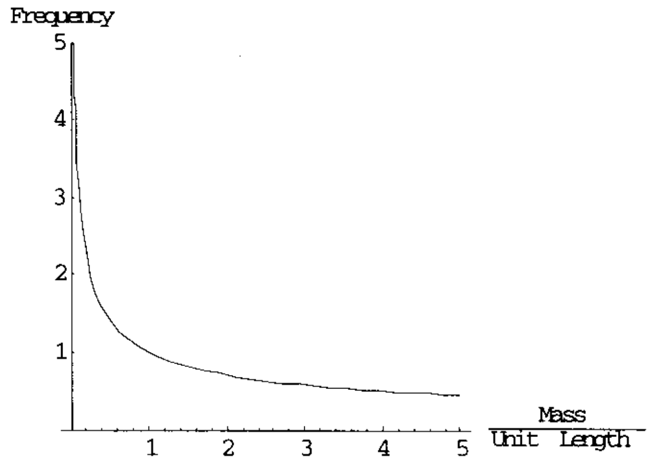
$$Frequency = \left(\frac{1}{2}Length\right)\left(\sqrt{\frac{Tension}{m.p.l.}}\right)$$

where 'm.p.l.' refers to the mass per unit length (Hall, pg. 673). A graph of this function has the same form as the following.



This graph has a much different shape than the graph of frequency as a function of string length. The reason for this specific non-linear shape in the second graph is the fact that the frequency changes as the square root of the tension. This means, as can be noted on the graph, that for frequency to double, tension must quadruple; and for the frequency to triple, the tension must increase by a factor of nine; and so on.

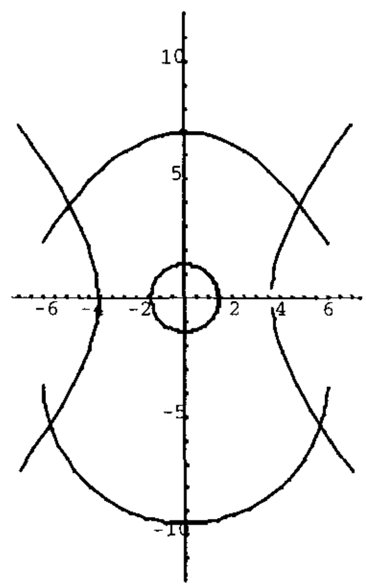
Using the same formula as in the last case, it should be noted that the frequency could also be changed by leaving the string length and tension constant, and changing instead the string's mass per unit length. This is not such an easy task with one string, but observations can be made of several strings of the same length with equal tensions, but each with a different mass per unit length (as is the case with a standard set of guitar strings). A graphical representation can be produced strictly through observations and plotting data points. However, the shape of the function can also be examined through plotting the function itself. According to the function, the frequency will vary according to the square root of the inverse of the m.p.l. For instance, if the m.p.l. is quadrupled, the result will be a decrease in frequency by a factor of two. For the frequency to double, the m.p.l. would have to decrease by a factor of four. The relationship between these two variables is represented in the following graph.



One interesting thing to note about this function and its graph is that it is, in a sense, a mix of the previous two types of functions. In this function the frequency changes as the square root of the inverse of the m.p.l., so that relationship could be looked at as an inverse relationship (like the frequency-length relation) and also as a square root relationship (like the frequency-tension relation). This fact is also reflected in the graphs as they are compared with one another. The fact that the last graph is more similar to the first than to the second implies which of the two aspects of the frequency-m.p.l. relationship (square root or inverse) has the most bearing on the function's overall shape.

Geometry provides another quite different way to look at the guitar from a mathematical perspective. There are some very basic ways of approaching the topic of the guitar from a geometric perspective that imply non-linearity. For instance, could the silhouette of the body of the average acoustic guitar (front view) be drawn using only straight lines? Though it is possible that Picasso would argue with the guitar's non-linearity, the typical artist would not attempt to represent a guitar with only straight lines. Since the guitar is a 'curvy' instrument, the shape of the guitar could best be modeled using non-linear functions. Now, a single function describing the entire shape of an acoustic guitar would be a very complex non-linear function. But a fairly good representation of a guitar can be created by looking at different parts of the guitar and estimating the shape of each part using portions of basic non-linear functions like hyperbolas, parabolas, and circles. (Admittedly, it would be hard to get around the problem of drawing the neck of the guitar without some straight lines. Linear functions are useful also.)

The following representation of a guitar was created using portions of some basic non-linear functions typically studied in algebra.



This particular representation is a combination of different non-linear functions, each used to represent part of the guitar. The top portion of the guitar can be modeled well with a parabola. The bottom appears to require a 'rounder' function, so a simple semi-circle is used. It can be observed in looking at the guitar that the two sides are symmetric to each other across the central vertical axis, so a function with that sort of symmetry is used, namely a simple hyperbola of the form $Ax^2 + y^2 = B$. The central 'sound hole' of the guitar is clearly modeled best by a circle.

The actual functions used in the above model of the guitar are as follows:

Top: $.13x^2 + y = 7$

Bottom: $y = -\sqrt{36 - x^2} - 3.5$

Sides: $-1.5x^2 + y^2 = 20$

Center: $x^2 + y^2 = 2$

Another way to look at the guitar from a geometric perspective is as an area bounded by portions of functions. In this way, the area of the guitar, or more specifically the area of the front of the body of the guitar, could be arrived at using integration techniques. The guitar face can be broken into different sections and the area of each section can easily be arrived at by integrating over the functions within the limits given by the intersections of those functions.

Teaching the Ideas

Many of the aforementioned ideas are quite pertinent to the secondary mathematics classroom because they relate solid mathematical ideas to an interesting real-world situation. It has long been thought that an integral part of effective mathematics teaching is applying mathematical concepts to real situations that students can relate to. Traditional story problems in textbooks do this to some degree, but there has been an even greater focus on this in recent years. Textbook companies have been attempting to answer the call of mathematics education specialists to offer effective real-world applications of mathematics, connections of mathematics to everyday life, and situations wherein students model real problems using mathematical concepts. The National Council of Teachers of Mathematics, for instance, in its 1989 *Curriculum and Evaluation Standards for School Mathematics*, calls for increased attention to be paid to real-world problems in many different subject areas. In the area of high school algebra, the *NCTM Standards* says that "the use of real-world problems to motivate and apply theory" should receive increased attention. In geometry, "real-world applications and modeling" should receive increased attention (NCTM, pg. 126).

The aforementioned ideas relating guitars to non-linear functions can be an effective integration of real-world applications in an engaging manner. One of the motives behind relating mathematics concepts to real-world problems is to show students how applicable mathematics can really be to their lives and to interest them more in the underlying content. The specific real-world applications in texts and classroom lessons must focus on applications in which students' interest lie. High school students tend to be quite interested in music, particularly rock 'n roll, and should find the integration of such a 'cool' instrument as a guitar into their mathematics curriculum to be an interesting and welcome change from their standard mathematics lessons.

The exploration of the relationship between string length and frequency could be looked at from several different perspectives as a potential classroom activity. A brief discussion of the relationship and its graph, along with a demonstration showing some of the characteristics of the relationship might be a nice way to integrate the idea into the classroom without demanding a great deal of time being spent on the idea. This discussion and demonstration would probably best accompany a lesson on inversely related variables or the function $f(x) = \frac{1}{x}$. An actual guitar could be brought into the classroom to act as an aural and visual accompaniment to the discussion, which could begin by telling students that there is a relationship between string length and frequency that acts like the relationships they have been studying.

A more in-depth exploration could be planned, though, which could allow the students personal experience finding appropriate models for a real-world problem. In this case, one or more guitars would need to be available to the students to explore the relationship between frequency and string length for themselves. This activity would require students to collect their own data on the relationship using either visual approximations of length and aural approximations of frequency (based on heard pitches), or more precise data collection using rulers and frequency measurement devices. Students would collect data and plot it to look at the shape of the function. It would be immediately apparent that the relationship is non-linear, but in order for students to determine the exact relationship, they would need to either look at the data more closely (possibly with some guidance from their teacher), or utilize a computer program that would allow them to do best-fit approximations of their data with several different types of non-linear relations.

The fact that the situation implies a restricted domain is another interesting point that could be taught in a number of different ways. The idea could be the basis for a class discussion, asking the students to think about what aspects of the real situation would put limits on the actual range of possibility for the variables. For a more personal exploration, students could instead be asked to think about and explore those limits themselves and arrive at specific values for those limits. They could then include their findings as the limits of their final graphs of the relationship between the variables.

In the other cases where frequency varies according to changes in tension or mass per unit length of the strings, explorations and discussions would be similar. The biggest difference between the frequency-length case and these two cases is the need in the latter two for a mechanism by which to determine tension on a string. It would be difficult to record data on the tension of a string that is already in its normal configuration on a regular guitar. A better way to determine the tension on a string is to simply apply a known force to it (using calibrated weights), which would require the use of some alternate device than the guitar's machine heads for applying the tension to the string. Suggestions for such devices can be found in science articles examining aspects of the guitar related to physics. One such device has the guitar placed on a table with its strings running past the machine heads on the guitar to a roller with grooves on it. The roller is attached to the corner of the table and the strings can hang from it over the edge of the table with weights hung from them (Hall, pg. 674).

Once some type of mechanism like the one explained is available for applying a known tension to a string, students can explore the relationship between frequency and both tension and mass per unit length. For the frequency-tension relationship, students would need to make observations, and graph their data, of the string's frequency while changing the amount of weight attached to the string. In the exploration of the frequency-m.p.l. relationship, students would not need to change the tension of the strings, but would need to have multiple different strings all with the same weight hanging from them. They could then make observations and plot data about the relationship between the frequency and mass per unit length. Weight could be changed on all of the strings to see if their findings were consistent for other tensions.

Again, these explorations lend themselves to an additional discussion or activity about limits on the range and domain of the function. Is there a minimum or maximum for any of the variables? A very poignant display of the upper limit for string tension would be found when a string snaps because its load is too heavy for it. Students could explore these types of limits on the different variables and discuss them, or could find the bounds and present them with their final graphs of the relationships.

With each of these explorations into the relationships between variables, there is a focus not only on "functions that are constructed as models of real-world problems," an important topic according to the *NCTM Standards*; but there is also a focus on the connections between the problem situation, the function in symbolic form, and the graph of that function, another area of emphasis in the *Standards*.

The geometric approaches to the non-linearity of the guitar also lend themselves to classroom activities that could draw on students' interests to apply mathematical concepts to real situations. Modeling the shape of the guitar using different functions may seem a fairly basic exercise, but it would be a good method of familiarizing students with those non-linear functions while maintaining their interest. The exercise, for which students could use most any basic equation-graphing software, would also be a way of allowing students' creativity to enter into their problem-solving, really challenging them to think for themselves and using the exact equation they deem best fit for each part of the guitar. The activity could also be extended to challenge students to think about how to change their graphical representations of guitars in order to move the guitar up, down, to the right or left, or to rotate the guitar. This would help solidify students' understandings of the relationship between their equations and the graphical representations thereof.

The more advanced idea of integrating to find the area of the face of the guitar bounded by the functions would be a good activity for an advanced calculus class. The activity would first necessitate that the students find their limits of integration by finding the intersections of the functions. Then they would have to use what they know about integration over two dimensions to break the figure into appropriate portions, then integrate to find the area of each portion. This activity would be relatively simple practice of calculus, but the connection to a real-world application should interest students more than integrating over shapes with no connection to real life.

Modeling aspects of the guitar is a creative exercise in problem-solving which would be beneficial to students' understandings of the functions they are dealing with. These ideas in general are examples of some of the many real-world applications of mathematics to topics in which students would likely be interested, thereby offering them more motivation to focus on the involved concepts. The guitar is something high school students find interesting. It also lends itself well to several explorations of non-linearity. Why not take advantage of student interests to help teach them? Thank goodness guitars are non-linear.

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